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MATURITY AND VOLATILITY EFFECTS ON SMILES

Or

Dying Smiling?

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Abstract

The "smile effect" is a result of an empirical observation of the options' implied volatility with the same expiration date, across different exercise prices. It describes a U-shape form showing high implied volatilities for in and out-of-the-money options and low volatility figures for at-the-money options.

We can find empirical evidence of this phenomenon. The reasons suggested in the literature were stochastic volatility, traders' behaviour, transaction costs, and the effect of dividends on pricing American options. But the most recent literature seems to conclude that the sophistication of financial modelling for option pricing is not enough for removing the "smile".

In this paper we used liquid equity options on 9 stocks traded on the *London International Financial Futures and Options Exchange* (LIFFE) between August 1990 and December 1991. We tested two different hypothesis trying to verify the existence of two different phenomena: (1) the increase of the "smile" as maturity approaches; (2) and the association between the smile and the volatility of the underlying stock.

In order to estimate implied volatilities for unavailable exercise prices, we modelled the smile using cubic B-spline curves.

We found empirical support for the smile intensification (the U-shape is more pronounced) as maturity approaches as well as when volatility rises. However, this increase in the curvature is asymmetric. As maturity approaches the implied volatility of out-of-the-money options tends to be higher than the implied volatility of in-the-money options and, as the volatility of the underlying increases, the implied volatility of in-the-money options tend to be higher than implied volatility of out-of-the-money options. We claim to have detected new empirical reasons for previous empirical findings where the smile was, for some authors, a symmetric smile, while for others it converts into a "wry grin" or a "reverse grin".

Key words: options, volatility, implied volatility, smile effect.

JEL classification: G13

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1. Introduction

The so called “smile effect” is a result of an empirical observation of the options’ implied volatility with the same expiration date, across different exercise prices. It typically describes a U-shape form showing high implied volatility patterns for in and out-of-the-money options and low volatility figures for at-the-money options.

We can find empirical evidences of this phenomena in Heynen [1994], Taylor and Xu [1994], Duque and Paxson [1994], Gemmill [1996], Dumas, Flemming and Whaley [1998] and Peña, Serna and Rubio [1997]. Some authors present theoretical reasons for such evidence as in Hull and White [1987], Taylor and Xu [1994] or Heynen [1994]. Among others, the suggested reasons were stochastic volatility, traders behaviour, transaction costs, and the effect of dividends on pricing American options. But the most recent literature seems to conclude that the sophistication of financial modelling for option pricing is not enough for removing the “smile”.

This paper refines the empirical analysis of the smile, trying to establish statistically significant links between some variables which impact on option valuation and the shape of the smile. We used cubic B-spline curves[2] in order to infer the implied volatility smile and observe how the smile changes with the approach of expiration as well as changes in volatility. We checked for the smile asymmetry and linked time to maturity, volatility and the smile U-shape pattern.

We believe that describing the smile and its association with other variables which impact on option pricing, may modestly contribute to the refinement of future option pricing models.

The paper is organised as follows. We start by reviewing the related literature and some theoretical relations between volatility, time to maturity and pricing bias. Next we describe the database and present the methodology in use. Finally we present the empirical results and a short conclusion.

[1] We thank Professor Dean A. Paxson, António Sousa Câmara, Ser-Huang Poon and the attendees of the 26th EFA Annual Conference for helpful comments on previous versions of this paper.

2. Literature review

When market prices are used to estimate implied volatilities using the Black and Scholes [1973] model, these implied values tend to differ across option series. Typically, when implied volatilities are plotted against a moneyness ratio, at-the-money options tend to have lower values than in or out-of-the-money options. This is the well known smile effect, usually described as showing a U-shape form and empirically detected by different authors for options on different underlying assets.

Although the smile effect as a research topic is a relatively recent one, early studies on option pricing theory had already noticed possible option model bias. In their seminal paper, Black and Scholes [1972] presented evidence for bias on option values when comparing market warrant prices to theoretical prices resulting from their model. Later, Macbeth and Merville [1979], studying stock options listed on the CBOE, found evidence that the Black and Scholes [1973] model was underpricing in-the-money options and overpricing out-of-the-money options. However, Rubinstein [1985], also studying stock options listed on the CBOE, found some confusing patterns. It seemed that the Black and Scholes [1973] model was overpricing out-of-the-money options and underpricing in-the-money options for a time period between August 1976 and October 1977. However, the same model was overpricing in-the-money options and underpricing out-of-the-money options for a time period between October 1977 and October 1978.

