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TIME TO BUILD AND AGGREGATE FLUCTUATIONS

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The equilibrium growth model is modified and used to explain the cyclical variances of a set of economic time series, the covariances between real output and the other series, and the autocovariance of output. The model is fitted to quarterly data for the post-war U.S. economy. Crucial features of the model are the assumption that more than one time period is required for the construction of new productive capital, and the non-time-separable utility function that admits greater intertemporal substitution of leisure. The fit is surprisingly good in light of the model's simplicity and the small number of free parameters.

1. INTRODUCTION

THAT WINE IS NOT MADE in a day has long been recognized by economists (e.g., Böhm–Bawerk [6]). But, neither are ships nor factories built in a day. A thesis of this essay is that the assumption of multiple-period construction is crucial for explaining aggregate fluctuations. A general equilibrium model is developed and fitted to U.S. quarterly data for the post-war period. The co-movements of the fluctuations for the fitted model are quantitatively consistent with the corresponding co-movements for U.S. data. In addition, the serial correlations of cyclical output for the model match well with those observed.

Our approach integrates growth and business cycle theory. Like standard growth theory, a representative infinitely-lived household is assumed. As fluctuations in employment are central to the business cycle, the stand-in consumer values not only consumption but also leisure. One very important modification to the standard growth model is that multiple periods are required to build new capital goods and only finished capital goods are part of the productive capital stock. Each stage of production requires a period and utilizes resources. Half-finished ships and factories are not part of the productive capital stock. Section 2 contains a short critique of the commonly used investment technologies, and presents evidence that single-period production, even with adjustment costs, is inadequate. The preference-technology-information structure of the model is presented in Section 3. A crucial feature of preferences is the non-time-separable utility function that admits greater intertemporal substitution of leisure. The exogenous stochastic components in the model are shocks to technology and imperfect indicators of productivity. The two technology shocks differ in their persistence.

The steady state for the model is determined in Section 4, and quadratic approximations are made which result in an "indirect" quadratic utility function that values leisure, the capital goods, and the negative of investments. Most of

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the relatively small number of parameters are estimated using steady state considerations. Findings in other applied areas of economics are also used to calibrate the model. For example, the assumed number of periods required to build new productive capital is of the magnitude reported by business, and findings in labor economics are used to restrict the utility function. The small set of free parameters imposes considerable discipline upon the inquiry. The estimated model and the comparison of its predictions with the empirical regularities of interest are in Section 5. The final section contains concluding comments.

2. A CRITIQUE OF CONVENTIONAL AGGREGATE INVESTMENT TECHNOLOGIES

There are two basic technologies that have been adopted in empirical studies of aggregate investment behavior. The first assumes a constant-returns-to-scale neoclassical production function F with labor L and capital K as the inputs. Total output $F(K,L)$ constrains the sum of investment and consumption, or $C + I \leq F(K,L)$, where $C, I, K, L \geq 0$. The rate of change of capital, $\dot{K}$, is investment less depreciation, and depreciation is proportional with factor δ to the capital stock, that is, $\dot{K} = I - \delta K$. This is the technology underlying the work of Jorgenson [**19**] on investment behavior.

An implication of this technology is that the relative price of the investment and consumption goods will be a constant independent of the relative outputs of the two goods.[2] It also implies that the shadow price of existing capital will be the same as the price of the investment good.[3] There is a sizable empirical literature that has found a strong association between the level of investment and a shadow price of capital obtained from stock market data (see [**26**]). This finding is inconsistent with this assumed technology as is the fact that this shadow price varies considerably over the business cycle.

The alternative technology, which is consistent with these findings, is the single capital good adjustment cost technology.[4] Much of that literature is based upon the problem facing the firm and the aggregation problem receives little attention. This has led some to distinguish between internal and external adjustment costs. For aggregate investment theory this is not an issue (see [**29**]) though for other questions it will be. Labor resources are needed to install capital whether the acquiring or supplying firm installs the equipment. With competitive equilibrium it is the aggregate production possibility set that matters. That is, if the Y_j are the production possibility sets of the firms associated with a given industrial organi-

[2] This, of course, assumes neither C nor I is zero. Sargent [**32**], within a growth context with shocks to both preferences and technology, has at a theoretical level analyzed the equilibrium with corners. Only when investment was zero did the price of the investment good relative to that of the consumption good become different from one and then it was less than one. This was not an empirical study and Sargent states that there currently are no computationally practical econometric methods for conducting an empirical investigation within that theoretical framework.

[3] The shadow price of capital has been emphasized by Brunner and Meltzer [**7**] and Tobin [**36**] in their aggregate models.

[4] See [**1, 17**] for recent empirical studies based on this technology.

zation and Y_j' for some other industrial organization, the same aggregate supply behavior results if $\sum Y_j = \sum Y_j'$.

The adjustment cost model, rather than assuming a linear product transformation curve between the investment and consumption goods, imposes curvature. This can be represented by the following technology:

$$G(C,I) \leq F(K,L), \qquad \dot{K} = I - \delta K,$$

where G like F is increasing, concave, and homogeneous of degree one. Letting the price of the consumption good be one, the price of the investment good q_t, the rental price of capital r_t, and the wage rate w_t, the firm's problem is to maximize real profits, $C_t + q_t I_t - w_t L_t - r_t K_t$, subject to the production constraint. As constant returns to scale are assumed, the distribution of capital does not matter, and one can proceed as if there were a single price-taking firm. Assuming an interior solution, given that this technology displays constant returns to scale and that the technology is separable between inputs and outputs, it follows that $I_t = F(K_t, L_t)h(q_t) \equiv Z_t h(q_t)$, where Z_t is defined to be aggregate output. The function h is increasing, so high investment-output ratios are associated with a high price of the investment good relative to the consumption good. Figure 1 depicts the investment-consumption product transformation curve and Figure 2 the function $h(q)$. For any I/Z, the negative of the slope of the transformation curve in Figure 1 is the height of the curve in Figure 2. This establishes that a higher q will be associated with higher investment for this technology. This restriction of the theory is consistent with the empirical findings previously cited.

There are other predictions of this theory, however, which are questionable. If we think of the q-investment curve h depicted in Figure 2 as a supply curve, the short- and the long-run supply elasticities will be equal. Typically, economists argue that there are specialized resources which cannot be instantaneously and costlessly transferred between industries and that even though short-run elasticities may be low, in the long run supply elasticities are high. As there are no specialized resources for the adjustment cost technology, such considerations are absent and there are no penalties resulting from rapid adjustment in the relative outputs of the consumption and investment good.

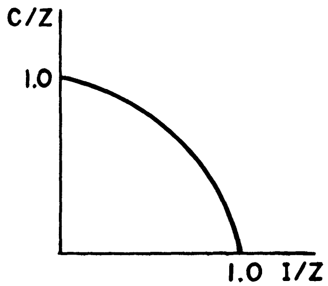

FIGURE 1.

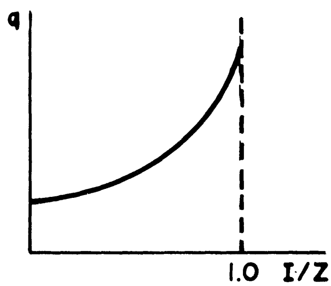

FIGURE 2.

