Counting Persistent Pitchforks

May 2, 2002

Sofia Castro  
Grupo de Matemática e Informática  
Faculdade de Economia  
Universidade do Porto  
Rua Dr. Roberto Frias  
4200 Porto  
sdcastro@fep.up.pt

Isabel Labouriau  
Departamento de Matemática Aplicada  
Faculdade de Ciências  
Universidade do Porto  
Rua das Taipas 135  
4050 Porto  
islabour@fc.up.pt

Centro de Matemática Aplicada  
Universidade do Porto  
Rua das Taipas 135  
4050 Porto  
Portugal

Abstract

We present an invariant for bifurcation problems with symmetry that counts the maximum number of pitchfork bifurcations, primary or not, arising in an unfolding.

1 introduction

In bifurcation problems

\[ g(X, \lambda) = 0 \quad X \in \mathbb{R}^n \quad \lambda \in \mathbb{R} \]

folds constitute the only nontrivial persistent class. In problems with symmetry, however, pitchforks arise as another class of generic objects. In this paper we present a method for counting the maximum number of pitchforks that may appear when a given bifurcation problem with symmetry is perturbed. A method for counting the maximum number of folds arising from perturbation of a given germ has been developed in [?].
Knowledge of the maximum number of generic objects in bifurcation is particularly useful in the study of high codimension problems, as we shall illustrate in section ??.

The complexity and abundance of pitchforks arising when a bifurcation problem is unfolded bears a strong relationship with the symmetries present. We explore this relationship in our method of counting, presented in algorithmic form in section ?? . The algorithm can be used in more than one way to provide different levels of information. In section ?? we prove the effectiveness of the algorithm.

We begin by defining the objects to be counted.

**Definition:** A pitchfork is a transverse intersection of two branches. Symmetry forces one of the branches to be folded and guarantees persistence.

Example: The standard pitchfork is defined by $f(x, \lambda) = x^3 - x\lambda = 0$, a $\mathbb{Z}_2$-equivariant bifurcation problem, with $\mathbb{Z}_2$ acting on $\mathbb{R}$ as $x \rightarrow -x$. The solutions of $f = 0$ near $(0, 0)$ consist of two branches: one being the handle (the trivial orbit $x = 0$) and the other being the prongs of the fork. Each prong is the reflection of the other on the mirror that coincides with the handle.

In higher dimensions persistent pitchforks maintain this structure: two branches meet in a fixed point subspace (mirror) of some isotropy subgroup containing one of them, the handle. The prongs are reflected images one another on this mirror.

## 2 algorithm

Let $\Gamma$ be a compact Lie group acting linearly on $\mathbb{R}^n$ and $\tilde{g}$ a $\Gamma$-equivariant bifurcation problem, i.e., $\tilde{g} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\tilde{g}(\gamma X, \lambda) = \gamma \tilde{g}(X, \lambda).$$

1. Fix a chain in the isotropy lattice of $\Gamma$:

$$\begin{align*}
\Gamma & \uparrow \\
\Sigma_1 & \\
\vdots & \\
\Sigma_{s-1} & \uparrow \\
\Sigma_s &
\end{align*}$$

and look at two consecutive subgroups in the chain:

$$\Sigma_{i+1} \longrightarrow \Sigma_i \quad \text{with} \quad \text{Fix}\Sigma_i \subset \text{Fix}\Sigma_{i+1}.$$

2. Define the germ $g$ as

$$g = \tilde{g}|_{\text{Fix}\Sigma_{i+1} \times \mathbb{R}} : \text{Fix}\Sigma_{i+1} \times \mathbb{R} \rightarrow \text{Fix}\Sigma_{i+1}.$$