More recently, other studies show empirical evidence for this exercise price bias. Clewlow and Xu [1993], studying options on stock index futures listed on the CME, found asymmetric patterns for the smile. Heynen [1994] also found empirical evidence for the exercise price bias when observing stock index options during 1989 on the EOE. Taylor and Xu [1994], studying currency options traded in Philadelphia between 1984 and 1992, found empirically that option bias was twice the size they were expecting, increasing its magnitude as maturity approaches. Duque and Paxson [1994] also found the smile effect for options traded on LIFFE during March 1991, and speculated that there is a possible empirical relation between time to maturity and the U-shape format of the smile. Gemmill [1996] also found the same effect for options on the FTSE 100 during a 5 year period (from 1985 to 1990), although the smile showed different patterns

[2] B-spline curves in their cubic version.

for different days extracted from the sample. Dumas, Fleming and Whaley [1998] also found empirical smile patterns for options on the S&P 500 stock index, but its shape seemed to be asymmetric and changing along time to maturity. Peña, Serna and Rubio [1997] also found empirical smiles for stock index options written on the Ibex 35 listed on MEFF, for a time period between 1994 and 1996, detecting a day of the week effect for the smile.

Other authors tried to extract superior information from the smile as in Bates [1991], concluding that persistence on the smile pattern could imply that the market expected the 1987 crash.

In order to accommodate this empirical bias, researchers have followed two different approaches: developing more sophisticated models trying to compensate for that effect; and modelling the bias itself. Different reasons have been suggested for option pricing bias such as the stochastic volatility of the underlying stock, the observed changes in interest rates or the difference between the observed and the assumed stock price path (like the jump processes). Merton [1976], Hull and White [1987], Scott [1987], Wiggins [1987], Johnson and Shanno [1987], Stein and Stein [1991], Bates [1996], Bakshi and Chen [1997], Cox [1996], Corrado and Su [1996], Madan, Carr and Chan [1998], are examples of models where the basic Black-Scholes assumptions are dropped. Gkamas and Paxson [1999], comparing several options pricing models assuming stochastic volatility, found that they remove a significant part of the bias, but they did not remove it completely. Some other authors impute the smile to the behaviour of traders or to their risk aversion, such as Bookstaber [1991], Grossman and Zhou [1996], Gemmill [1996], Dumas, Fleming and Whaley [1998] and Gemmill and Kamiyama [1997]. Some others attribute the smile to transaction costs, like Clewlow and Xu [1992] and [1993], Constantinides [1997] and Peña, Serna and Rubio [1997]. Finally, Geske and Roll [1984] claim that smiles are a result of errors in valuing American options on stocks paying dividends prior to maturity.

We conclude that the literature is not unanimous in finding causes for the smile effect and the models developed in order to cover this bias have only partially solved the problem.

When discussing the model for pricing options on stocks, assuming both stock price returns and stock return volatility follow a geometric Brownian motion process, Hull and White [1987] found that the theoretical prices would generally differ from the

Black and Scholes solution. These differences could be based on several variables. However, some were discussed in detail: the volatility, the time to maturity and the correlation between the underlying price return and the instantaneous stock price return volatility. The same idea was later pursued in Hull and White [1988] when a different stochastic process was proposed to govern the instantaneous stock price return volatility. In this paper Hull and White [1988] derived an approximate equation for the pricing bias when the Black and Scholes [1973] equation is used.

