"Rational" Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule

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Alternative monetary policies are analyzed in an ad hoc macroeconomic model in which the public's expectations about prices are rational. The ad hoc model is one in which there is long-run neutrality, since it incorporates the aggregate supply schedule proposed by Lucas. Following Poole, the paper studies whether pegging the interest rate or pegging the money supply period by period minimizes an ad hoc quadratic loss function. It turns out that the probability distribution of output—dispersion as well as mean—is independent of the particular deterministic money supply rule in effect, and that under an interest rate rule the price level is indeterminate.

This paper analyzes the effects of alternative ways of conducting monetary policy within the confines of an ad hoc macroeconomic model. By ad hoc we mean that the model is not derived from a consistent set of assumptions about individuals' and firms' objective functions and the information available to them. Despite this deplorable feature of the model, it closely resembles the macroeconomic models currently in use, which is our excuse for studying it. Following Poole (1970), we compare two alternative strategies available to the monetary authority. One is to peg the interest rate period by period, letting the supply of money be whatever it must be to satisfy the demand for it. The other is to set the money supply period by period, accepting whatever interest rate equilibrates the system. We study the effects of such policies for two versions of the model: an autoregressive version in which the public's expectations are assumed formed via fixed autoregressive schemes on the variables being forecast, and a rational-expectations version in which the public's expectations are

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assumed equal to objective (mathematical) expectations that depend on, among other things, what is known about the rules governing monetary and fiscal policy (see Muth 1961).

The two versions have radically different policy implications. In the rational-expectations version, (*a*) the probability distribution of output is independent of the deterministic money supply rule in effect, (*b*) if the loss function includes quadratic terms in the price level, then the optimal deterministic money supply rule is that which equates the expected value of next period's price level to the target value, and (*c*) a unique equilibrium price level does not exist if the monetary authority pegs the interest rate period by period, regardless of how its value varies from period to period. None of these results emerges from the autoregressive version. It, instead, exhibits all the usual exploitable tradeoffs between output and inflation, implies that minimization of the above loss function is a well-defined nontrivial dynamic problem giving rise to a unique optimal deterministic feedback rule either for the money supply or for the interest rate, and has a unique period-by-period equilibrium if the interest rate is pegged. Thus, in the autoregressive version of the model, which in principle is merely a variant of Poole's model, whether an interest rate feedback rule or a money supply feedback rule is superior depends, just as Poole asserted, on most of the parameters of the model including the covariance matrix of the disturbances.

In the rational-expectations version of the model, one deterministic money supply rule is as good as any other, insofar as concerns the probability distribution of real output. In this weak sense, an X percent growth rule for the money supply is optimal in this model, from the point of view of minimizing the variance of real output. Thus, switching from the assumption of autoregressive expectations to that of rational expectations has drastic policy implications. In particular, making that change transforms the model in which following Friedman's X percent growth rule would in general be foolish into one in which such a rule can be defended as being the best the authority can do.

1. The Ad Hoc Model

We assume a structure described by the following equations:[1]

aggregate supply schedule:

$$y_t = a_1 k_{t-1} + a_2(p_t - {}_t p^*_{t-1}) + u_{1t}, \qquad a_i > 0, i = 1, 2; \tag{1}$$

aggregate demand schedule or "*IS*" curve:

$$\begin{aligned} y_t = {} & b_1 k_{t-1} + b_2[r_t - ({}_{t+1}p^*_{t-1} - {}_t p^*_{t-1})] \\ & + b_3 Z_t + u_{2t}, \qquad b_1 > 0, b_2 < 0; \end{aligned} \tag{2}$$

[1] The structure closely resembles the model used by Sargent (1973).

portfolio balance or "*LM*" schedule:

$$m_t = p_t + c_1 y_t + c_2 r_t + u_{3t}, \qquad c_1 > 0, c_2 < 0; \tag{3}$$

determination of productive capacity:

$$k_t = d_1 k_{t-1} + d_2[r_t - ({}_{t+1}p^*_{t-1} - {}_t p^*_{t-1})] + d_3 Z_t + u_{4t}, \quad d_2 < 0; \tag{4}$$

evolution of the exogenous variables:

$$\begin{aligned} Z_t &= \sum_{j=1}^{q} \rho_j Z_{t-j} + \xi_t, \\ u_{it} &= \sum_{j=1}^{q} \rho_{ij} u_{i,t-j} + \xi_{i,t}. \end{aligned} \tag{5}$$