To test whether the theory is a reasonable approximation, we examined cross-section state data. The correlations between the ratios of commercial construction to either state personal income or state employment and price per square foot[5] are both -0.35. With perfectly elastic supply and uncorrelated supply and demand errors, this correlation cannot be positive. To explain this large negative correlation, one needs a combination of high variability in the cross-sectional supply relative to cross-sectional demand plus a positive slope for the supply curve. Our view is that, given mobility of resources, it seems more plausible that the demand is the more variable. Admitting potential data problems, this cross-sectional result casts some doubt upon the adequacy of the single capital good adjustment cost model.

At the aggregate level, an implication of the single capital good adjustment cost model is that when the investment-output ratio is regressed on current and lagged q, only current q should matter.[6] The findings in **[26]** are counter to this prediction.

In summary, our view is that neither the neoclassical nor the adjustment cost technologies are adequate. The neoclassical structure is inconsistent with the positive association between the shadow price of capital and investment activity. The adjustment cost technology is consistent with this observation, but inconsistent with cross-sectional data and the association of investment with the lagged as well as the current capital shadow prices. In addition, the implication that long- and short-run supply elasticities are equal is one which we think a technology should not have.

Most destructive of all to the adjustment-cost technology, however, is the finding that the time required to complete investment projects is not short relative to the business cycle. Mayer **[27]**, on the basis of a survey, found that the average time (weighted by the size of the project) between the decision to undertake an investment project and the completion of it was twenty-one months. Similarly, Hall **[13]** found the average lag between the design of a project and when it becomes productive to be about two years. It is a thesis of this essay that periods this long or even half that long have important effects upon the serial correlation properties of the cyclical components of investment and total output as well as on certain co-movements of aggregate variables.

The technological requirement that there are multiple stages of production is not the delivery lag problem considered by Jorgenson **[19]**. He theorized at the firm level and imposed no consistency of behavior requirement for suppliers and demanders of the investment good. His was not a market equilibrium analysis and there was no theory accounting for the delivery lag. Developing such a market theory with information asymmetries, queues, rationing, and the like is a challenging problem confronting students of industrial organization.

[5] The data on commercial construction and price per square foot were for 1978 and were obtained from F. W. Dodge Division of McGraw-Hill.

[6] This observation is due to Fumio Hayashi.

Our technology assumes that a single period is required for each stage of construction or that the time required to build new capital is a constant. This is not to argue that there are not alternative technologies with different construction periods, patterns of resource use, and total costs. We have found no evidence that the capital goods are built significantly more rapidly when total investment activity is higher or lower. Lengthening delivery lags (see [9]) in periods of high activity may be a matter of longer queues and actual construction times may be shorter. Premiums paid for earlier delivery could very well be for a more advanced position in the queue than for a more rapidly constructed factory. These are, of course, empirical questions, and important cyclical variation in the construction period would necessitate an alternative technology.

Our time-to-build technology is consistent with short-run fluctuations in the shadow price of capital because in the short run capital is supplied inelastically. It also implies that the long-run supply is infinitely elastic, so on average the relative price of the investment good is independent of the investment-output ratio.

3. THE MODEL

Technology

The technology assumes time is required to build new productive capital. Let s_{jt} be the number of projects j stages or j periods from completion for $j = 1, \dots, J-1$, where J periods are required to build new productive capacity. New investment projects initiated in period t are s_{Jt}. The recursive representation of the laws of motion of these capital stocks is

$$k_{t+1} = (1-\delta)k_t + s_{1t}, \tag{3.1}$$

$$s_{j,t+1} = s_{j+1,t} \qquad (j = 1, \dots, J-1). \tag{3.2}$$

Here, k_t is the capital stock at the beginning of period t, and δ is the depreciation rate. The element s_{Jt} is a decision variable for period t.

The final capital good is the inventory stock y_t inherited from the previous period.[7] Thus, in this economy, there are $J+1$ types of capital: inventories y_t, productive capital k_t, and the capital stocks j stages from completion for $j = 1, \dots, J-1$. These variables summarize the effects of past decisions upon current and future production possibilities.

Let φ_j for $j = 1, \dots, J$ be the fraction of the resources allocated to the investment project in the jth stage from the last. Total non-inventory investment in period t is $\sum_{j=1}^{J} \varphi_j s_{jt}$. Total investment, i_t, is this amount plus inventory

[7] All stocks are beginning-of-the-period stocks.

investment $y_{t+1} - y_t$, and consequently

$$i_t = \sum_{j=1}^{J} \varphi_j s_{jt} + y_{t+1} - y_t . \tag{3.3}$$

Total output, that is, the sum of consumption c_t and investment, is constrained as follows:

$$c_t + i_t \leq f(\lambda_t, k_t, n_t, y_t), \tag{3.4}$$

where n_t is labor input, λ_t a shock to technology, and f is a constant-returns-to-scale production function to be parameterized subsequently.

Treating inventories as a factor of production warrants some discussion. With larger inventories, stores can economize on labor resources allocated to restocking. Firms, by making larger production runs, reduce equipment down time associated with shifting from producing one type of good to another. Besides considerations such as these, analytic considerations necessitated this approach. If inventories were not a factor of production, it would be impossible to locally approximate the economy using a quadratic objective and linear constraints. Without such an approximation no practical computational method currently exists for computing the equilibrium process of the model.

The production function is assumed to have the form

$$f(\lambda, k, n, y) = \lambda n^{\theta}\left[(1-\sigma)k^{-\nu} + \sigma y^{-\nu}\right]^{-(1-\theta)/\nu} \tag{3.5}$$

where $0 < \theta < 1$, $0 < \sigma < 1$, and $0 < \nu < \infty$. This form was selected because, among other things, it results in a share θ for labor in the steady state. The elasticity of substitution between capital and inventory is $1/(1+\nu)$. This elasticity is probably less than one which is why ν is required to be positive.

Preferences

The preference function, whose expected value the representative household maximizes, has the form $\sum_{t=0}^{\infty} \beta^t u(c_t, \alpha(L)l_t)$, where $0 < \beta < 1$ is the discount factor, l_t leisure, L the lag operator, and $\alpha(L) = \sum_{i=0}^{\infty} \alpha_i L^i$. Normalizing so that one is the endowment of time, we let $n_t = 1 - l_t$ be the time allocated to market activity. The polynomial lag operator is restricted so that the α_i sum to one, and $\alpha_i = (1-\eta)^{i-1}\alpha_1$ for $i \geq 1$, where $0 < \eta \leq 1$. With these restrictions,

$$\alpha(L)l_t = 1 - \alpha(L)n_t = 1 - \alpha_0 n_t - (1-\alpha_0)\eta \sum_{i=1}^{\infty} (1-\eta)^{i-1} n_{t-i} .$$

By defining the variable $a_t = \sum_{i=1}^{\infty} (1-\eta)^{i-1} n_{t-i}$, the distributed lag has the following recursive representation:

$$\alpha(L)l_t = 1 - \alpha_0 n_t - \eta(1-\alpha_0)a_t, \quad \text{and}$$

$$a_{t+1} = (1-\eta)a_t + n_t . \tag{3.6}$$

The variable a_t summarizes the effects of all past leisure choices on current and future preferences. If $n_s = n_t$ for all $s \leq t$, then $a_t = n_t/\eta$, and the distributed lag is simply $1 - n_t$.

The parameters α_0 and η determine the degree to which leisure is intertemporally substitutable. We require $0 < \eta \leq 1$ and $0 < \alpha_0 \leq 1$. The nearer α_0 is to one, the less is the intertemporal substitution of leisure. For α_0 equal to one, time-separable utility results. With η equal to one, a_t equals n_{t-1}. This is the structure employed in [33]. As η approaches zero, past leisure choices have greater effect upon current utility flows.