Let us assume that both, the stock price returns and its volatility follow stochastic processes, such as

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz \qquad \textbf{Eq. 1}$$

$$d\sigma^2 = \left(a + b\sigma^2\right)dt + \xi\, \sigma\, dw \qquad \textbf{Eq. 2}$$

where S stands for the underlying stock price, σ represents the instantaneous stock price return volatility and ξ represents the volatility of the volatility. The parameters a, b and μ are constants and dz and dw are Wiener processes with an instantaneous correlation ρ. The variance in the mean reversion model reverts to the long-term variance $\hat{\sigma}^2 = -\frac{a}{b}$ at a reversion rate of -b. The model suggested by equation 2 is quite general, since it admits several reduced forms for the variance:

Parameter	Model	Consequence
$\xi = 0$	$d\sigma^2 = \left(a + b\sigma^2\right)dt$	the variance is not constant but deterministic
a = 0	$d\sigma^2 = b\sigma^2 dt + \xi\, \sigma\, dw$	the variance is stochastic with a constant proportional drift
b = 0	$d\sigma^2 = a\, dt + \xi\, \sigma\, dw$	the variance is stochastic with a constant drift
a = 0 and b = 0	$d\sigma^2 = \xi\, \sigma\, dw$	the variance is stochastic with no drift
a > 0 and b < 0	$d\sigma^2 = \left(a + b\sigma^2\right)dt + \xi\, \sigma\, dw$	the variance is stochastic following, a mean reverting process

In the last model, as volatility changes, the expected variance at time s conditional on the actual instantaneous variance observed at time t (s≥t) is:

$$\mathrm{E}\left[\sigma^2(s,t)\right] = \left[\sigma_t^2 - \hat{\sigma}^2\right]e^{b(s-t)} + \hat{\sigma}^2 \qquad \textbf{Eq. 3}$$

where $\hat{\sigma}^2$ represents the long term mean to which the process reverts and σ_t^2 represents the actual variance observed at time t. For stock prices, a range of [-10, -1] is reasonable for b and a range of [0.05, 0.1] is reasonable for $\hat{\sigma}^2$.

Assuming the stochastic processes given by equations 1 and 2, Hull and White [1998] showed that the Black and Scholes model should bias the call option theoretical price. The bias was defined as the difference of the observed option premium (calculated according to the stochastic volatility model based on equations 1 and 2) and the Black and Scholes theoretical price:

$$B = c - C_{BS} \qquad \textbf{Eq. 4}$$

The bias can be expressed in terms of an expanded equation as a function of ξ which represents the volatility of the volatility:

$$B = f_0 + f_1\xi + f_2\xi^2 + \dots \qquad \textbf{Eq. 5}$$

If we assume ξ is small, we may approach the bias value by dropping the third and following terms of equation 5 without losing too much precision. Therefore, the bias equation comes:

$$B(\xi) \approx f_0 + f_1\xi \qquad \textbf{Eq. 6}$$

where:

$$f_0 = C\left(\overline{\sigma}^2\right) - C\left(\sigma_t^2\right) \qquad \textbf{Eq. 7}$$

$$f_1 = \frac{\rho}{b^2\delta}\left[\left(a + b\sigma^2\right)\left(1 - e^{\delta} + \delta e^{\delta}\right) + a\left(1 + \delta - e^{\delta}\right)\right] S\,\frac{\partial^2 C\left(\overline{\sigma}^2\right)}{\partial S\,\partial\overline{\sigma}^2} \qquad \textbf{Eq. 8}$$

$$\frac{\partial^2 C\left(\overline{\sigma}^2\right)}{\partial S\,\partial\overline{\sigma}^2} = -\frac{N'(d_1)\,d_2}{2\overline{\sigma}^2}$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\overline{\sigma}^2}{2}\right)(T - t)}{\overline{\sigma}\sqrt{T - t}}$$

$$d_2 = d_1 - \overline{\sigma}\sqrt{T - t}$$

$$\delta = b(T - t)$$

$C\left(\sigma_t^2\right)$ is the Black and Scholes option price for a given instantaneous variance σ_t^2, and $\overline{\sigma}^2$ is the average expected variance rate in the interval (T-t) at time t. This can be expressed as:

$$\overline{\sigma}^2 = \frac{1}{T-t}\int_t^T \mathrm{E}\left[\sigma^2(s,t)\right] \quad ds = \frac{\sigma_t^2 - \hat{\sigma}^2}{b(T-t)}\left(e^{b(T-t)} - 1\right) + \hat{\sigma}^2 \qquad \textbf{Eq. 9}$$