Here y_t, p_t, and m_t are the natural logarithms of output, the price level, and the money supply, respectively; r_t is the nominal rate of interest itself (not its logarithm); while Z_t is the vector of exogenous variables. The variable ${}_{t+i}p^*_{t-j}$ is the public's psychological expectation of the log of the price level to prevail at $t + i$, the expectation being held as of the end of period $t - j$. The variable k_{t-1} is a measure of productive capacity, such as the logarithm of the stock of capital or labor or some linear combination of the logarithms of those stocks at the end of period $t - 1$.

Equation (1) is an aggregate supply schedule relating output directly to productive capacity and the gap between the current price level and the public's prior expectation of the current price level. Unexpected increases in the price level thus boost aggregate supply, the reason being that suppliers of labor and goods mistakenly interpret surprise increases in the aggregate price level as increases in the relative prices of the labor and goods they are supplying. This happens because suppliers receive information about the prices of their own goods faster than they receive information about the aggregate price level. This is the kind of aggregate supply schedule that Lucas (1973) has used to explain the inverse correlation between observed inflation and unemployment depicted by the Phillips curve.

Equation (2) is an aggregate demand or "*IS*" schedule showing the dependence of aggregate demand on the real rate of interest and capacity, a measure of wealth. The real rate of interest equals the nominal rate r_t minus the rate of inflation between t and $t + 1$ expected by the public as of the end of period $t - 1$, namely, ${}_{t+1}p^*_{t-1} - {}_t p^*_{t-1}$. The rate r_t is assumed to be the yield to maturity on a one-period bond. Aggregate demand also depends on a vector of exogenous variables Z_t which includes government expenditures and tax rates.

Equation (3) summarizes the condition for portfolio balance. Owners of bonds and equities (assumed to be viewed as perfect substitutes for one another) are satisfied with the division of their portfolios between money, on the one hand, and bonds and equities, on the other hand, when equation (3) is satisfied. Equation (3) posits that the demand for real balances depends directly on real income and inversely on the nominal rate of interest.

Equation (4) determines productive capacity for the next period, while equation (5) describes autoregressive processes for the exogenous variables. The ξ's, which are sometimes called the "innovations" in the Z and u processes, are serially uncorrelated random variables.

To complete the model, we need equations describing ${}_{t+1}p^*_{t-1}$ and ${}_tp^*_{t-1}$. Adding those equations to (1)–(5) then results in a system capable of determining the evolution over time of y_t, p_t, r_t, ${}_{t+1}p^*_{t-1}$, and ${}_tp^*_{t-1}$ and k_t.

2. The Stabilization Problem

In order to discuss policy within the context of an ad hoc model, we must adopt an ad hoc loss function. The most familiar such function is the quadratic loss function

$$L = E_0 \sum_{t=1}^{\infty} \delta^{t-1}\left[(y_t, p_t)K(y_t, p_t)' + (y_t, p_y)(K_1, K_2)' + \frac{K_1^2}{4} + \frac{K_2^2}{4}\right],$$

where K is diagonal with elements $K_{ii} > 0$, $i = 1, 2$; K_1 and K_2 are parameters; and $0 < \delta < 1$. This function is separately quadratic in y and p and implies that $L = 0$, its lower bound, at particular constant values of y and p, the target values $-K_1/2K_{11}$ for y and $-K_2/2K_{22}$ for p. This function is easy to work with because it is quadratic, additive over time, and stationary in that the function of y and p whose expectation is to be minimized does not depend on t.

To minimize L, the monetary authority compares two strategies. The first is to peg r_t via a deterministic linear feedback rule

$$r_t = G\theta^*_{t-1}, \tag{6}$$

where θ^*_t represents the set of current and past values of all of the endogenous and exogenous variables in the system as of the end of period t, and G is a vector of parameters conformable to θ^*_{t-1}. The monetary authority chooses the parameters in G to minimize L. It must then compare the minimum loss associated with an interest rate rule having those G's with the loss associated with the best money supply feedback rule of the form

$$m_t = H\theta^*_{t-1}. \tag{7}$$

Whichever rule delivers the lower loss is the one that should be used.

