INTRINSIC COMPLETE TRANSVERSALS AND THE RECOGNITION OF EQUIVARIANT BIFURCATIONS

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We show how intrinsic complete transversals simplify both classification and recognition of equivariant bifurcations.

1. Intrinsic complete transversals

Intrinsic complete transversals provide a systematic way both to classify map-germs with respect to natural equivalences and to solve the recognition problem with respect to such a classification. We illustrate with an example from equivariant bifurcation theory.

Our methods are based on the following:

Theorem 1.1. Let $A$ be an affine space, $U$ a unipotent algebraic group acting almost affinely on $A$, $LU$ its Lie algebra, $T \subset V_A$ a subspace, and $x_0 \in A$ a point such that

1. $T + LU.x_0 = V_A$;
2. $l.(x_0 + t) - l.x_0 \in T$ for all $t \in T$ and all $l \in LU$.

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Then $x_0 + T$ meets every $U$-orbit in $A$ transversally. Furthermore, there is a subgroup $U_T$ of $U$ preserving $x_0 + T$ such that the intersection of any $U$-orbit with $x_0 + T$ is a $U_T$-orbit.

The affine space $x_0 + T$ appearing above is called an intrinsic complete transversal. The proof of this result will appear elsewhere.

The groups to which it will be applied are finite-dimensional quotients of subgroups of the group of contact equivalences. Such a group is unipotent if and only if the 1-jets of its elements are the direct sum of two unipotent linear isomorphisms.

These groups act on quotients of affine subspaces of jet-spaces; these actions are always "almost affine", so the definition of this concept may safely be omitted here.

2. Weight filtrations

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a sequence of positive integers. A monomial $x_1^{k_1} \cdots x_n^{k_n}$ has weight $\alpha_1 k_1 + \cdots + \alpha_n k_n$ with respect to $\alpha$.

The ideal in the ring $E_n$ of smooth function-germs $(R^n, 0) \to R$ generated by the monomials of weight at least $r$ with respect to $\alpha$ is denoted $F^r_{\alpha}E_n$; these ideals define a filtration of the ring $E_n$.

Now let $\beta = (\beta_1, \ldots, \beta_p)$ be another sequence of positive integers. The submodule of the $E_n$-module $E(n, p)$ of smooth map-germs $f : (R^n, 0) \to R^p$ generated by map-germs $f = (f_1, \ldots, f_p)$ with $f_i \in F^r_{\alpha+i}E_n$ for $1 \leq i \leq p$, is denoted $F^r_{\alpha, \beta}E(n, p)$; these submodules define a filtration of $E(n, p)$.

The group $R$ of germs of diffeomorphisms of $(R^n, 0)$ is filtered by the subgroups $F^r_{\alpha}R = (1 + F^r_{\alpha, \alpha}E(n, n)) \cap R$; in a similar way the subgroup $C$ of the group of diffeomorphism-germs of $(R^{n+p}, 0)$ is filtered by the subgroups $F^r_{\alpha, \beta}C = (1 + F^r_{\alpha+i, \alpha+i}E(n + p, n + p)) \cap C$.

The group $K = R \cdot C$ of contact equivalences is filtered by the subgroups $F^r_{\alpha, \beta}K = F^r_{\gamma}R \cdot F^r_{\alpha, \beta}C$; intersecting with a subgroup, e.g. the group $B$ of bifurcation equivalences, filters the subgroup also.

We note that the 1-jet of an element of $F^1K$ is the sum of two unipotent linear transformations: for if the coordinates in $R^n$, $R^p$ are re-ordered so that the corresponding weights are non-decreasing, then the matrices of these linear transformations with respect to the re-ordered coordinates are lower unitriangular, that is, triangular with 1’s in the main diagonal.

To simplify the notation, subscripts $\alpha, \beta$ indicating systems of weights will be discarded, and spaces of map-germs will be denoted simply $M$.