Equation 9 tells us that when the actual instantaneous volatility (σ_t) equals the long-term average volatility ($\hat{\sigma}$), its expected average conditional on the present level equals the long-term average volatility. In such a case $\overline{\sigma}^2 = \hat{\sigma}^2 = \sigma_t^2$ and $C\left(\overline{\sigma}^2\right) = C\left(\sigma_t^2\right)$. Then, equation 6 reduces to:

$$B(\xi) \approx f_1\xi \qquad \textbf{Eq. 10}$$

Another particular situation may be explored assuming that $\rho = 0$. In such a case $f_1 = 0$ and, in order to improve the approximation, we may need an extra term in equation 6.

$$B(\xi) \approx f_0 + f_2\xi^2 \qquad \textbf{Eq. 11}$$

where

$$f_2 = \frac{1}{2b^4(T-t)^2}\left[\left(a + b\sigma^2\right)\left(e^{2\delta} - 2\delta e^{\delta} - 1\right) - \frac{1}{2}a\left(e^{2\delta} - 4e^{\delta} + 2\delta + 3\right)\right]\frac{\partial^2 C\left(\overline{\sigma}^2\right)}{\partial\overline{\sigma}^2} \qquad \textbf{Eq. 12}$$

$$\frac{\partial^2 C\left(\overline{\sigma}^2\right)}{\partial\overline{\sigma}^2} = -\frac{S\sqrt{T-t}}{4\overline{\sigma}^3}N'(d_1)\left(d_1 d_2 - 1\right) \qquad \textbf{Eq. 13}$$

and the rest defined as previously. But, in general, for all other situations, equation 6 is appropriated:

$$B(\xi) \approx C\left(\overline{\sigma}^2\right) - C\left(\sigma_t^2\right) + \frac{\rho}{b^2\delta}\left[\left(a + b\sigma^2\right)\left(1 - e^{\delta} + \delta e^{\delta}\right) + a\left(1 + \delta - e^{\delta}\right)\right] S\left(-\frac{N'(d_1)d_2}{2\overline{\sigma}^2}\right)\xi \qquad \textbf{Eq. 14}$$

From equation 14, we observe that there are several variables and parameters with a direct influence on the bias. As noted by Hull and White [1987] and [1988], the bias is affected by the correlation between the stochastic processes (ρ), the time to maturity $(T-t)$, the instantaneous volatility (σ_t) and the exercise price (X). We should now be in place to remove the bias when pricing options using the Black and Scholes equation and, therefore, the so-called "smile effect" should not be observed.

Assuming that the model suggested by Hull and White [1988] captures the market premium of equity options, the bias obtained by equation 14 should differ according to the moneyness in order to accommodate the empirical finding known as the smile effect.

Table 1 shows the results of the computed bias when pricing options with a time to maturity ranging from 1 month to 1 year, an underlying stock price with an instantaneous volatility ranging from 22.4% to 44.7% ($\sigma_t^2 = 0.05, \sigma_t^2 = 0.1$ and $\sigma_t^2 = 0.2$), for five different stock price levels ($S_t = 70.00$, $S_t = 94.00$, $S_t = 100.00$, $S_t = 106.00$ and $S_t = 130.00$). For the remaining assumptions we assumed $a = 0.5$, $b = -5$, $\rho = -0.5$, $r_f = 4\%$ and $X = 100.00$. We assumed a negative ρ as a result of previous empirical research found in Christie [1982] and Kon [1984].

Table 1 - Bias incurred when pricing options according to the Black and Scholes equation if call premiums are computed with a mean reverting stochastic volatility process

BIAS			S/X				
			0.7	0.94	1.0	1.06	1.3
Time to Maturity	1/12	$\sigma^2_t = 0.05$	0,0001	0,1126	0,2248	0,1882	0,0004
		$\sigma^2_t = 0.10$	-0,0001	-0,0197	-0,0003	0,0187	0,0017
		$\sigma^2_t = 0.20$	-0,0049	-0,2177	-0,2401	-0,2109	-0,0252
	1/2	$\sigma^2_t = 0.05$	0,1969	1,5856	1,7170	1,6583	0,6593
		$\sigma^2_t = 0.10$	-0,0541	-0,0354	-0,0028	0,0287	0,0734
		$\sigma^2_t = 0.20$	-0,9522	-2,0793	-2,1409	-2,1093	-1,4547
	1	$\sigma^2_t = 0.05$	1,0854	2,8284	2,9281	2,8717	1,8237
		$\sigma^2_t = 0.10$	-0,0909	-0,0340	-0,0049	0,0232	0,0891
		$\sigma^2_t = 0.20$	-2,4231	-3,7780	-3,8597	-3,8503	-3,2190

Then we computed the instantaneous stock price return volatility that would result in an insignificant bias $B(\xi) < \frac{1}{10^{10}}$ (see Table 2).