3. The Autoregressive Expectations Version

Here we assume that the psychological expectations ${}_t p_{t-1}^*$ and ${}_{t+1}p_{t-1}^*$ are governed by the distributed-lag or "adaptive" schemes

$$ {}_{t+1}p_t^* = \sum_{i=0}^{q} v_{1i} p_{t-i}, \tag{8}$$

$$ {}_{t+2}p_t^* = \sum_{i=0}^{q} v_{2i} p_{t-i}, \tag{9}$$

where the v_{1i}'s and v_{2i}'s are fixed numbers. Given that the money supply is used as the monetary instrument, the system formed by equations (1)–(5), (8), and (9) can be reduced to a difference equation of the form

$$ Y_{1t} = \sum_{i=1}^{q'} A_i Y_{1t-i} + \sum_{i=0}^{q'} B_i m_{t-i} + \phi_{1t}, \tag{10}$$

where $Y'_{1t} = (y_t, p_t, r_t, k_{t-1}, Z_t)$ and ϕ_{1t} is a vector of serially uncorrelated random variables, the components of which are functions of the ξ_t's in equations (5). The A_i's are vectors conformable with Y_{1t} and the B_i's are scalars; both the A_i's and B_i's depend on the parameters of equations (1)–(5), (8), and (9). To find the best money-supply feedback rule, the monetary authority chooses the parameters H of the rule (7) to minimize the loss L subject to (10). Where loss is quadratic and the model is linear with known coefficients, rules of the linear form of (7) are known to be optimal.[2]

To find the optimal interest rate rule, the system formed by equations (1)–(5), (8), and (9) is written as

$$ Y_{2t} = \sum_{i=1}^{q'} C_i Y_{2t-i} + \sum_{i=0}^{q'} D_i r_{t-i} + \phi_{2t}, \tag{11}$$

where $Y'_{2t} = (y_t, p_t, m_t, k_{t-1}, Z_t)$. The optimal interest rate rule is the one with the G's of (6) chosen so as to minimize loss L subject to (11).[3]

To show that (1)–(5), (8), and (9) yield versions of (10) and (11) that give rise to well-defined, nontrivial dynamic problems, it is enough to examine the one-period reduced forms for y_t and p_t.

With the money supply as the monetary instrument, we solve (1)–(3) for y, r, and p and get as a reduced form for p_t

$$ p_t = J_0({}_t p_{t-1}^*) + J_1({}_{t+1}p_{t-1}^*) + J_2 m_t + X_t, \tag{12}$$

[2] See Chow (1970).

[3] Chow (1970) describes how optimal rules of the form (6) or (7) are found for a system like (10) or (11).

where X_t is a linear function (involving the parameters of [1]–[3]) of k_{t-1}, Z_t, and the u_{it}'s and where

$$J_0 = [a_2(1 + b_2c_2^{-1}) + b_2]/[a_2(1 + b_2c_2^{-1}) + b_2c_2^{-1}] < 1,$$
$$J_1 = (1 - J_0)/(1 - c_2^{-1}),$$
$$J_2 = -c_2^{-1}J_1.$$

Substitution of p_t from equation (12) into equation (1) gives the one-period reduced form for y_t. Taking E_{t-1} of p_t and y_t from these reduced forms and eliminating m_t gives the set of pairs $(E_{t-1}y_t, E_{t-1}p_t)$ attainable by choice of m_t. The set is a line whose slope is neither infinity nor zero. Its position, obviously, depends on lagged values of p, via the p^* variables, and on lagged values of other endogenous variables, the distributions of which depend on lagged values of m. In other words, the choice for the deterministic part of m_t has effects in future periods, which is what we mean when we say that (10) gives rise to a nontrivial dynamic problem.

With the interest rate as the monetary instrument, equation (2) is the one-period reduced form for y_t while that for p_t is obtained by substituting the solution for y_t into equation (1) and solving for p_t. The solution for p_t is

$$\begin{aligned} a_2p_t = {} & (a_2 + b_2)\,{}_tp^*_{t-1} - b_2({}_{t+1}p^*_{t-1}) + b_2r_t \\ & + (b_1 - a_1)k_{t-1} + b_3Z_t - u_{1t} + u_{2t}. \end{aligned} \tag{13}$$

Again, if we take E_{t-1} of equation (2) and equation (13) and eliminate r_t, we find the set of pairs $(E_{t-1}y_t, E_{t-1}p_t)$ attainable by choice of r_t. That set again depends on lagged values of p, which shows that (11) also gives rise to a nontrivial dynamic problem.