The action of $K$ on $M$ is compatible with weight filtrations:
Lemma 2.1. Let $f \in F^0 M$, $m \in F^k M$, $l \in L(F^r K)$. Then $l.(f+m) - l.f \in F^{k+r} M$.

For proof see $^2$, lemma 2.24(iii); for filtrations see $^2$, section 2.3.

A weighted determinacy result for $B$ now follows from $^5$, theorem 1.3:

Theorem 2.1. If $f \in F^0 M$ is a germ of bifurcation $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $T$ is a subspace of $F^k M$ such that

$$F^k M \subset L(F^1 B).f$$

then $f$ is $F^k M$-equivalent to $f + T + F^{k+1} M$.

An application of theorem 1.1 yields a complete transversal:

Theorem 2.2. If $f \in F^0 M$ is a germ of bifurcation $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $T$ is a subspace of $F^k M$ such that

$$F^k M \subset T + L(F^1 B).f + F^{k+1} M$$

then any $g \in F^0 M$ with $g - f \in F^k M$ is $F^1 B$-equivalent to $f + t + \phi$ for some $t \in T$ and $\phi \in F^{k+1} M$.

Proof. It follows from lemma 2.1 that $F^0 B$ acts on $M/F^k M$. Let $G$ be the subgroup fixing $f + F^k M$, and let $U = G \cap F^1 B/F^{k+1} B$; then $U$ is a unipotent algebraic group acting affinely on $f + F^k M/F^{k+1} M$. According to theorem 1.1 and lemma 2.1, $f + T + F^{k+1} M/F^{k+1} M$ is an intrinsic complete transversal to the action, and the result follows.

This result may also be obtained using the less general results on complete transversals in $^2$ (see theorem 2.25 there).

In the presence of a symmetry group $\Gamma$ acting on $\mathbb{R}^n$ we define a filtration $F^r M_\Gamma$ of the space $M_\Gamma$ of equivariant bifurcation-germs $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ by $F^r M_\Gamma = F^r M \cap M_\Gamma$, and a filtration $F^r B_\Gamma$ of the group $B_\Gamma$ of equivariant equivalences by $F^r B_\Gamma = F^r B \cap B_\Gamma$. For more information about $M_\Gamma$ and $B_\Gamma$ see $^7$, ch. XIV, §1.

The arguments already given adapt easily (adding subscript $\Gamma$ as appropriate) to yield $\Gamma$-equivariant versions of theorem 2.1 (applying $^5$, theorem 1.15) and theorem 2.2.
3. $D_4$-symmetric bifurcations

Let $D_4$ act on $\mathbb{R}^2$ as symmetries of the square, so that the action is generated by $(x, y) \mapsto (x, -y)$ and $(x, y) \mapsto (-y, x)$.

The invariant theory for this action of is described in Golubitsky et. al. 7 and Golubitsky and Roberts 6. We write

$$E_1(x, y) = (x, y), \quad E_2(x, y) = \delta(x, y)(x, -y),$$

where $\delta(x, y) = y^2 - x^2$; these generate the equivariant maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ as a module over the ring $\mathcal{E}_{D_4}$ of invariant functions, itself generated by $N(x, y) = x^2 + y^2$ and $\Delta(x, y) = \delta(x, y)^2$.

A $D_4$-equivariant bifurcation is thus a map-germ

$$g(x, y, \lambda) = p(N, \Delta, \lambda)E_1(x, y) + r(N, \Delta, \lambda)E_2(x, y),$$

at $(0, 0)$ with $dg(0,0) = 0$.

The classification of $D_4$-equivariant bifurcations up to $B_{D_4}$ has been carried out up to topological codimension two by Golubitsky and Roberts 6 (topological codimension is the difference between the orbit codimension and the number of moduli involved).