Table 2 - Implied volatilities computed in order to remove the bias when pricing options according to the Black and Scholes equation if call premiums are computed with a mean reverting stochastic volatility process

IMPLIED VOLATILITY			S/X				
			0.7	0.94	1.0	1.06	1.3
Time to Maturity	1/12	$\sigma^2_t = 0.05$	31,28%	23,85%	24,32%	24,76%	26,00%
		$\sigma^2_t = 0.10$	29,69%	31,62%	31,62%	31,82%	32,43%
		$\sigma^2_t = 0.20$	41,99%	42,54%	42,63%	42,71%	42,98%
	1/2	$\sigma^2_t = 0.05$	27,56%	28,40%	28,57%	28,73%	29,27%
		$\sigma^2_t = 0.10$	30,88%	31,49%	31,61%	31,73%	32,13%
		$\sigma^2_t = 0.20$	36,50%	36,88%	36,96%	37,03%	37,28%
	1	$\sigma^2_t = 0.05$	29,47%	29,91%	30,00%	30,09%	30,39%
		$\sigma^2_t = 0.10$	31,15%	31,53%	31,61%	31,68%	31,94%
		$\sigma^2_t = 0.20$	34,26%	34,54%	34,60%	34,66%	34,85%

As observed by Hull and White [1988] when ρ is negative instead of a symmetric smile we observe, in most of the cases, a constant bias where implied volatilities of in-the-money options are higher than implied volatilities of out-of-the-money options. However, when maturity approaches, we clearly observe a decreasing U-shape smile as volatility increases. Therefore, apart the correlation effect, we should expect that the smile effect increases as the maturity approaches and decreases as volatility increases. We felt that some empirical work could be done in terms of showing how some of these variables could affect observed volatility patterns, in particular for options written on equities. This paper explores a data set of equity options and tests whether time and volatility confirms with the expected patterns suggested in the literature.

3. Data

The original database consists of 37,338 daily call option quotes on nine heavily traded stocks[3], traded on the *London International Financial Futures and Options Exchange* (LIFFE) between August 1990 to December 1991. Then we selected a smaller sample of 9739 call options using a filter rule requiring at least 5 exercise prices per maturity. The firms selected were Amstrad, British Airways, British Gas, British Petroleum, British Telecom, Forte, General Electric, Hanson and Rolls Royce. The option premiums were collected from Datastream, the underlying stock prices simultaneous with the closing quotations were retrieved from the Daily Official List published by the Stock Exchange. The option premiums are average closing quoted prices (bid + ask)/2.

The dividends were assumed known and the necessary adjustments were done using the Black [1975] model. As call options on stocks traded in LIFFE are American style, we excluded any options where early exercise could be optimal.

The database was, on average, an at-the-money option (with a moneyness degree of 1) and with a time to maturity of 125 days.

With the available data we could, theoretically, draw 1,693 smiles, which gives, on average, 5.75 exercise prices per maturity. However, as the available exercise prices are not centred on the at-the-money option and, as required by the methodology, we wanted to select options with a moneyness of 0.94 and 1.06, some observations were dropped for some regressions. Therefore, we obtained 714 in-the-money “half-smiles”, 900 out-of-the-money “half-smiles” and 547 “complete” smiles.

4. Methodology

As a result of the analysis of equation 14, this paper explores two main possible empirical hypotheses. <u>First</u> we conjecture that the smile has an exercise price bias permanently observed until options mature. However its shape changes as expiration approaches. <u>Secondly</u> we believe that the smile shape is also related to the volatility of the underlying asset. <u>Finally</u> we combine both independent variables (time and volatility) to observe how significant is the residual bias on the smile after extracting the maturity and the volatility effect.