The monetary authority is supposed to solve each of the two dynamic problems, minimizing loss first under an m rule and then under an r rule. Which policy is superior depends on which delivers the smaller loss, which in turn depends on all of the parameters of the model, including the covariance matrix of the disturbances. Which rule is superior is therefore an empirical matter, an outcome which is completely consistent with Poole's analysis.

4. The Rational-Expectations Version under a Money Supply Rule

Here we impose the requirement that the public's expectations be rational by requiring that

$${}_{t+i}p^*_{t-j} = E_{t-j}p_{t+i}, \tag{14}$$

where $E_{t-j}p_{t+i}$ is the mathematical expectation of p_{t+i} calculated using the model (i.e., the probability distribution of p_{t+i}) and all information assumed to be available as of the end of period $t - j$. The available information is assumed to consist of data on current and past values of all endogenous and exogenous variables observed as of the end of period $t - j$, that is, θ^*_{t-j}.

To begin, we again solve the system (1)–(3) for y, r, and p given m. With expectations given by (14), what is now a pseudo-reduced-form equation for p is

$$p_t = J_0 E_{t-1} p_t + J_1 E_{t-1} p_{t+1} + J_2 m_t + X_t. \tag{15}$$

Computing $E_{t-1}p_t$ from (15) and subtracting the result from (15) we get

$$\begin{aligned} p_t - E_{t-1}p_t &= J_2(m_t - E_{t-1}m_t) + X_t - E_{t-1}X_t \\ &= X_t - E_{t-1}X_t, \end{aligned} \tag{16}$$

where the last equality follows from the assumption that a deterministic rule of the form (7) is being followed. But since $X_t - E_{t-1}X_t$ is a linear combination of the innovations in the exogenous processes, it follows that $p_t - E_{t-1}p_t$ is an exogenous process, unaffected by the rule chosen for determining the money supply.

Using (14) and (16), we can write equation (1) as

$$y_t = a_1 k_{t-1} + a_2(X_t - E_{t-1}X_t) + u_{1t}. \tag{17}$$

If we substitute the right-hand side for y_t in equation (2), we can obtain the real interest rate as a function of k_{t-1} and exogenous processes. Substituting that function into equation (4), we get a difference equation in k driven by exogenous processes. This proves that k is an exogenous process, which by (17) implies that y is an exogenous process, that is, has a distribution independent of the deterministic rule for the money supply. So we have proved assertion (*a*) above: the distribution of output does not depend on the parameters of the feedback rule for the money supply.

To prove assertion (*b*), we write the tth term of the loss function L as

$$L_t = E_0[E_{t-1}(K_2 p_t + K_{22} p_t^2 + K_1 y_t + K_{11} y_t^2)],$$

where the insertion of E_{t-1} is valid for $t > 0$. Using $E(x^2) = E[(x - Ex)^2] + (Ex)^2$, we have

$$L_t = E_0[K_{0t} + K_2 E_{t-1}p_t + K_{22}(E_{t-1}p_t)^2],$$

where

$$K_{0t} = E_{t-1}[K_{22}(p_t - E_{t-1}p_t)^2 + K_1 y_t + K_{11} y_t^2]$$

and where, given the exogeneity of y_t and $p_t - E_{t-1}p_t$ proved above, K_0 is an exogenous process. Moreover, it is possible, as we shall show below, to find a rule for m that implies choosing $E_{t-1}p_t$ to minimize

$$K_{0t} + K_2 E_{t-1}p_t + K_{22}(E_{t-1}p_t)^2.$$

And, because K_{0t} is unaffected by settings for the money supply at any other time, a rule which minimizes L_t also minimizes L.