Non-time-separable utility functions are implicit in the empirical study of aggregate labor supply in [25]. Grossman [12] and Lucas [24] discuss why a non-time-separable utility function is needed to explain the business cycle fluctuations in employment and consumption. A micro justification for our hypothesized structure based on a Beckerian household production function is as follows.[8] Time allocated to non-market activities, that is l_t, is used in household production. If there is a stock of household projects with varying output per unit of time, the rational household would allocate l_t to those projects with the greatest returns per time unit. If the household has allocated a larger amount of time to non-market activities in the recent past, then only projects with smaller yields should remain. Thus, if a_t is lower, the marginal utility value of l_t should be smaller.

Cross-sectional evidence of households' willingness to redistribute labor supply over time is the lumpiness of that supply. There are vacations and movements of household members into and out of the labor force for extended periods which are not in response to large movements in the real wage. Another observation suggesting high intertemporal substitutability of leisure is the large seasonal variation in hours of market employment. Finally, the failure of Abowd and Ashenfelter [2] to find a significant wage premium for jobs with more variable employment and earnings patterns is further evidence. In summary, household production theory and cross-sectional evidence support a non-time-separable utility function that admits greater intertemporal substitution of leisure—something which is needed to explain aggregate movements in employment in an equilibrium model.

The utility function in our model is assumed to have the form

$$u(c_t, \alpha(L)l_t) = \left[c_t^{1/3}(\alpha(L)l_t)^{2/3}\right]^{\gamma}/\gamma,$$

where $\gamma < 1$ and $\gamma \neq 0$. If the term in the square brackets is interpreted as a composite commodity, then this is the constant-relative-risk-aversion utility function with the relative degree of risk aversion being $1 - \gamma$. We thought this composite commodity should be homogeneous of degree one as is the case when there is a single good. The relative size of the two exponents inside the brackets is

[8] We thank Nasser Saïdi for suggesting this argument.

motivated by the fact that households' allocation of time to nonmarket activities is about twice as large as the allocation to market activities.

Information Structure

We assume that the technology parameter is subject to a stochastic process with components of differing persistence. The productivity parameter is not observed but the stand-in consumer does observe an indicator or noisy measure of this parameter at the beginning of the period. This might be due to errors in reporting data or just the fact that there are errors in the best or consensus forecast of what productivity will be for the period. On the basis of the indicator and knowledge of the economy-wide state variables, decisions of how many new investment projects to initiate and of how much of the time endowment to allocate to the production of marketed goods are made. Subsequent to observing aggregate output, the consumption level is chosen with inventory investment being aggregate output less fixed investment and consumption.

Specifically, the technology shock, λ_t, is the sum of a permanent component, λ_{1t}, and a transitory component,[9] λ_{2t}:

$$\lambda_t = \lambda_{1t} + \lambda_{2t} + \bar{\lambda}. \tag{3.7}$$

In the spirit of the Friedman-Muth permanent-income model, the permanent component is highly persistent so

$$\lambda_{1,t+1} = \rho\lambda_{1t} + \zeta_{1t}, \tag{3.8}$$

where ρ is less than but near one and ζ_{1t} is a permanent shock.[10] The transitory component equals the transitory shock so

$$\lambda_{2,t+1} = \zeta_{2t}. \tag{3.9}$$

The indicator of productivity, π_t, is the sum of actual productivity λ_t and a third shock ζ_{3t}:

$$\pi_t = \lambda_t + \zeta_{3t} = \lambda_{1t} + \lambda_{2t} + \zeta_{3t} + \bar{\lambda}. \tag{3.10}$$

The shock vectors $\zeta_t = (\zeta_{1t}, \zeta_{2t}, \zeta_{3t})$ are independent multivariate normal with mean vector zero and diagonal covariance matrix.

The period-t labor supply decision n_t and new investment project decision s_{Jt} are made contingent upon the past history of productivity shocks, the λ_k for $k < t$, the indicator of productivity π_t, the stocks of capital inherited from the past, and variable a_t. These decisions cannot be contingent upon λ_t for it is not

[9] The importance of permanent and transitory shocks in studying macro fluctuations is emphasized in [**8**].

[10] The value used for ρ in this study was 0.95. The reason we restricted ρ to be strictly less than one was technical. The theorem we employ to guarantee the existence of competitive equilibrium requires stationarity of the shock.

observed or deducible at the time of these decisions. The consumption-inventory investment decision, however, is contingent upon λ_t for aggregate output is observed prior to this decision and λ_t can be deduced from aggregate output and knowledge of inputs.

The state space is an appropriate formalism for representing this recursive information structure. Because of the two-stage decision process, it is not a direct application of Kalman filtering. Like that approach the separation of estimation and control is exploited. The general structure assumes an unobservable state vector, say x_t, that follows a vector autoregressive process with independent multivariate normal innovations:

$$x_{t+1} = Ax_t + \epsilon_{0t}, \qquad \text{where} \quad \epsilon_{0t} \sim N(0, V_0). \tag{3.11}$$

Observed prior to selecting the first set of decisions is

$$p_{1t} = B_1 x_t + \epsilon_{1t}, \qquad \text{where} \quad \epsilon_{1t} \sim N(0, V_1). \tag{3.12}$$

The element B_1 is a matrix and the ϵ_{1t} are independent over time. Observed prior to the second set of decisions and subsequent to the first set is

$$p_{2t} = B_2 x_t + \epsilon_{2t}, \qquad \text{where} \quad \epsilon_{2t} \sim N(0, V_2). \tag{3.13}$$

Equations (3.11)–(3.13) define the general information structure.

To map our information structure into the general formulation, let $x_t' = (\lambda_{1t}, \lambda_{2t})$, $B_1 = [1\ 1]$, $B_2 = [1\ 1]$,

$$A = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix}, \qquad V_0 = \begin{bmatrix} \text{var}(\zeta_1) & 0 \\ 0 & \text{var}(\zeta_2) \end{bmatrix},$$

$V_1 = [\text{var}(\zeta_3)]$, and $V_2 = [0]$. With these definitions, the information structure (3.7)–(3.10) viewed as deviations from the mean and the representation (3.11)–(3.13) are equivalent.

Let m_{0t} be the expected value and Σ_0 the covariance of the distribution of x_t conditional upon the $p_k = (p_{1k}, p_{2k})$ for $k < t$. Using the conditional probability laws for the multivariate normal distribution (see [**28**, p. 208]) and letting m_{1t} and Σ_1 be the mean and covariance of x_t conditional upon p_{1t} as well, we obtain

$$m_{1t} = m_{0t} + (B_1\Sigma_0)'(B_1\Sigma_0 B_1' + V_1)^{-1}(p_{1t} - B_1 m_{0t}), \quad \text{and} \tag{3.14}$$

$$\Sigma_1 = \Sigma_0 - (B_1\Sigma_0)'(B_1\Sigma_0 B_1' + V_1)^{-1} B_1 \Sigma_0 . \tag{3.15}$$

Similarly, the mean vector m_{2t} and covariance matrix Σ_2 conditional upon p_{2t} as well are

$$m_{2t} = m_{1t} + (B_2\Sigma_1)'(B_2\Sigma_1 B_2' + V_2)^{-1}(p_{2t} - B_2 m_{1t}), \quad \text{and} \tag{3.16}$$

$$\Sigma_2 = \Sigma_1 - (B_2\Sigma_1)'(B_2\Sigma_1 B_2' + V_2)^{-1} B_2 \Sigma_1 . \tag{3.17}$$

Finally, from (3.11),

(3.18) $\quad m_{0,t+1} = Am_{2t}$, and

(3.19) $\quad \Sigma_0 = A\Sigma_2 A' + V_0$.