The first hypothesis is based on Clewlow and Xu [1993], Heynen [1994], Taylor and Xu [1994] and Kratka [1998]. According to the discussion of equation 14 and according to these authors, implied volatilities should increase in exercise price bias, as options approach maturity. Therefore we set up the first hypothesis stating that the smile magnitude increases as maturity approaches. However, in other studies, no special attention is given to the symmetry of the smile, which we explore.

In order to measure the smile magnitude independent of the exercise price bias, we would like to estimate a series of the implied volatility for options at fixed moneyness degrees. We define the moneyness degree of an option (Mnss) as:

$$\mathrm{Mnss} = \frac{S - \sum \mathrm{NPV}(\mathrm{Div})}{X e^{-r(T-t)}} \qquad \textbf{Eq. 15}$$

where S represents the underlying stock price, X is represents the exercise price, r the risk free interest rate, T-t represents the time to maturity and NPV(Div) the net present value of the dividends paid on the stock until expiration. The moneyness degree of at-the-money options equals 1, for out-of-the-money call options it will be lower than 1 and for in-the-money options it will be greater than 1. We are particularly interested in estimating the implied volatility of three specific moneyness degrees, such as: Mnss = 1.06; Mnss = 1.00 and Mnss = 0.94.

As the underlying stock price moves stochastically and the exercise prices are constant, daily observed moneyness of quoted options for the available exercise prices change. Therefore, it is impossible to extract daily implied volatilities with those exact moneyness degrees, directly from the available option premiums. In order to overcome this problem, we used a B-spline curve in its cubic version to estimate the desired implied volatilities[4].

B-spline curves are helpful instruments to draw non-linear functions from an irregular data set. In order to draw a piece of the desired function we need 4 data points. With these 4 data points, the cubic version of the B-spline enables us to draw a curve for the interval contained between the 2nd and the 3rd point. Taking n data points, where

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[3] Please see Duque [1994] for aditional details on this database.

[4] Clewlow and Xu [1994] had already used these functions for drawing smoothed curves when studying the smile effect.

each point $p_i = (x_i, y_i)$ with $i = 0,1,...,n$, the cubic B-spline for the interval (p_i, p_{i+1}) with $i = 1,...,n-1$, is

$$B_i(u) = \sum_{k=-1}^{2} b_k\, p_{i+k} \qquad \text{where}$$

$$b_{-1} = \frac{(1-u)^3}{6},$$

$$b_0 = \frac{u^3}{2} - u^2 + \frac{2}{3},$$

$$b_1 = -\frac{u^3}{2} + \frac{u^2}{2} + \frac{u}{2} + \frac{1}{6},$$

$$b_2 = \frac{u^3}{6}, \qquad 0 \le u \le 1 \qquad \textbf{Eq. 16}$$

For each, x- and y-coordinates, we obtain a transformed value which becomes:

$$x_i(u) = \frac{1}{6}(1-u)^3 x_{i-1} + \frac{1}{6}\left(3u^3 - 6u^2 + 4\right)x_i + \frac{1}{6}\left(-3u^3 + 3u^2 + 3u + 1\right)x_{i+1} + \frac{1}{6}u^3 x_{i+2} \qquad \textbf{Eq. 17}$$

$$y_i(u) = \frac{1}{6}(1-u)^3 y_{i-1} + \frac{1}{6}\left(3u^3 - 6u^2 + 4\right)y_i + \frac{1}{6}\left(-3u^3 + 3u^2 + 3u + 1\right)y_{i+1} + \frac{1}{6}u^3 y_{i+2} \qquad \textbf{Eq. 18}$$

When studying the smile effect, the x- and y-coordinates become, respectively, the moneyness degree and the implied volatility. From a set of five or more available exercise prices, we computed the estimated implied volatility according to the cubic B-spline function for three different moneyness degrees, for each day and for each maturity. Assuming that

$\hat{\sigma}_{\mathrm{imp,i}}(\mathrm{in-the-money})$ - denotes the estimated in-the-money implied volatility of option i, using a cubic B-spline curve for a moneyness degree of 0.94,

$\hat{\sigma}_{imp,i}(\text{at}-\text{the}-\text{money})$ - denotes the estimated at-the-money implied volatility of option i, using a cubic B-spline curve for a moneyness degree of 1,

$\hat{\sigma}_{imp,i}(\text{out}-\text{of}-\text{the}-\text{money})$ - denotes the estimated out-of-the-money implied volatility of option i, using a cubic B-spline curve for a moneyness degree of 1.06.