To show that there exists such a rule for m, we must solve the model. Again, we take $E_{t-1}p_t$ in (15) and write the result as

$$(1 - J_0)E_{t-1}p_t = J_1E_{t-1}p_{t+1} + J_2E_{t-1}m_t + E_{t-1}X_t. \tag{18}$$

Since this holds for all t, it follows that

$$\begin{aligned}(1 - J_0)E_{t-1}p_{t+j} = J_1E_{t-1}p_{t+j+1} &+ J_2E_{t-1}m_{t+j} \\ &+ E_{t-1}X_{t+j}.\end{aligned} \tag{19}$$

By repeated substitution from (19) into (18), we obtain

$$\begin{aligned}(1 - J_0)E_{t-1}p_t = \sum_{j=0}^{n} &[J_1/(1 - J_0)]^j(E_{t-1}X_{t+j} + J_2E_{t-1}m_{t+j}) \\ &+ [J_1/(1 - J_0)]^{n+1}E_{t-1}p_{t+n+1},\end{aligned} \tag{20}$$

where

$$0 < J_1/(1 - J_0) = 1/(1 - c_2^{-1}) < 1.$$

We assume that the limit as $n \to \infty$ of the second term on the right-hand side of (20) is zero, which is a terminal condition that has the effect of ruling out "speculative bubbles." Then, from (20),

$$(1 - J_0)E_{t-1}p_t = \sum_{j=0}^{\infty} [J_1/(1 - J_0)]^jE_{t-1}(X_{t+j} + J_2m_{t+j}). \tag{21}$$

Since this holds for all t, we may replace t by $t + 1$ and compute E_{t-1} of the result to get

$$\begin{aligned}(1 - J_0)E_{t-1}p_{t+1} = \sum_{j=0}^{\infty} &[J_1/(1 - J_0)]^j \\ &\times E_{t-1}(X_{t+j+1} + J_2m_{t+j+1}).\end{aligned} \tag{22}$$

For an arbitrary money supply rule of the form (7), substituting (21) and (22) into (15) gives the solution for p_t; substituting (21) and (22) into (2) gives the solution for r_t. This assumes that the rule is not such as to imply too explosive a process for $X_{t+j} + J_2m_{t+j}$.[4]

[4] A workable "reduced form" for p_t can be obtained by substituting (20) into (15) and then by using (5) and (7) to replace $E_{t-1}m_{t+j}$ and $E_{t-1}X_{t+j}$ with the linear functions of past variables that they equal. These linear functions are easily calculated from the feedback rule for m_t and the autoregressions governing components of X_t. While the resulting "reduced form" for p_t formally resembles the corresponding equation in the system with "adaptive" expectations, there is a crucial difference. Now changes in the parameters of the feedback rule for m_t produce changes in the parameters of the reduced form for p_t. This feature of the system is what renders Poole's results inapplicable. For an explicit illustration of the dependence of the reduced-form parameters on the form of the policy rule, see Sargent and Wallace (1973, pp. 332–33).

To find the optimal money supply rule, multiply (22) by $J_1/(1 - J_0)$ and subtract the result from (21) to get

$$(1 - J_0)E_{t-1}p_t - J_1E_{t-1}p_{t+1} = E_{t-1}X_t + J_2m_t. \tag{23}$$

The value of $E_{t-1}p_t$ that minimizes L_t for all t is

$$E_{t-1}p_t = -K_2/2K_{22}, \tag{24}$$

so that

$$E_{t-1}p_{t+1} = -K_2/2K_{22}. \tag{25}$$

The optimal rule for the money supply is obtained by substituting (24) and (25) into (23). The resulting expression for m_t is a feedback rule of the form (7).

Thus, in the rational-expectations version of our model, the choice among deterministic rules for the money supply is a trivial problem. One argument of the loss function, y, is unaffected by the rule, so the problem becomes the simplest kind of one target–one instrument problem. Moreover, a definite rule emerges only because we have assumed in specifying L that there is a target value for the price level. If, instead, loss were made dependent only on the variance of the price level, then one deterministic rule would be as good as any other.