The covariances Σ_0, Σ_1, and Σ_2 are defined recursively by (3.15), (3.17), and (3.19). The matrix V_0 being of full rank along with the stability of A are sufficient to insure that the method of successive approximations converges exponentially fast to a unique solution.

The covariance elements Σ_0, Σ_1, and Σ_2 do not change over time and are therefore not part of the information set. The m_{0t}, m_{1t}, and m_{2t} do change but are sufficient relative to the relevant histories for forecasting future values of both the unobserved state and the observable p_τ, $\tau > t$, and for estimating the current unobserved state.

Equilibrium

To determine the equilibrium process for this model, we exploit the well-known result that, in the absence of externalities, competitive equilibria are Pareto optima. With homogeneous individuals, the relevant Pareto optimum is the one which maximizes the welfare of the stand-in consumer subject to the technology constraints and the information structure. Thus, the problem is to

$$\text{maximize}\, E\sum_{t=0}^{\infty} \beta^t u[c_t, 1-\alpha_0 n_t - \eta(1-\alpha_0)a_t]$$

subject to constraints (3.1)–(3.4), (3.6), and (3.11)–(3.13), given k_0, $s_{10}, \dots, s_{J-1,0}$, a_0, and that $x_0 \sim N(m_0, \Sigma_0)$. The decision variables at time t are n_t, s_{Jt}, c_t, and y_{t+1}. Further, n_t and s_{Jt} cannot be contingent upon p_{2t} for it is observed subsequent to these decisions.

This is a standard discounted dynamic programming problem. There are optimal time-invariant or stationary rules of the form

$$n_t = n(k_t, s_{1t}, s_{2t}, \dots, s_{J-1,t}, y_t, a_t, m_{1t}),$$

$$s_{Jt} = s(k_t, s_{1t}, s_{2t}, \dots, s_{J-1,t}, y_t, a_t, m_{1t}),$$

$$c_t = c(k_t, s_{1t}, s_{2t}, \dots, s_{Jt}, y_t, a_t, n_t, m_{2t}),$$

$$y_{t+1} = y(k_t, s_{1t}, s_{2t}, \dots, s_{Jt}, y_t, a_t, n_t, m_{2t}).$$

It is important to note that the second pair of decisions are contingent upon m_{2t} rather than m_{1t} and that they are contingent also upon the first set of decisions s_{Jt} and n_t.

The existence of such decision rules and the connection with the competitive allocation is established in **[31]**. But, approximations are necessary before equilibrium decision rules can be computed. Our approach is to determine the steady state for the model with no shocks to technology. Next, quadratic approximations are made in the neighborhood of the steady state. Equilibrium decision rules for the resulting approximate economy are then computed. These rules are linear, so in equilibrium the approximate economy is generated by a system of stochastic difference equations for which covariances are easily determined.

4. STEADY STATE, APPROXIMATION, AND COMPUTATION OF EQUILIBRIUM

Variables without subscript denote steady state values. The steady state interest rate is $r = (1 - \beta)/\beta$, and the steady state price of (non-inventory) capital $q = \sum_{j=1}^{J}(1 + r)^{j-1}\varphi_j$. The latter is obtained by observing that φ_1 units of consumption must be foregone in the current period, φ_2 units the period before, etc., in order to obtain one additional unit of capital for use next period.

Two steady state conditions are obtained by equating marginal products to rental rates, namely $f_y = r$ and $f_k = q(r + \delta)$. These imply $f_k/f_y = q(r + \delta)/r$. For production function (3.5), this reduces to

$$(4.1) \qquad y = \left[\frac{r + \delta}{r} q \frac{\sigma}{1 - \sigma} \right]^{1/(\nu+1)} k \equiv b_1 k.$$

Differentiating the production function with respect to capital, substituting for y from (4.1), and equating to the steady-state rental price, one obtains

$$(1 - \theta)(1 - \sigma)b_2^{-(1-\theta-\nu)/\nu}\lambda n^{\theta}k^{-\theta} = q(r + \delta),$$

where $b_2 = 1 - \sigma + \sigma b_1^{-\nu}$. Solving for k as a function of n yields

$$(4.2) \qquad k = \left[\frac{(1 - \theta)(1 - \sigma)}{q(r + \delta)} b_2^{-(1-\theta-\nu)/\nu} \right]^{1/\theta} \lambda^{1/\theta} n \equiv b_3 \lambda^{1/\theta} n.$$

Steady-state output as a function of n is $f = b_2^{-(1-\theta)/\nu} b_3^{1-\theta} \lambda^{1/\theta} n \equiv b_4 \lambda^{1/\theta} n$. In the steady state, net investment is zero, so

$$(4.3) \qquad c = b_4 \lambda^{1/\theta} n - \delta k = (b_4 - \delta b_3)\lambda^{1/\theta} n.$$

The steady-state values of c, k, and y are all proportional to $\lambda^{1/\theta} n$. We also note that the capital-output ratio is b_3/b_4, and that consumption's share to total steady-state output is $1 - (\delta b_3/b_4)$.

Turning now to the consumer's problem and letting μ be the Lagrange multiplier for the budget constraint and w_t the real wage, first-order conditions are

$$\frac{1}{3} c_t^{(\gamma/3)-1} (\alpha(L) l_t)^{2\gamma/3} = \mu, \quad \text{and}$$

$$\frac{2}{3} \sum_{i=0}^{\infty} \beta^i \alpha_i c_{t+i}^{\gamma/3} (\alpha(L) l_{t+i})^{(2\gamma/3)-1} = \mu w_t.$$

In the steady state, $c_t = c$, $l_t = l$, and $w_t = w$ for all t. Making these substitutions and using the fact that the α_i sum to one, these expressions simplify to

$$\frac{1}{3}\left(c^{1/3}l^{2/3}\right)^{\gamma} = \mu c, \quad \text{and} \quad \frac{2}{3}\left(c^{1/3}l^{2/3}\right)^{\gamma}\sum_{i=1}^{\infty}\beta^i\alpha_i = \mu wl.$$

Eliminating μ from these equations yields $2c\sum_{i=0}^{\infty}\beta^i\alpha_i = wl$. Since $\sum_{i=0}^{\infty}\beta^i\alpha_i = \alpha_0 + (1-\alpha_0)\eta/(r+\eta)$ and $l = 1 - n$, this in turn implies

$$2c(\alpha_0 + (1-\alpha_0)\eta/(r+\eta)) = w(1-n). \tag{4.4}$$

Returning to the production side, the marginal product of labor equals the real wage:

$$w = f_n = \frac{\theta}{n}f = \theta b_4\lambda^{1/\theta}. \tag{4.5}$$

Using (4.3) and (4.5), we can solve (4.4) for n:

$$n = \left[1 + 2\frac{\alpha_0 r + \eta}{\theta(r+\eta)}\left(1 - (\delta b_3/b_4)\right)\right]^{-1}.$$

That n does not depend upon average λ matches well with the American experience over the last thirty years. During this period, output per man-hour has increased by a few hundred per cent, yet man-hours per person in the 16–65 age group has changed but a few per cent.

Approximation About the Steady State

If the utility function u were quadratic and the production function f linear, there would be no need for approximations. In equilibrium, consumption must be equal to output minus investment. We exploit this fact to eliminate the nonlinearity in the constraint set by substituting $f(\lambda, k, n, y) - i$ for c in the utility function to obtain $u(f(\lambda, k, n, y) - i, n, a)$. The next step is to approximate this function by a quadratic in the neighborhood of the model's steady state. As investment i is linear in the decision and state variables, it can be eliminated subsequent to the approximation and still preserve a quadratic objective.