Note that $\Gamma$ acts on $\text{Fix}\Sigma_{i+1}$ as $N(\Sigma_{i+1})/\Sigma_{i+1}$ and that $\text{Fix}\Sigma_i \subset \text{Fix}\Sigma_{i+1}$ is $g$-invariant.
3. Choose coordinates \((x, y)\) in \(\text{Fix}\Sigma_{i+1}\) using the decomposition

\[
\text{Fix}\Sigma_{i+1} = \text{Fix}\Sigma_i \oplus (\text{Fix}\Sigma_i)^\perp
\]

then

\[
g(x, y, \lambda) = (g_0(x, y, \lambda), \bar{g}(x, y, \lambda))
\]

with

\[
g_0 : \text{Fix}\Sigma_{i+1} \times \mathbb{R} \rightarrow \text{Fix}\Sigma_i
\]
\[
\bar{g} : \text{Fix}\Sigma_{i+1} \times \mathbb{R} \rightarrow (\text{Fix}\Sigma_i)^\perp.
\]

By invariance of \(\text{Fix}\Sigma_i\) we have \(\bar{g}(x, 0, \lambda) \equiv 0\) and therefore

\[
\bar{g}(x, y, \lambda) = (y \cdot g_1(x, y, \lambda), \ldots, y \cdot g_m(x, y, \lambda))
\]

where \(m = \dim(\text{Fix}\Sigma_i)^\perp\) and \(\cdot\) is the scalar product.

4. For each \(j = 1, \ldots, m\) define

\[
X_j = (x, 0, \ldots, 0, y_{j+1}, \ldots, y_m, \lambda) \in \text{Fix}\Sigma_i \oplus (\text{Fix}\Sigma_i)^\perp \times \mathbb{R}
\]

\[
Y_j = (0, \ldots, 0, 1, y_{j+1}, \ldots, y_m) \in (\text{Fix}\Sigma_i)^\perp
\]

and let

\[
f_j = (g_0(X_j), Y_j \cdot g_1(X_j), \ldots, Y_j \cdot g_m(X_j))
\]

5. For each \(j = 1, \ldots, m\) define

\[
\varphi_{ij} = \text{cod} (g_0(X_j), Y_j \cdot g_1(X_j), \ldots, Y_j \cdot g_m(X_j))_{\mathcal{E}X_j}
\]

Compute

\[
\varphi_{\Sigma_i}(\bar{g}) = \sum_{j=1}^m \varphi_{ij}
\]

3 main result

**Theorem 3.1** Let \(\tilde{g}\) be a \(\Gamma\)-equivariant bifurcation problem of finite codimension and let \(\Sigma_{i+1} \subset \Sigma_i\) be consecutive isotropy subgroups of \(\Gamma\). Then any germ in the unfolding of \(\tilde{g}\) has at most \(\varphi_{\Sigma_i}\) pitchfork bifurcations from \(\text{Fix}\Sigma_i\) to \(\text{Fix}\Sigma_{i+1}\).

**Proof:**

The first \?? steps in the algorithm are just the restriction of \(\tilde{g}\) to \(\text{Fix}\Sigma_i\) with a suitable choice of coordinates. After step \??, define \(M_1\) and \(M_2 \subset \text{Fix}\Sigma_{i+1} \times \mathbb{R}\) as the germs of the sets of solutions of

\[
g_0(x, y, \lambda) = 0 \quad \text{and} \quad \frac{\bar{g}(x, y, \lambda)}{|y|} = 0,
\]

respectively. After unfolding, the set \(M_1 \cap \text{Fix}\Sigma_i\) corresponds to the handle of the fork, and \(M_2 \cap M_1\) will give rise to the prongs. Thus the number of pitchforks from \(\text{Fix}\Sigma_i\) to \(\text{Fix}\Sigma_{i+1}\)
in the unfolding of $\tilde{g}$ is at most the multiplicity of $(M_2 \cap M_1) \cap \text{Fix}\Sigma_i$, i.e. the number of common zeros (counted with multiplicity) of

$$g_0(x, y, \lambda) \quad \text{and} \quad \frac{\tilde{g}(x, y, \lambda)}{|y|} \quad \text{in} \quad \text{Fix}\Sigma_i.$$ 

We claim that these zeros are counted in step $??$. Results of Damon & Galligo [?] show that $\varphi_{ij}$ is the maximum number of zeros of germs in the unfolding of $f_j$. It remains to see that these zeros are the pitchforks.