The simplest of these bifurcations have codimension one and one modulus; their orbits are represented by the normal forms

$$(\epsilon_0 \lambda + mN)E_1 + \epsilon_1 E_2, \quad \epsilon_0, \epsilon_1 = \pm 1, \quad m \neq 0, \epsilon_1, \quad (I)$$

where $m$ is the modulus. The more degenerate bifurcations are listed in Golubitsky et. al. 7, pp.342-3; we single out the following bifurcations of codimension two with one modulus, represented by the normal forms

$$(\epsilon_0 \lambda^2 + mN + \epsilon_1 \lambda N)E_1 + \epsilon_2 E_2, \quad \epsilon_0, \epsilon_1, \epsilon_2 = \pm 1, \quad m \neq 0, \epsilon_2, \quad (V)$$

where $m$ is the modulus.

We will illustrate how the use of intrinsic complete transversals simplifies both the classification process and the question of recognising which orbit a given bifurcation is contained in, by considering orbits of types I and V.

It is convenient to introduce weights $1, 1$ for $x, y, \lambda$ in the source, and weights $3, 3$ in the target. We note that all equivariant monomial vectors are of even weight; so $F^{2r-1}M_{D_4} = F^{2r}M_{D_4}$ for $r \geq 1$. Thus, we have

$$F^0M_{D_4} = \mathbb{R}.\{\lambda E_1, NE_1, E_2\} + F^2M_{D_4},$$

$$F^2M_{D_4} = \mathbb{R}.\{\lambda^2 E_1, \lambda NE_1, \Delta E_1, N^2 E_1, \lambda E_2, NE_2\} + F^4M_{D_4},$$

$$F^4M_{D_4} = \mathbb{R}.\{\lambda^3 E_1, \lambda^2 NE_1, \lambda \Delta E_1, N^3 E_1, N \Delta E_1, \lambda^2 E_2, \lambda NE_2, N^2 E_2, \Delta E_2\} + F^6M_{D_4}.$$
We will write $\mathcal{U} = F^1 \mathcal{B}_{D_4}$; we note that $\mathcal{U}$ is the subgroup of $\mathcal{B}_{D_4}$ of equivalences whose 1-jets are the identities.

Consider now a bifurcation $h \in M_{D_4}$; the terms of weight zero in $h$ form a bifurcation

$$h_0(x, y, \lambda) = (a \lambda + bN)E_1 + cE_2, \quad a, b, c \in \mathbb{R},$$

differing from $h$ by terms of (relative) weight at least 2. To see whether $h$ is $\mathcal{U}$-equivalent to $h_0$, we seek to apply theorem 2.1, and must thus calculate whether $F^2 M_{D_4} \subset L\mathcal{U}. h_0$.

The lowest-weight terms $(X, \Lambda, S) \in L\mathcal{U}$ are $\Lambda(\lambda) \in \mathbb{R}. \lambda^2$, $X(x, y, \lambda) \in \mathbb{R}. \{\lambda E_1, NE_1, E_2\}$, and $S(x, y, \lambda) \in \mathbb{R}. \{\lambda S_1, NS_1, \Delta S_1, S_2, S_3, S_4\}$, where $S_i$ is one of the $D_4$-equivariant matrices computed by Golubitsky and Roberts (see also 7, chapter XVII).

It is easy to see that $F^2 M_{D_4}$ is generated by $\lambda^2 E_1$, $\Delta E_1$, $\lambda N E_1$, $N^2 E_1$, $\lambda E_2$, $N E_2$, $\Delta E_2$ over $\mathcal{E}_{D_4}$. The elements of $L\mathcal{U}. h_0$ obtained from the $\Lambda, X, S$ listed above may be written as $\mathcal{E}_{D_4}$-linear combinations of these generators; the rows of the following matrix $A$ give the coefficients:

$$\begin{bmatrix}
  a & 0 & 0 & 0 & 0 & 0 & 0 \\
  a & 3b & 0 & 0 & 3c & 0 & 0 \\
  0 & a & 3b & 0 & 0 & 3c & 0 \\
  0 & 0 & 0 & c - 2b & a & b - 2c & 0 \\
  a & b & 0 & 0 & c & 0 & 0 \\
  0 & a & b & 0 & 0 & c & 0 \\
  0 & 0 & 0 & a \lambda + bN & 0 & 0 & c \\
  0 & a & b & -c & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & a & b - c & 0 \\
  \frac{1}{4}aN & \frac{1}{4}bN & -\frac{1}{4}(a \lambda + bN) & 0 & -\frac{1}{4}N & \frac{1}{4} & 0
\end{bmatrix}$$