Consider the general problem of approximating function $u(x)$ near $\bar{x}$. The approximate quadratic function is

$$U(x) = u(\bar{x}) + b'(x-\bar{x}) + (x-\bar{x})'Q(x-\bar{x}),$$

where $x, b \in \mathsf{R}^n$ and Q is an $n \times n$ symmetric matrix. We want an approximation that is good not only at $\bar{x}$ but also at other x in the range experienced during the sample period. Let z^i be a vector, all of whose components are zero except for $z_i^i > 0$. Our approach is to select the elements b_i and q_{ii} so that the approximation error is zero at the $\bar{x}+z^i$ and $\bar{x}-z^i$, where the z_i^i selected correspond to the approximate average deviations of the x_i from their steady state values $\bar{x}_i$. The values of $z_i^i/\bar{x}_i$ used for λ, k, y, n, i, and a were 3, 1, 2, 3, 8, and 0.5 per cent, respectively.[11]

The approximation errors being zero at the $\bar{x}+z^i$ and $\bar{x}-z^i$ requires that

$$b_i=\left[u(\bar{x}+z^i)-u(\bar{x}-z^i)\right]/2z_i, \quad \text{and}$$

$$q_{ii}=\left[u(\bar{x}+z^i)-u(\bar{x})+u(\bar{x}-z^i)-u(\bar{x})\right]/2z_i^2.$$

The elements q_{ij}, $i\neq j$, are selected to minimize the sum of the squared approximation errors at $\bar{x}+z^i+z^j$, $\bar{x}+z^i-z^j$, $\bar{x}-z^i+z^j$, and $\bar{x}-z^i-z^j$. The approximation error at the first point is

$$u(\bar{x}+z^i+z^j)-u(\bar{x})-b_iz_i-b_jz_j-q_{ii}z_i^2-q_{jj}z_j^2-2q_{ij}z_iz_j.$$

Summing over the square of this error and the three others, differentiating with respect to q_{ij}, setting the resulting expression equal to zero and solving for q_{ij}, we obtain

$$q_{ij}=\left[u(\bar{x}+z^i+z^j)-u(\bar{x}+z^i-z^j)-u(\bar{x}-z^i+z^j)\right.$$
$$\left.+u(\bar{x}-z^i-z^j)\right]/8z_iz_j$$

for $i\neq j$.

Computation of Equilibrium

The equilibrium process for the approximate economy maximizes the welfare of the representative household subject to the technological and informational constraints as there are no externalities. This simplifies the determination of the equilibrium process by reducing it to solving a linear-quadratic maximization problem. For such mathematical structure there is a separation of estimation and control. Consequently, the first step in determining the equilibrium decision rules for the approximate economy is to solve the following deterministic problem:

$$\max\sum_{t=0}^{\infty}\beta^tU(k_t,n_t,y_t,\lambda_t,i_t,a_t)$$

[11] We experimented a little and found that the results were essentially the same when the second order Taylor series approximation was used rather than this function. Larry Christiano [**10**] has found that the quadratic approximation method that we employed yields approximate solutions that are very accurate, even with large variability, for a structure that, like ours, is of the constant elasticity variety.

subject to

$$k_{t+1} = (1-\delta)k_t + s_{1t}, \tag{4.6}$$

$$s_{j,t+1} = s_{j+1,t} \qquad (j = 1, \ldots, J-1), \tag{4.7}$$

$$x_{t+1} = Ax_t, \tag{4.8}$$

$$a_{t+1} = (1-\eta)a_t + n_t, \tag{4.9}$$

$$i_t = \sum_{j=1}^{J} \varphi_j s_{jt} + y_{t+1} - y_t, \tag{4.10}$$

$$\lambda_t = x_{1t} + x_{2t}. \tag{4.11}$$

At this stage, the fact that there is an additive stochastic term in the equation determining x_{t+1} is ignored as is the fact that x_t is not observed for our economy. Constraints (4.6)–(4.9) are the laws of motion for the state variables. The free decision variables are n_t, s_{Jt}, and y_{t+1}. It was convenient to use inventories taken into the subsequent period, y_{t+1}, as a period t decision variable rather than i_t because the decisions on inventory carry-over and consumption are made subsequent to the labor supply and new project decisions n_t and s_{Jt}.

For notational simplicity we let the set of state variables other than the unobserved x_t be $z_t = (k_t, y_t, a_t, s_{1t}, \ldots, s_{J-1,t})$ and the set of decision variables $d_t = (n_t, s_{Jt}, y_{t+1})$. The unobserved state variables $x_t = (x_{1t}, x_{2t})$ are the permanent and transitory shocks to technology. Finally, $v(x,z)$ is the value of the deterministic problem if the initial state is (x,z). It differs from the value function for the stochastic problem by a constant.

Using constraints (4.10) and (4.11) to substitute for i_t and λ_t in the utility function, an indirect utility function $U(x,z,d)$ is obtained. The value function, $v(x,z)$, was computed by the method of successive approximations or value iteration. If $v_j(x,z)$ is the jth approximation, then

$$v_{j+1}(x_t, z_t) = \max_{d_t} \left[U(x_t, z_t, d_t) + \beta v_j(x_{t+1}, z_{t+1}) \right]$$

subject to constraints (4.6)–(4.9). The initial approximation, $v_0(x,z)$, is that function which is identically zero.

The function U is quadratic and the constraints are linear. Then, if v_j is quadratic, v_{j+1} must be quadratic. As v_0 is trivially quadratic, all the v_j are quadratic and therefore easily computable. We found that the sequence of quadratic functions converged reasonably quickly.[12]

[12] The limit of the sequence of value functions existed in every case and, as a function of z, was bounded from above, given x. This, along with the stability of the matrix A, is sufficient to ensure that this limit is the optimal value function and that the associated policy function is the optimal one (see [**30**]).

The next step is to determine the optimal inventory carry-over decision rule. It is the *linear* function $y_{t+1} = y(x_t, z_t, n_t, s_{Jt})$ which solves

$$\max_{y_{t+1}} \left[U(x_t, z_t, n_t, s_{Jt}, y_{t+1}) + \beta v(x_{t+1}, z_{t+1}) \right] \tag{4.12}$$

subject to (4.6)–(4.9) and both n_t and s_{Jt} given. Finally, the solution to the program

$$\max_{s_{Jt}, n_t} v_2(x_t, z_t, n_t, s_{Jt}),$$

where v_2 is the value of maximization of (4.12), is determined. The linear functions $s_{Jt} = s(x_t, z_t)$ and $n_t = n(x_t, z_t)$ which solve the above program are the optimal decision rules for new projects and labor supply.

Because of the separation of estimation and control in our model, these decision rules can be used to determine the motion of the stochastic economy. In each period t, a conditional expectation, m_{0t}, is formed on the basis of observations in previous periods. An indicator of the technology shock is observed, which is the sum of a permanent and a transitory component as well as an indicator shock. The conditional expectation, m_{1t}, of the unobserved x_t is computed according to equation (3.14), and s_{Jt} and n_t are determined from

$$s_{Jt} = s(m_{1t}, z_t), \tag{4.13}$$

$$n_t = n(m_{1t}, z_t), \tag{4.14}$$

where x_t has been replaced by m_{1t}. Then the technology shock, λ_t, is observed, which changes the conditional expectation of x_t. From (3.16), this expectation is m_{2t}, and the inventory carry-over is determined from

$$y_{t+1} = y(m_{2t}, z_t, s_{Jt}, n_t). \tag{4.15}$$

To summarize, the equilibrium process governing the evolution of our economy is given by (3.1)–(3.3), (3.6), (3.11)–(3.14), (3.16), (3.18), and (4.13)–(4.15).