The reason that the distribution of real output is independent of the systematic money supply rule can be summarized as follows. In order for the monetary authority to induce fluctuations in real output, it must induce unexpected movements in the price level by virtue of the aggregate supply curve (1). But by virtue of the assumption that expectations about the price level are rational, the unexpected part of price movements is independent of the systematic part of the money supply, as long as the authority and the public share the same information. There is no systematic rule that the authority can follow that permits it to affect the unexpected part of the price level. Of course, the authority could add an unpredictable random term to the systematic part of the money supply, so that (7) would be amended to become

$$m_t = H\theta^*_{t-1} + \psi_t, \tag{7'}$$

where ψ_t is a random variable obeying $E\psi_t \mid \theta^*_{t-1} = 0$. Then the distribution of unexpected price movements and of real output will depend on the distribution of ψ_t. But clearly, there is no way the authority can base a countercyclical policy on this particular nonneutrality, since there is no way the authority can regularly choose ψ_t in response to the state of economic affairs in order to offset other disturbances in the system.

This follows since ψ_t is that part of the money supply obeying $E\psi_t \mid \theta^*_{t-1} = 0$. Furthermore, in our model it is optimal to set $\psi_t = 0$ for all t.

5. The Rational-Expectations Version under an Interest Rate Rule

Above we showed that a certain terminal condition implied the existence of a unique equilibrium price level for the rational-expectations version under a money supply rule that is not too explosive. That analysis took as a starting point the difference equation (18). With the interest rate determined by the feedback rule (6), a seemingly analogous difference equation is obtained by imposing rationality, equation (14), in (13) and taking E_{t-1} of the result

$$0 = b_2(E_{t-1}p_t - E_{t-1}p_{t+1}) + b_2 r_t + (b_1 - a_1)k_{t-1} + b_3 E_{t-1}(Z_t - u_{1t} + u_{2t}). \tag{26}$$

If we solve (26) by recursion, proceeding as we did in deriving (20) from (18), we find

$$E_{t-1}p_t = -\sum_{j=0}^{n} E_{t-1}\{r_{t+j} + [(b_1 - a_1)/b_2]k_{t+j-1} + (b_3/b_2) \times (Z_{t+j} - u_{1t+j} + u_{2t+j})\} + E_{t-1}p_{t+n+1}. \tag{27}$$

To obtain a particular solution for $E_{t-1}p_t$ from (27) requires imposing a terminal condition in the form of taking as exogenous a value of $E_{t-1}p_{t+j}$ for some $j \geq 0$. This is obviously a very much stronger terminal condition than we had to impose on (20), a consequence of (26) being a non-convergent difference equation. Thus, when the interest rate is pegged, the model cannot determine a path of expected prices $E_{t-1}p_{t+j}$, $j = 0, 1, \ldots,$ and by implication cannot determine the price level p_t. Neither can it determine the money supply.

The economics behind the underdetermined expected price level is pretty obvious. Under the interest rate rule (6), the public correctly expects that the monetary authority will accommodate whatever quantity of money is demanded at the pegged interest rate. The public therefore expects that, *ceteris paribus,* any increase in p_t will be met by an equal increase in m_t. But that means that one $E_{t-1}p_t$ is as good as any other from the point of view of being rational. There is nothing to anchor the expected price level. And this is not simply a matter of choosing the "wrong" level or rule for the interest rate. There is no interest rate rule that is associated with a determinate price level.

At least since the time of Wicksell it has been known that, in the context of a static analysis of a full-employment model with wages and prices that are flexible instantaneously, it can happen that the price level is indeterminate if the monetary authority pegs the interest rate.[5] In such a static analysis, the indeterminacy of the price level depends critically on output and employment being exogenous with respect to shocks to aggregate demand or portfolio balance; that is, the Phillips curve must be vertical. In our model, however, the Phillips curve is not vertical, but Wicksell's indeterminacy still arises.

6. An Information Advantage for the Monetary Authority

Here we shall examine some consequences of the monetary authority having more information than the public. We shall first show that if the monetary authority follows the money supply rule that is optimal if there is no information discrepancy, then the loss attained is the same as attained when there is no information discrepancy. Then we consider whether that rule is optimal given an information discrepancy.

