Approximate Solutions of Second Kind Integral Equations

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Abstract

Integral equations are used as mathematical models in many physical situations. Two important problems associated with integral equations are solutions of second kind operator equations and of the associated eigenvalue problem. As these problems can rarely be solved exactly, approximate solutions are obtained by replacing the integral operator by a continuous finite rank operator. The original operator equation problem is then replaced by a sustem of linear equations and the eigenvalue problem for the integral operator is replaced by a matrix eigenvalue problem. Here we restrict ourselves to a relatively simpler problem of solution of operator equation.

We consider the following operator equation

$$x - Tx = f, \tag{1}$$

where T is a compact linear operator defined on an infinite dimensional space. Integral operator

$$Tx(s) = \int_a^b k(s,t)x(t)dt$$

with a continuous kernel $k(.,.)$ and defined on $C[a,b]$ is an important example of a compact operator.

It is assumed that $(I - T)$ is invertible so that the above problem has a unique solution for any right hand side f.

There are two major ways of obtaining a sequence of continuos finite rank operators T_n converging to T in an appropriate sense. The Nyström method is obtained by replacing the integral in the integral operator by a numerical quadrature formula, whereas a sequence of projections converging pointwise to the identity operator gives rise to various projection methods. While the classical Galerkin method is

associated with the orthogonal projections, the interpolatory projections are used to define collocation methods. The iterated versions of these methods were introduced by Sloan.

The operator equation (1) is then approximated by

$$x_n - T_n x_n = f (\text{or } f_n).$$

It is important that the sequence T_n should be such that

- $(I - T_n)$ is invertible for n large enough so that the above equation is uniquely solvable.
- The approximate solution x_n should converge to x as $n \to \infty$.
- x_n should be computable (with the help of a computer).

The performance of a particular method is judged by the rate of convergence of x_n to x and the number of computations.

In these lectures precise orders of convergence in the Nyström method, Galerkin and collocation methods along with their iterated versions will be obtained. The implementation details for these methods will also be discussed.