We defined two measures for the smile magnitude. The first measure (U_{in}) accounts for the in-the-money bias while the second (U_{out}) accounts for the out-of-the-money bias:

$$U_{in} = \left| \hat{\sigma}_{imp,i}(\text{in}-\text{the}-\text{money}) - \hat{\sigma}_{imp,i}(\text{at}-\text{the}-\text{money}) \right| \qquad \textbf{Eq. 19}$$

$$U_{out} = \left| \hat{\sigma}_{imp,i}(\text{out}-\text{of}-\text{the}-\text{money}) - \hat{\sigma}_{imp,i}(\text{at}-\text{the}-\text{money}) \right| \qquad \textbf{Eq. 20}$$

Both measures give the absolute difference between what we would expect to be the implied volatility of the at-the-money option with a moneyness degree of 1 and what we would expect to be the implied volatility of an in(out)-the-money option, assuming the exercise price bias follows a cubic B-spline. As both measures are partial measures of the smile, only accounting for one of its sides, we call these measures <u>reduced measures of magnitude</u>. As previously explained, we have a database of 714 U_{in} (in-the-money "half-smiles") and 900 U_{out} (out-of-the-money "half-smiles").

As an illustrative example, we selected the option series quoted on Forte on the 3rd of January of 1991, expiring on the 26th of March, with 82 days to maturity. There were 7 exercise prices with active quotes and positive open interest. Table 3 shows the figures of the implied volatilities extracted from the quoted options for all available exercise prices.

Table 3 - Moneyness degree and observed implied volatilities for stock options on Forte, expiring on March, 1991 and calculated on January 3, 1991

Moneyness degree	1.26	1.15	1.05	0.97	0.90	0.84	0.76
Implied Volatility	0.3033	0.2236	0.2557	0.2688	0.2944	0.2978	0.4301

As we previously noticed, these are unlikely to be options with a moneyness degrees of exactly 1.06, 1 and 0.94! In such a situation implied volatilities for those moneyness degrees should be estimated from the 7 observed implied volatilities. Therefore, the estimated implied volatilities using a cubic B-spline curve adjustment are:

$$\hat{\sigma}_{imp,i}(1.06) = 0.2509$$

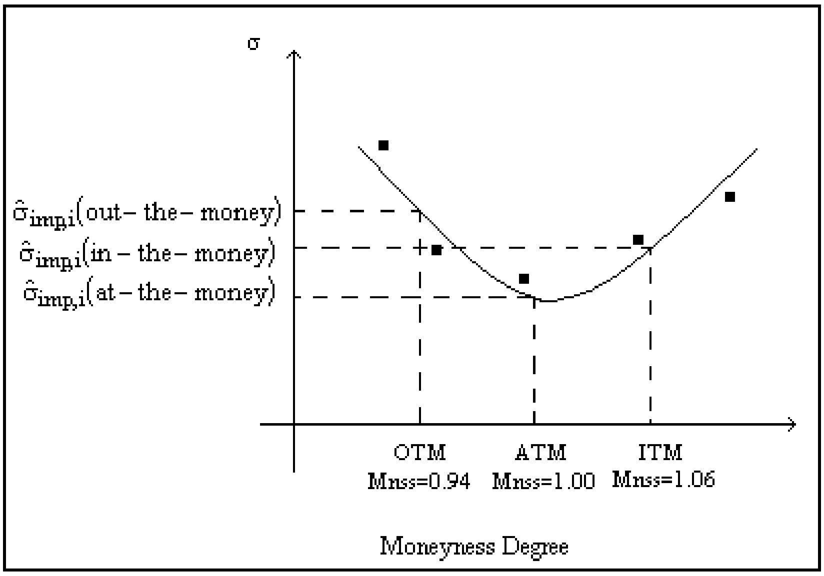
$$\hat{\sigma}_{imp,i}(1.00) = 0.2645$$

$$\hat{\sigma}_{imp,i}(0.94) = 0.2802.$$

This example also shows how at-the-money implied volatility may not be the absolute minimum of the smile curve.