We shall write E_{t-1} for the expectation conditional on what the monetary authority knows at the end of period $t - 1$ and $E_{\theta,t-1}$ for the expectation conditional on what the public knows at the end of period $t - 1$, where θ is a subset of what the monetary authority knows. Then in place of (14) we impose

$$ {}_{t+i}p^*_{t-j} = E_{\theta,t-j}p_{t+i}, \tag{28} $$

so that in place of (15) we have

$$ p_t = J_0 E_{\theta,t-1}p_t + J_1 E_{\theta,t-1}p_{t+1} + J_2 m_t + X_t. \tag{29} $$

Then, taking $E_{\theta,t-1}$ of p_t and subtracting from p_t, we have

$$ p_t - E_{\theta,t-1}p_t = J_2(m_t - E_{\theta,t-1}m_t) + (X_t - E_{\theta,t-1}X_t). \tag{30} $$

The rule that we found to be optimal without an information discrepancy is

$$ J_2 m_t = -(K_2/2K_{22})(1 - J_0 - J_1) - E_{t-1}X_t. \tag{31} $$

From this it follows that

$$ J_2(m_t - E_{\theta,t-1}m_t) = -E_{t-1}X_t + E_{\theta,t-1}X_t. \tag{32} $$

Substituting into (30), we have

$$ p_t - E_{\theta,t-1}p_t = X_t - E_{t-1}X_t. \tag{33} $$

[5] See Olivera (1970). In both our model and the standard static model, the aggregate demand schedule must exclude any components of real wealth that vary with the price level if Wicksell's indeterminacy is to arise. For example, if the anticipated rate of capital gains on real (outside) money balances is included in the aggregate demand schedule, the price level is determinate with a pegged interest rate. However, such a system has peculiar stability characteristics, since stability hinges on the sign of the expected rate of inflation.

Upon substituting from (33) into equation (1) of the structure, we get equation (17). And if we substitute for y_t in equation (2) the right-hand side of (17), we obtain $[r_t - E_{\theta,t-1}(p_{t+1} - p_t)]$ as a function of k_{t-1} and the exogenous processes, the same function that we previously found for $r_t - E_{t-1}(p_{t+1} - p_t)$. Then, substituting this function into equation (5) of the structure, we get the same first-order difference equation in k as we had without an information discrepancy. This proves that under the rule given by (31), the distribution of k does not depend on the information discrepancy. It follows then from equation (17) that the same is true for y.

To find the distribution of p_t, we proceed to solve the difference equation

$$(1 - J_0)E_{\theta,t-1}p_t = J_1E_{\theta,t-1}p_{t+1} + J_2E_{\theta,t-1}m_t + E_{\theta,t-1}X_t, \tag{34}$$

which is obtained by taking $E_{\theta,t-1}$ of (29). Then, proceeding as we did for (18), we obtain expressions exactly like (21) and (22) except that in place of E_{t-1} on the left and right we have $E_{\theta,t-1}$.

But from (31)

$$X_{t+j} + J_2m_{t+j} = -(K_2/2K_{22})(1 - J_0 - J_1) + X_{t+j} - E_{t-1}X_{t+j},$$

so for $j > 0$

$$\begin{aligned} E_{\theta,t-1}(X_{t+j} + J_2m_{t+j}) &= -(K_2/2K_{22})(1 - J_0 - J_1) \\ &= E_{t-1}(X_{t+j} + J_2m_{t+j}). \end{aligned}$$

Thus, use of the rule given by (31) implies $E_{\theta,t-1}p_{t+j} = E_{t-1}p_{t+j}$, $j = 0, 1$, which by (29) implies that under the rule given by (31) the distribution of p does not depend on θ, that is, does not depend on the information discrepancy. It follows that the loss attained under the rule given by (31) does not depend on the information discrepancy.

This shows that the monetary authority can do as well given an information discrepancy as it can do if there is none. But can it do better? Can it, as it were, take advantage of the presence of an information discrepancy? We are not sure. But within our structure, the answer seems to be that it can take advantage of a discrepancy, although necessarily in a limited and rather subtle way.

To indicate why, let us focus first on how the distribution of y depends on the rule for m. Under present assumptions, equation (1) of the structure is

$$y_t = a_1k_{1-t} + a_2(p_t - E_{\theta,t-1}p_t) + u_{1t}. \tag{35}$$

It follows that as of the end of $t - 1$, $E_{\theta,t-1}y_t$ is unaffected by the choice of m_t, since

$$E_{\theta,t-1}y_t = a_1E_{\theta,t-1}k_{t-1} + E_{\theta,t-1}u_{1t}. \tag{36}$$