5. TEST OF THE THEORY

The test of the theory is whether there is a set of parameters for which the model's co-movements for *both* the smoothed series and the deviations from the smoothed series are *quantitatively* consistent with the observed behavior of the corresponding series for the U.S. post-war economy. An added requirement is that the parameters selected not be inconsistent with relevant micro observations, including reported construction periods for new plants and cross-sectional observations on consumption and labor supply. The closeness of our specification of preferences and technology to those used in many applied studies facilitates such comparisons.

The model has been rigged to yield the observations that smoothed output, investment, consumption, labor productivity, and capital stocks all vary roughly proportionately while there is little change in employment (all variables are in per-household terms) when the technology parameter λ grows smoothly over time. These are just the steady state properties of the growth model with which we began.

Quantitatively explaining the co-movements of the deviations is the test of the underlying theory. For want of better terminology, the deviations will be referred to as the cyclical components even though, with our integrated approach, there is no separation between factors determining a secular path and factors determining deviations from that path. The statistics to be explained are the covariations of the cyclical components. They are of interest because their behavior is stable and is so different from the corresponding covariations of the smoothed series. This is probably why many have sought separate explanations of the secular and cyclical movements.

One cyclical observation is that, in percentage terms, investment varies three times as much as output does and consumption only half as much. In sharp contrast to the secular observations, variations in cyclical output are principally the result of variations in hours of employment per household and not in capital stocks or labor productivity.

The latter observation is a difficult one to explain. Why does the consumption of market produced goods and the consumption of leisure move in opposite directions in the absence of any apparent large movement in the real wage over the so-called cycle? For our model, the real wage is proportional to labor's productivity, so the crucial test is whether most of the variation in cyclical output arises from variations in employment rather than from variations in labor's productivity.

We chose not to test our model versus the less restrictive vector autoregressive model.[13] This most likely would have resulted in the model being rejected, given the measurement problems and the abstract nature of the model. Our approach is to focus on certain statistics for which the noise introduced by approximations and measurement errors is likely to be small relative to the statistic. Failure of the theory to mimic the behavior of the post-war U.S. economy with respect to these stable statistics with high signal-noise ratios would be grounds for its rejection.

Model Calibration

There are two advantages of formulating the model as we did and then constructing an approximate model for which the equilibrium decision rules are linear. First, the specifications of preferences and technology are close to those used in many applied studies. This facilitates checks of reasonableness of many parameter values. Second, our approach facilitates the selection of parameter values for which the model steady-state values are near average values for the American economy during the period being explained. These two considerations

[13] Sims [34] has estimated unrestricted aggregate vector autoregressive models.

reduce dramatically the number of free parameters that will be varied when searching for a set that results in cyclical covariances near those observed. In explaining the covariances of the cyclical components, there are only seven free parameters, with the range of two of them being severely constrained a priori.

Capital for our model reflects all tangible capital, including stocks of plant and equipment, consumer durables and housing. Consumption does not include the purchase of durables but does include the services from the stock of consumer durables. Different types of capital have different construction periods and patterns of resource requirements. The findings summarized in Section 2 suggest an average construction period of nearly two years for plants. Consumer durables, however, have much shorter average construction periods. Having but one type of capital, we assume, as a compromise, that four quarters are required, with one-fourth of the value put in place each quarter. Thus $J = 4$ and $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0.25$.

Approximately ten per cent of national income account GNP is the capital consumption allowance and another ten per cent excise tax. To GNP should be added the depreciation of consumer durables which has the effect of increasing the share of output going to owners of capital. In 1976, compensation to employees plus proprietary income was approximately 64 per cent of GNP plus consumer durables depreciation less indirect business tax, while owners of capital received about 36 per cent. As labor share is θ, we set $\theta = 0.64$.

Different types of capital depreciate more rapidly than others, with durables depreciating more rapidly than plant and housing, and land not depreciating at all. As a compromise, we set the depreciation rate equal to 10 per cent per year. We assume a subjective time discount rate of four per cent and abstract from growth. This implies a steady-state capital to annual output ratio of 2.4. Of total output 64 per cent is wages, 24 per cent depreciation, and 12 per cent return on capital which includes consumer durables.

The remaining parameters of technology are average λ, which we normalize to one by measuring output in the appropriate units, and parameters σ and ν, which determine the shares of and substitution between inventories and capital. Inventories are about one-fourth of annual GNP so we require ν and σ to be such that $k/y = 10$. A priori reasoning indicates the substitution opportunities between capital and inventory are small, suggesting that ν should be considerably larger than zero. We restricted it to be no less than two, but it is otherwise a free parameter in our search for a model to explain the cyclical covariances and autocovariances of aggregate variables. Given ν and the value of $b_1 = y/k$, σ is implied. From (4.1) it is $\sigma = [1 + q(r + \delta)/(rb_1^{\nu+1})]^{-1}$. For purposes of explaining the covariances of the percentage deviation from steady state values, ν is the only free parameter associated with technology.

The steady state real interest rate r is related to the subjective time discount rate, $\rho = \beta^{-1} - 1$, and the risk aversion parameter, γ, by the equation $r = \rho + (1 - \gamma)(\dot{c}/c)$, where $\dot{c}/c$ is the growth rate of per capita consumption. We have assumed ρ is four per cent per year (one per cent per quarter). As the growth rate of per capita consumption has been about two per cent and the real return on physical capital six to eight per cent, the risk aversion parameter, γ, is constrained to be between minus one and zero.[14]

The parameters α_0 and η which affect intertemporal substitutability of leisure will be treated as free parameters for we could find no estimate for them in the labor economics literature. As stated previously, the steady-state labor supply is independent of the productivity parameter $\bar{\lambda}$. The remaining parameters are those specifying the process on λ_t and the variance of the indicator. These three parameters are var(ζ_1), var(ζ_2), and var(ζ_3). Only two of these are free parameters, however. We restricted the sum of the three variances to be such that the estimate of the variance of cyclical output for the model equalled that of cyclical output for the U.S. economy during the sample period.

In summary, the parameters that are estimated from the variance-covariance properties of the model are these variances plus the parameter ν determining substitutability of inventories and capital, the parameters α_0 and η determining intertemporal substitutability of leisure, and the risk aversion parameter γ. For each set of parameter values, means and standard deviations were computed for several statistics which summarize the serial correlation and covariance properties of the model. These numbers are compared with those of the actual U.S. data for the period 1950 : 1 to 1979 : 2 as reported in Hodrick and Prescott **[18]**. A set of parameter values is sought which fits the actual data well. Having only six degrees of freedom to explain the observed covariances imposes considerable discipline upon the analysis.

The statistics reported in **[18]** are not the only way to quantitatively capture the co-movements of the deviations.[15] This approach is simple, involves a minimum of judgment, and is robust to slowly changing demographic factors which affect growth, but are not the concern of this theory.[16] In addition, these statistics are robust to most measurement errors, in contrast to, say, the correlations between the first differences of two series. It is important to compute the same statistics for the U.S. economy as for the model, that is, to use the same function of the data. This is what we do.

A key part of our procedure is the computation of dynamic competitive equilibrium for each combination of parameter values. Because the conditional forecasting can be separated from control in this model, the dynamic equilibrium decision rules need only be computed for each new combination of the parame-

[14] Estimates in **[16]** indicate γ is near zero.