If $m = 1$ the variable $y$ defined in step $??$ is 1-dimensional, $\tilde{g}(x, y, \lambda) = yg_1(x, y, \lambda)$ and $M_2 = \{(x, y, \lambda) : g_1(x, y, \lambda) = 0\}$. In step $??$, $X_j = (x, 0, \lambda)$, $Y_j = 1$ and $f_j = (g_0(x, 0, \lambda), g_1(x, 0, \lambda))$. Setting $y = 0$ corresponds to being in $\text{Fix}\Sigma_i$ since the $y$ are coordinates in $(\text{Fix}\Sigma_i)\perp$ thus proving our claim in this case.

For larger values of $m$ we replace the division by $|y|$ by a directional blow-up followed by a restriction to the subspace not covered by the blow-up. After $m$ steps we have replaced the counting of pitchforks by that of zeros of all $f_j$. Thus in step $??$ of the algorithm we obtain the desired number of pitchforks.

**Corollary 1** Let $\tilde{g}$ be a $\Gamma$-equivariant bifurcation problem of finite codimension and consider a chain on the isotropy lattice of $\Gamma$ as in step $??$ of the algorithm. Then for any germ in the unfolding of $\tilde{g}$ the number of pitchforks bifurcating from any of the invariant subspaces $\text{Fix}\Sigma_i$ is at most

$$\sum_i \varphi_{\Sigma_i}.$$ 

**4 examples**

In this section we present two examples of how the counting works. In both cases the maximum number of pitchfork bifurcations counted is realized in an unfolding of the germ.

In the first case, the pitchforks are non-planar as they occur in a problem equivariant under a non-irreducible action of $\mathbb{Z}_2$ on $\mathbb{R}^2$. The second example is that of a codimension 2 $D_4$-equivariant problem and we count both primary and secondary pitchforks, illustrating how many pitchforks are counted at each step in the algorithm. With this we show that the algorithm can be used in several ways providing different levels of information.

**4.1 $\mathbb{Z}_2$ acting on $\mathbb{R}^2$**

Consider the following action of $\mathbb{Z}_2$ on $\mathbb{R}^2$

$$\kappa(x, y) = (x, -y) \quad \forall \ (x, y) \in \mathbb{R}^2.$$ 

Bifurcation problems with this symmetry have been studied in [?], chapter XIX.

The isotropy lattice for this problem consists of only one simple chain as follows

$$\begin{array}{c}
\mathbb{Z}_2 \\
\uparrow \\
1 \\
4
\end{array}$$
with \( \text{Fix}(\mathbb{Z}_2) = \{(x, 0) : x \in \mathbb{R}\} \) and \( \text{Fix}(1) = \mathbb{R}^2 \). The germs \( \tilde{g} \) up to \( \mathbb{Z}_2 \)-topological codimension 2 can be found in Table 3.1 in [?] and coincide with the germs \( g \) in step 2 of the algorithm. The coordinates in step 3 can be chosen to be the standard coordinates in \( \mathbb{R}^2 \) and the decomposition of \( \mathbb{R}^2 \) is

\[
\mathbb{R}^2 = \{(x, 0) : x \in \mathbb{R}\} \oplus \{(0, y) : y \in \mathbb{R}\}.
\]

In steps 4 and 5, \( m = 1 \) and

\[
X_1 = (x, 0) \quad \text{and} \quad Y_1 = 1.
\]

Since there is only a simple chain in the isotropy lattice, \( \wp_\Sigma(\tilde{g}) = \wp_{ij} \).

The results of the counting are presented in Table 1.