To find the variance of y_t, we subtract (36) from (35) and obtain $\tilde{y}_t = a_1\tilde{k}_{t-1} + a_2\tilde{p}_t + \tilde{u}_{1t}$, where $\tilde{x}_t \equiv x_t - E_{\theta,t-1}x_t$. The variance of

y_t around $E_{\theta,t-1}y_t$ is, therefore,

$$E_{\theta,t-1}(\tilde{y}_t^2) = E_{\theta,t-1}[(a_1\hat{k}_{t-1} + \hat{u}_{1t})\hat{p}_t] + E_{\theta,t-1}(\hat{p}_t^2) + \text{other terms,} \tag{37}$$

where $\hat{x}_t \equiv E_{t-1}x_t - E_{\theta,t-1}x_t$ and where the omitted terms are unaffected by the setting for the deterministic part of m_t. Thus, setting m_t according to (31) (i.e., setting $\hat{p}_t = 0$) minimizes $E_{\theta,t-1}(\tilde{y}_t^2)$ only if the first term on the right-hand side of (37) cannot be made negative. That term can be made negative by a rule different from (31) if $a_1\hat{k}_{t-1} + \hat{u}_{1t} \neq 0$, that is, if the monetary authority knows more about either the k_{t-1} or the u_1 process than does the public. Of course, to take advantage of this information discrepancy, the monetary authority must know precisely how the public's information differs from its own.

Similar conclusions hold for the distribution of k_t. The expectation $E_{\theta,t-1}k_t$ is unaffected by the setting for m_t, but, in general, the variance $E_{\theta,t-1}(\tilde{k}_t^2)$ depends on it and is not minimized by use of the rule given by (31).[6] And since the setting for m_t affects the distribution of (y_{t+j}, p_{t+j}) for $j > 0$ by way of its effect on the distribution of k_t, this means that, given an information discrepancy, our structure gives rise to a nontrivial dynamic problem.

But this should not be taken to mean that we are back in the setting produced by the assumption that expectations are formed on the basis of fixed autoregressive schemes. The information discrepancy assumption does not produce any simple tradeoff between the means of output and the price level. The fact that $E_{\theta,t-1}y_t$ and $E_{\theta,t-1}k_t$ are unaffected by m_t is very limiting if θ contains, say, as little as $(1, p_{t-1}, y_{t-1})$. Second, to exploit the information discrepancy, the monetary authority must know what it is. To assume that it does seems farfetched. Indeed, we suspect that estimating the discrepancy is a very subtle and perhaps intractable econometric problem.

For these reasons, we think some comfort can be taken from the first result established in this section. Use of the rule given by (31) is optimal if the public is as well informed as the monetary authority. The loss attained under that rule does not depend on how well informed the public is, and implementation of the rule does not required knowledge of how well informed the public is.

This does not, of course, deny that there is a gain from learning more about the exogenous processes. Settings for the money supply under the rule given by (31) depend on what the monetary authority knows. Operating under that rule, loss is smaller the more the monetary authority knows about the exogenous processes.

[6] The reader may verify this by finding k_t as a function of k_{t-1} and $p_t - E_{\theta,t-1}p_t$ using (35) and equations (2) and (5) of the structure.

7. Concluding Remarks

Given that our conclusions are derived from an ad hoc model, should they be taken seriously? In one sense, they should not be. Because of their ad hoc nature, neither the structure set out in section 1 nor the loss function of section 2 should be accepted as providing a suitable context within which to study macroeconomic policy. Nevertheless, some aspects of our model cannot be dismissed so easily. First, the hypothesis that expectations are rational must be taken seriously, if only because its alternatives, for example, various fixed-weight autoregressive models, are subject to so many objections. Second, the aggregate supply hypothesis is one that has some microeconomic foundations,[7] and it has proved difficult to dispose of empirically.[8] It is precisely these two aspects of our model—rational expectations in conjunction with Lucas's aggregate supply hypothesis—that account for most of our results. We believe that the results concerning systematic countercyclical macroeconomic policy are fairly robust to alterations of other features of the model, such as the aggregate demand schedule and the portfolio balance condition. In particular, the dramatically different implications associated with assuming rational expectations, on the one hand, or fixed autoregressive expectations, on the other hand, will survive such alterations.

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[7] For example, see Lucas (1973).

[8] Tests of the aggregate supply hypothesis are reported by Lucas (1973) and Sargent (1973).