[15] With the Hodrick-Prescott method, the smooth path $\{s_t\}$ for each series $\{y_t\}$ minimized

$$\sum_{t=1}^{T} (y_t - s_t)^2 + 1600 \sum_{t=1}^{T} [(s_{t+1} - s_t) - (s_t - s_{t-1})]^2.$$

The deviations for series $\{y_t\}$ are $\{y_t - s_t\}$. The number of observations, T, was 118. The solution to the above program is a linear transformation of the data. Thus, the standard deviations and correlations reported are well-defined statistics.

[16] See, for example, **[11]**.

TABLE I

MODEL PARAMETERS[a]

Preference Parameters:	$\alpha_0 = 0.50$, $\eta = 0.10$, $\gamma = -0.50$, $\beta = 0.99$
Technology Parameters:	$\nu = 4.0$, $\theta = 0.64$, $\sigma = 0.28 \times 10^{-5}$, $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0.25$, $\delta = 0.10$, $\bar{\lambda} = 1.0$
Shock Variances:	$\text{var}(\zeta_1) = 0.0090^2$, $\text{var}(\zeta_2) = 0.0018^2$, $\text{var}(\zeta_3) = 0.0090^2$

[a] For parameters with a time dimension, the unit of time is a quarter of a year.

ters ν, α_0, η, and γ. Similarly, the conditional expectations of the permanent and transitory shocks which enter the decision rules depend only on the variances of the three shocks and not upon the parameters of preferences and technology.

For each set of parameter values the following statistics are computed: the autocorrelation of cyclical output for up to six periods, standard deviations of the cyclical variables of interest, and their correlations with cyclical output. In **[18]** the variables (except interest rates) are measured in logs while we use the levels rather than the logs. This is of consequence only in the measurement of amplitudes, so in order to make our results comparable to theirs, our standard deviations (except for interest rates) are divided by the steady states of the respective variables. One can then interpret the cyclical components essentially as percentage deviations as in **[18]**.

The parameter values that yielded what we considered to be the best fit are reported in Table I. They were determined from a grid search over the free parameters. In the case of ν, we tried the values 2, 3, 4, and 5. The parameters α_0 and η were just constrained to be between zero and one. Only the values -1, -0.5, and -0.1 were considered for the risk aversion parameter γ. The last value is close to the limiting case of $\gamma = 0$ which would correspond to the logarithmic utility function.

Results

All reported statistics refer to the cyclical components for both the model and the U.S. economy. Estimated autocorrelations of real output for our model along with sample values for the U.S. economy in the post-war period are reported in Table II. The fit is very good, particularly in light of the model's simplicity.

Table III contains means of standard deviations and correlations with output for the model's variables. Table IV contains sample values of statistics for the post-war U.S. economy as reported in **[18]**.

The variables in our model do not correspond perfectly to those available for the U.S. economy so care must be taken in making comparisons. A second problem is that there may be measurement errors that seriously bias the estimated correlations and standard deviations. A final problem is that the estimates for the U.S. economy are subject to sampling error. As a guide to the magnitude

TABLE II
AUTOCORRELATIONS OF OUTPUT[a]

Order of Autocorrelations	Model Means (Standard Deviations) of Sample Distribution	U.S. Economy Sample Values for 1950 : 1–1979 : 2
1	.71 (.07)	.84
2	.45 (.12)	.57
3	.28 (.13)	.27
4	.19 (.12)	−.01
5	.02 (.11)	−.20
6	−.13 (.12)	−.30

[a]The length of the sample period both for the model and for the U.S. economy is 118 quarters.

of this variability, we report the standard deviations of sample distributions for the model's statistics which, like the estimates for the U.S. economy, use only 118 observations. These are the numbers in the parentheses in Tables II and III.

The model is consistent with the large (percentage) variability in investment and low variability in consumption and their high correlations with real output. The model's negative correlation between the capital stock and output is consistent with the data though its magnitude is somewhat smaller.

Inventories for our model correspond to finished and nearly finished goods while the inventories in Table IV refer to goods in process as well. We added half

TABLE III
MODEL'S STANDARD DEVIATIONS AND CORRELATIONS WITH REAL OUTPUT[a]

Variable	Standard Deviations: Means (Standard Deviations) of Sample Distribution[b]	Correlations with Output: Means (Standard Deviations) of Sample Distribution
Real Output	1.80 (.23)	—
Consumption	.63 (.09)	.94 (.01)
Investment	6.45 (.62)	.80 (.04)
Inventories	.89 (.06)	−.15 (.11)
Inventories plus	2.00 (.20)	.39 (.06)
Capital Stock	.63 (.08)	−.07 (.06)
Hours	1.05 (.13)	.93 (.01)
Productivity	.90 (.10)	.90 (.02)
Real Interest Rate (Annual)	.23 (.02)	.47 (.10)

[a]The length of the sample period both for the model and for the U.S. economy is 118 quarters.
[b]Measured in per cent.

TABLE IV

SAMPLE STANDARD DEVIATIONS AND CORRELATIONS WITH REAL OUTPUT
U.S. ECONOMY 1950 : 1–1979 : 2

	Standard Deviations (per cent)	Correlations with Real Output
Output	1.8	—
Total Consumption	1.3	.74
Services	0.7	.62
Non-Durables	1.2	.71
Durables	5.6	.57
Investment Fixed	5.1	.71
Capital Stock		
Durable Mfg.	1.2	−.21
Non-durable Mfg.	0.7	−.24
Inventories	1.7	.51
Hours	2.0	.85
Productivity	1.0	.10

the value of uncompleted capital goods to the model's inventory variable to obtain what we call inventories plus. This corresponds more closely to the U.S. inventory stock variable, with its standard deviation and correlation with real output being consistent with the U.S. data.

In Table III we include results for the implicit real interest rate given by the expression $r_t = (\partial u/\partial c_t)/(\beta E(\partial u/\partial c_{t+1})) - 1$. The expectation is conditional on the information known when the allocation between consumption and inventory carry-over is made.

The model displays more variability in hours than in productivity, but not by as much as the data show. In light of the difficulties in measuring output and, in particular, employment, we do not think this discrepancy is large. For example, all members of the household may not be equally productive, say due to differing stocks of human capital. If there is a greater representation in the work force of the less productive, for example less experienced youth, when output is high, hours would be overestimated. The effects of such errors would be to bias the variability of employment upwards. It also would bias the correlation between productivity and output downwards, which would result in the model being nearly consistent with the data. Measurement errors in employment that are independent of the cycle would have a similar effect on the correlation between output and productivity.

Another possible explanation is the oversimplicity of the model. The shocks to technology, given our production function, are pure productivity shocks. Some shocks to technology alter the transformation between the consumption and investment goods. For example, investment tax credits, accelerated depreciation, and the like, have such effects, and so do some technological changes. Further, some technological change may be embodied in new capital, and only after the capital becomes productive is there the increment to measured productivity. Such shocks induce variation in investment and employment without the variability in productivity. This is a question that warrants further research.

We also examined lead and lag relationships and serial correlation properties of aggregate series other than output. We found that, both for the post-war U.S. economy and the model, consumption and non-inventory investment move contemporaneously with output and have serial correlation properties similar to output. Inventory and capital stocks for the model lag output, which also matches well with the data. Some of the inventory stock's cross-serial correlations with output deviate significantly, however, from those for the U.S. economy. The one variable whose lead-lag relationship does not match with the data is productivity. For the U.S. economy it is a leading indicator, while there is no lead or lag in the model. This was not unexpected in view of our discussion above with regard to productivity. Thus, even though the overall fit of the model is very good, it is not surprising, given the level of abstraction, that there are elements of the fine structure of dynamics that it does not capture.