### 4.2 \( D_4 \) acting on \( \mathbb{R}^2 \)

We consider the standard action of \( D_4 \) on \( \mathbb{R}^2 \) generated by

\[
\kappa(x, y) = (x, -y) \\
\zeta(x, y) = (-y, x).
\]

The isotropy subgroups of \( D_4 \) for this action are \( \mathbb{Z}_2(\kappa) \) and \( \mathbb{Z}_2(\zeta \kappa) \) generated by \( \kappa \) and \( \zeta \kappa \), respectively. The fixed-point spaces for these isotropy subgroups are as follows:

\[
\text{Fix}(\mathbb{Z}_2(\kappa)) = \{(x, 0) : x \in \mathbb{R}\},
\]

\[
\text{Fix}(\mathbb{Z}_2(\zeta \kappa)) = \{(x, x) : x \in \mathbb{R}\}.
\]

Both isotropy subgroups are maximal and the isotropy lattice is

\[
\begin{array}{ccc}
D_4 & \rightarrow & Z_2(\kappa) \\
\downarrow & & \downarrow \\
Z_2(\kappa) & \leftarrow & Z_2(\zeta \kappa)
\end{array}
\]

Bifurcation problems equivariant under this action of \( D_4 \) are studied in [?], chapter XVII.

We present the calculations for the germ

\[
\tilde{g}(x, y, \lambda) = \left( \frac{\lambda^2 + \epsilon_1(x^2 + y^2) + n(y^2 - x^2)^3 + \lambda(y^2 - x^2)x}{(\lambda^2 + \epsilon_1(x^2 + y^2) - n(y^2 - x^2)^3 - \lambda(y^2 - x^2)y} \right).
\]
normal form XIII in \[\text{?}\], which has topological codimension 2 and a modal parameter \(n\), with \(\epsilon_1 = \pm 1\). Next we calculate the maximum number of pitchfork bifurcations using our algorithm.

**Step 1** We fix the chain on the left of the isotropy lattice, with fixed point spaces

\[
\begin{align*}
1 & \rightarrow Z_2(\kappa) \rightarrow D_4 \\
\mathbb{R}^2 & \\& \supset (x,0) \supset (0,0)
\end{align*}
\]

and start counting from right to left in this chain.

**Step 2.1**

\[
g : \text{Fix}(Z_2(\kappa)) \times \mathbb{R} \rightarrow \text{Fix}(Z_2(\kappa)) \\
(x,0,\lambda) \rightarrow \tilde{g}(x,0,\lambda)
\]

**Step 3.1** Write

\[
\text{Fix}(Z_2(\kappa)) = \text{Fix}(D_4) \oplus \text{Fix}(Z_2(\kappa)).
\]

Then \(g_0(x,0,\lambda) \equiv 0\) and \(\tilde{g}\) is the projection of \(\tilde{g}\) into \(\text{Fix}(Z_2(\kappa))\).

**Step 4.1** We have \(m = 1, X_1 = (0,0,\lambda)\) and \(Y_1 = 1\). Thus \(f_1 = (\lambda^2,0)\).

**Step 5.1** We obtain \(\varphi_{\Sigma_1}(\tilde{g}) = 2\).

**Step 2.2**

\[
g : \text{Fix}(1) \times \mathbb{R} \rightarrow \text{Fix}(1) \\
(x,y,\lambda) \rightarrow \tilde{g}(x,y,\lambda)
\]

**Step 3.2** Write

\[
\text{Fix}(1) = \text{Fix}(Z_2(\kappa)) \oplus \{(0,y) : y \in \mathbb{R}\}.
\]

Note that \(g \equiv \tilde{g}\) and therefore \(g_0\) and \(\tilde{g}\) are the components of \(\tilde{g}\).

**Step 4.2** Again \(m = 1\). We have \(X_1 = (x,0,\lambda)\), \(Y_1 = 1\) and

\[
f_1 = ((\lambda^2 + \epsilon_1 x^2 - nx^6 - \lambda x^2)x, \lambda^2 + \epsilon_1 x^2 + nx^6 + \lambda x^2).
\]

**Step 5.2** We obtain \(\varphi_{\Sigma_1}(\tilde{g}) = 8\).

A few remarks are now in order. The two pitchforks counted in step 5.1 bifurcate from the trivial solution, for two different values of \(\lambda\), into the \((\lambda,0)\)-plane. We call them primary pitchforks. They can be seen in the bifurcation diagrams in \([\text{?}]\), page 357. Looking at these bifurcation diagrams we expect to see two pitchforks bifurcating from the primary branch (only half of the branch is drawn in \([\text{?}]\)). The information in step 5.2 is that there are 8 pitchforks. Hence, there are 6 hidden pitchforks. This is not surprising if we recall that the pitchforks counted in step 5.2 are those branching from the \((\lambda,0)\)-plane into
the 3-dimensional space. Therefore the counting includes pitchforks bifurcating from any fixed-point space contained in the \( \{(\lambda, x, 0)\} \)-plane. Recall that \( \{(\lambda, 0, 0)\} \) is \( \mathbb{R} \times \text{Fix}(D_4) \).