Thus $h_0$ is $F^2 M_{D_4}$-determined if and only if $L\mathcal{U}. h_0 = F^2 M_{D_4}$, if and only if $A$ has rank 7; so if and only if $a, b, c \neq 0$ and $c \neq -b$. Equivariant scaling shows that for such $a, b, c h_0$, and so $h$ also, lie in an orbit of type I, solving the recognition problem for this orbit, Golubitsky and Roberts.

Now we consider the case $a = 0$. The first column of $A$ is now zero, and the second and fifth columns are linearly dependent. Thus we have

$$F^2 M_{D_4} = T + L\mathcal{U}. h_0$$

where $T = \mathbb{R}. \{\lambda^2 E_1, \lambda N E_1, \lambda E_2\}$; of course only a two-dimensional space is required here, but it is not obvious which is to be preferred at this point.
Applying theorem 1.1, $T$ projects to an intrinsic complete transversal in $h_0 + F^2 M_{D_4}/P$, where $P$ is any intrinsic submodule of $F^2 M_{D_4}$ of finite codimension; we suppose $P \subset F^0 M_{D_4}$.

We note $dh_0 \lambda E_1, (\lambda S_1)h_0 \in T$, identifying $\lambda E_1, \lambda S_1$ as elements of $LU_T$. These two elements correspond to the change of coordinates

$$ x = X + a_2 t\lambda X, y = Y + a_2 t\lambda Y, $$

and the matrix transformation $S = I + a_1 t\lambda I$. Applying these to the germ $g(x, y, \lambda) = h_0(x, y, \lambda) + \alpha \lambda^2 E_1 + \beta \lambda NE_1 + \gamma \lambda E_2$ produces (modulo higher-order terms, which we can assume to be contained in $P$)

$$ bNE_1 + cE_2 + \alpha \lambda^2 E_1 + ((3a_2 + a_1)bt + \beta)\lambda NE_1 + ((3a_2 + a_1)ct + \gamma)\lambda E_2 $$

which is equivalent to $h_0$ modulo $P$ if and only if $\alpha = 0$ and $\gamma b - c\beta = 0$; these yield the non-degeneracy conditions $p_{\lambda\lambda} \neq 0$ and $r_{\lambda P N} - r_{P N} \neq 0$ of $^6$, see $^7$, pp.342-3.

In fact, we can suppose $\gamma = 0$. For let $f = h_0 + \alpha \lambda^2 E_1 + \beta \lambda NE_1$ and $T_1 = \mathbb{R}\{\lambda^2 E_1, \lambda NE_1\}$. Then $LU_T.f + T_1 = T$, so by theorem 1.1, $f$ is equivalent to $g$; here we take $P = \mathcal{P}(f)$, the higher-order terms for $f$ (according to $^5$, theorem 1.17, this is the intrinsic part of $LU.f$).

Equivariant rescaling shows that $f$ lies in an orbit of type V. (A similar argument shows that we could reduce to the case $\beta = 0$ instead of $\gamma = 0$, giving different, but equivalent, normal forms.) Summing up, we see that any bifurcation with $p_{\lambda} = 0$ (i.e. $a = 0$) but satisfying the other non-degeneracy conditions obtained above, lies in an orbit of type V. We have thus solved the recognition problem for these orbits. The conclusion is of course that of $^6$, but the arguments given here are both much shorter and more systematic.

References

4. J. Damon, “The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$”, *Proceedings of Symposia in Pure Mathematics* 40, 233 (1983).