The Smoothed Series

The smoothed output series for the U.S. post-war data deviated significantly from the linear time trend. During the 118-quarter sample period this difference had two peaks and two troughs. The times between such local extremes were 30, 31, and 32 quarters, and the corresponding differences in values at adjacent extremes were 5.00, 7.25, and 5.90 per cent, respectively.

These observations match well with the predictions of the model. The mean of the model's sampling distribution for the number of peaks and troughs in a 118-quarter period is 4.0—which is precisely the number observed. The mean of the number of quarters between extremes is 26.1 with standard deviation 9.7, and the mean of the vertical difference in the values at adjacent extremes is 5.0 with standard deviation 2.9. Thus, the smoothed output series for the U.S. economy is also consistent with the model.

Sensitivity of Results to Parameter Selection

With a couple of exceptions, the results were surprisingly insensitive to the values of the parameters. The fact that the covariations of the aggregate variables in the model are quite similar for broad ranges of many of the parameters suggests that, even though the parameters may differ across economies, the nature of business cycles can still be quite similar.

We did find that most of the variation in technology had to come from its permanent component in order for the serial correlation properties of the model to be consistent with U.S. post-war data. We also found that the variance of the indicator shock could not be very large relative to the variance of the permanent technology shock. This would have resulted in cyclical employment varying less than cyclical productivity which is inconsistent with the data.

Of particular importance for the model is the dependence of current utility on past leisure choices which admits greater intertemporal substitution of leisure. The purpose of this specification is not to contribute to the persistence of output changes. If anything, it does just the opposite. This element of the model is crucial in making it consistent with the observation that cyclical employment fluctuates substantially more than productivity does. For the parameter values in Table I, the standard deviation of hours worked is 18 per cent greater than the deviation of productivity. The special case of $\alpha_0 = 1$ corresponds to a standard time-separable utility function. For this case, with the parameters otherwise the same as in Table I, the standard deviation of hours is 24 per cent less than the deviation of productivity.

Importance of Time to Build

Of particular interest is the sensitivity of our results to the specification of investment technology. The prominent alternative to our time-to-build technology is the adjustment-cost structure. If only one period is required for the construction of new productive capital, we can write the law of motion for the single capital good as $k_{t+1} = (1 - \delta)k_t + s_t$, where s_t is the amount of investment in productive capital in period t. We can then introduce cost of adjustment into the model by modifying the resource constraint (3.4) as follows:

$$c_t + i_t + \xi(s_t - \delta k_t)^2 \leq f(\lambda_t, k_t, n_t, y_t),$$

where the parameter ξ is nonnegative. The model in Section 3 implied that the price of investment goods, i_t, relative to consumption goods, c_t, must be one. This price will now of course generally not equal one, but our cost-of-adjustment formulation insures that it is one when net investment is zero.

The magnitude of the adjustment cost can probably best be judged in terms of the effect it has on this relative price of investment goods which differs from one by the amount $2\xi(s_t - \delta k_t)$. If, for example, the parameter ξ is 0.5, and the economy is near its steady state, a one per cent increase in the relative price of the investment good would be associated with a four per cent increase in gross investment which is approximately one per cent of GNP.

Even when the adjustment cost is of this small magnitude, the covariance properties of the model are grossly inconsistent with the U.S. data for the post-war period. In particular, most of the fluctuation of output in this model is caused by productivity changes rather than changes in work hours. The standard deviation of hours is 0.60, while the standard deviation of productivity is 1.29. This is just the opposite of what the U.S. data show.

Further evidence of the failure of the cost-of-adjustment model is that, relative to the numbers reported in Table III for our model, the standard deviation is nearly doubled for consumption and reduced by a factor of two for investment expenditures, making the amplitudes of these two output components much too close as compared with the data. In addition, the standard deviation of capital stock was reduced by more than one half. The results were even worse for larger values of ξ.

The extreme case of $\xi = 0$ corresponds to the special case of $J = 1$ in our model. Thus, neither time to build nor cost of adjustment would be an element of the model. The biggest changes in the results for this version as compared with Table III are that the correlation between capital stock and output becomes positive and of sizable magnitude (0.43 if the parameters are otherwise the same as in Table I), and that the correlation between inventory stock and output becomes negative (-0.50 for our parameter values). Both of these correlations are inconsistent with the observations. Also, the persistence of movements in investment expenditures as measured by the autocorrelations was substantially reduced.

For our model with multiple periods required to build new capital, the results are not overly sensitive to the number of periods assumed. With a three or five-quarter construction period instead of four, the fit is also good.

6. CONCLUDING COMMENTS

A competitive equilibrium model was developed and used to explain the autocovariances of real output and the covariances of cyclical output with other aggregate economic time series for the post-war U.S. economy. The preference-technology environment used was the simplest one that explained the *quantitative* co-movements and the serial correlation properties of output. These results indicate a surprisingly good fit in light of the model's simplicity.

A crucial element of the model that contributed to persistence of output movements was the time-to-build requirement.[17] We experimented with adjustment costs, the standard method for introducing persistence (e.g., [**4**, **33**]), and found that they were not a substitute for the time-to-build assumption in explaining the data.[18] One problem was that, even with small adjustment costs, employment and investment fluctuations were too small and consumption fluctuations too large to match with the observations.

There are several refinements which should improve the performance of the model. In particular, we conjecture that introducing as a decision variable the hours per week that productive capital is employed, with agents having prefer-

[17] Capital plays an important role in creating persistence in the analysis of Lucas [**23**] as well as in those of Blinder and Fischer [**5**] and Long and Plosser [**22**]. In [**23**] gradual diffusion of information also plays a crucial role. This is not the case in our model, however, as agents learn the value of the shock at the end of the period. Townsend [**37**] analyzes a model in which decision makers forecast the forecasts of others, which gives rise to confounding of laws of motion with forecasting problems, and results in persistence in capital stock and output movements.

[18] An alternative way of obtaining persistence is the use of long-term staggered nominal wage contracts as in [**35**].

ences defined on hours worked per week, should help. Introducing more than a single type of productive capital, with different types requiring different periods for construction and having different patterns of resource requirement, is feasible. It would then be possible to distinguish between plant, equipment, housing, and consumer durables investments. This would also have the advantage of permitting the introduction of features of our tax system which affect transformation opportunities facing the economic agents (see, e.g., [**14**]). Another possible refinement is in the estimation procedure. But, in spite of the considerable advances recently made by Hansen and Sargent [**15**], further advances are needed before formal econometric methods can be fruitfully applied to testing this theory of aggregate fluctuations.

Models such as the one considered in this paper could be used to predict the consequence of a particular policy rule upon the operating characteristics of the economy.[19] As we estimate the preference-technology structure, our structural parameters will be invariant to the policy rule selected even though the behavioral equations are not. There are computational problems, however, associated with determining the equilibrium behavioral equations of the economy when feedback policy rules, that is, rules that depend on the aggregate state of the economy, are used. The competitive equilibrium, then, will not maximize the welfare of the stand-in consumer, so a particular maximization problem cannot be solved to find the equilibrium behavior of the economy. Instead, methods such as those developed in [**20**] to analyze policy rules in competitive environments will be needed.

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[19] Examples of such policy issues are described in [**21**]. See also Barro (e.g., [**3**]), who emphasizes the differences in effects of temporary and permanent changes in government expenditures.

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