The 3-dimensional space contains several other fixed-point spaces, namely

- \( \{(0, y) : y \in \mathbb{R}\} \), the fixed-point space conjugated to that of \( Z_2(\kappa) \);
- \( \{(x, x) : x \in \mathbb{R}\} \), the fixed-point space of \( Z_2(\zeta \kappa) \) and
- \( \{(x, -x) : x \in \mathbb{R}\} \), the fixed-point space conjugated to that of \( Z_2(\zeta \kappa) \).

By conjugacy, there will be 2 primary pitchforks in \( \{(0, y) : y \in \mathbb{R}\} \). The 4 remaining pitchforks must bifurcate into one or both of the last two fixed-point spaces above. We must have 2 pitchforks bifurcating into each since they are conjugated to each other and therefore must have the same number of bifurcating branches. Note that we have neither counted any secondary pitchforks in the other chain of the isotropy lattice nor secondary pitchforks from the conjugated fixed-point space of \( Z_2(\kappa) \). The latter we know must exist by conjugacy and are in number of 2. To decide whether there are secondary pitchforks bifurcating from \( \text{Fix}(Z_2(\zeta \kappa)) \) we must count along the other chain of the isotropy lattice. This is what we do next.

**Step 1** We fix the chain on the right of the isotropy lattice, with fixed point spaces

\[
\begin{align*}
1 \rightarrow & \quad Z_2(\zeta \kappa) \rightarrow D_4 \\
\mathbb{R}^2 \supset & \quad \{(x, x)\} \supset \{(0, 0)\}
\end{align*}
\]

and start counting from right to left in this chain.

**Step 2.1**

\[
g : \text{Fix}(Z_2(\zeta \kappa)) \times \mathbb{R} \rightarrow \text{Fix}(Z_2(\zeta \kappa))
\]

\[
(x, x, \lambda) \rightarrow \tilde{g}(x, x, \lambda)
\]

**Step 3.1** Change coordinates as follows

\[
X = x + y \quad \text{and} \quad Y = y - x
\]

and write

\[
\text{Fix}(Z_2(\zeta \kappa)) = \text{Fix}(D_4) \oplus \text{Fix}(Z_2(\zeta \kappa))
\]

where \( \text{Fix}(Z_2(\zeta \kappa)) = \{(X, 0) : X \in \mathbb{R}\} \). Rename the new variables with small letters. Then \( g_0(x, 0, \lambda) \equiv 0 \) and \( \tilde{g} \) is the projection of \( \tilde{g} \) into \( \text{Fix}(Z_2(\zeta \kappa)) \).

The counting proceeds exactly as in the previous chain and we obtain the same numbers. Again when counting secondary pitchforks along this chain, we shall recover the primary pitchforks in the first chain. We shall neither recover secondary pitchforks in the first chain nor count secondary pitchforks bifurcating from the fixed-point space conjugated to \( \text{Fix}(Z_2(\zeta \kappa)) \) (these we know must be 2 by conjugacy). Therefore the new information we obtain from counting along this chain is the existence of 2 secondary pitchforks bifurcating from \( \text{Fix}(Z_2(\zeta \kappa)) \) and 2 conjugated ones. Hence, the counting along this second chain could have been simplified and only done for

\[
1 \rightarrow Z_2(\zeta \kappa).
\]

We obtain a total number of 16 pitchforks, 8 primary and 8 secondary.
5 acknowledgements

This work was carried out under the auspices of a grant from “Fundação para a Ciência e Tecnologia”, Portugal and of grant 2/2.1/MAT/407/94 of program PRAXIS XXI.

References