Figure 1, represents the differences, just noted, between smoothed estimated implied volatilities obtained from the cubic B-spline adjustment and the observed implied volatilities, extracted from market data. The curve represents the estimated points using a cubic B-spline curve drawn from some known data points. From the curve, it is possible to infer implied volatilities for options with specific moneyness ratios.

Figure 1 – Observed and Estimated Implied Volatilities



The dots represent the observed implied volatilities while the points of the curve represent estimated implied volatilities using a B-spline curve fitting.

We also computed two additional estimates for the smile magnitude:

$$U_{at} = \left| \frac{\hat{\sigma}_{imp,i}(1.06) + \hat{\sigma}_{imp,i}(0.94)}{2} - \hat{\sigma}_{imp,i}(1.00) \right| \qquad \textbf{Eq. 21}$$

$$U_G = \frac{\hat{\sigma}_{imp,i}(1.06) - \hat{\sigma}_{imp,i}(0.94)}{\hat{\sigma}_{imp,i}(1,06)} x100 \qquad \textbf{Eq. 22}$$

Equation 21 gives a measure for the average depth of the smile, taking into account both the in-the-money and the out-of-the-money estimated implied volatilities, and was computed as an additional measure for the smile magnitude. As it takes into consideration both sides of the smile, we call this, the <u>complete measure of magnitude</u>. Equation 22 was computed in order to study the smile skewness[5]. If $U_G = 0$ the smile is symmetric, otherwise, it will be either asymmetric positive (when $U_G > 0$) or asymmetric negative (when $U_G < 0$).

The second set of hypothesis was also based on the theoretical implications of equation 14. Accordingly, we raised the hypothesis that volatility has an impact on the smile effect. This means that when the underlying stock price becomes more unstable, the exercise price bias will vary.

Finally, we hypothesised that both time to maturity and the volatility of the underlying stock explain the U-shape format of the smile.

5. Empirical Results

5.1. Maturity and Smile Effect

We started by testing whether time to maturity is related to the smile. In order to do so, we regressed time on all measures of magnitude previously defined by equations 19, 20 and 21. These regressions are expressed by equations 23, 24 and 25.

$$U_{out,i} = \beta_0 + \beta_1 (T - t)_i \qquad \textbf{Eq. 23}$$

$$U_{in,i} = \beta_0 + \beta_1 (T - t)_i \qquad \textbf{Eq. 24}$$

$$U_{at,i} = \beta_0 + \beta_1 (T - t)_i \qquad \textbf{Eq. 25}$$

[5] We chose U_G since this measure was used by Gemmill [1996].

The results are presented in Table 4. T-tests presented in Table 4 are already corrected for heteroscedasticity according to White [1980] estimator for the variance matrix of the least squares estimators.

Table 4 - The smile magnitude and time to maturity

	$U_{out,i} = \beta_0 + \beta_1 (T-t)_i$	$U_{in,i} = \beta_0 + \beta_1 (T-t)_i$	$U_{at,i} = \beta_0 + \beta_1 (T-t)_i$
β_0	0.015829	0.012078	0.007331
t-ratio	17.479850	12.525440	13.089540
P-value	0.000000	0.000000	0.000000
β_1	-4.49E-05*	-1.64E-05*	-2.51E-05*
t-ratio	-7.351227	-2.313366	-6.498290
P-value	0.000000	0.021000	0.000000
F–Stat.	68.341770*	7.329401*	62.005520*
R^2	0.070722	0.010189	0.102150

* - These results are statistically significant at a 95% confidence level.

This table shows that all coefficients are statistically significant with $\beta_1 < 0$ for all the tests. As time to maturity reduces, the magnitude of the smile increases. This pattern is common to the three measures of magnitude. Additionally, we conclude that the bias for out-of-the-money options increases faster with the decrease of maturity than for in-the-money options. This can be obtained from testing whether β_1 from regression equations 23 and 24 statistically differ from each other. As a conclusion, we found empirical support for the theoretical developments reported in the literature, where options seem to die smiling.

However, as just noticed, the exercise price bias seems to have a different behaviour according to the options' time to maturity. Therefore in order to determine whether the asymmetry is related to maturity, we regressed time to maturity on the symmetry smile measure given by U_G.

$$U_{G,i} = \beta_0 + \beta_1 (T-t)_i \qquad \textbf{Eq. 26}$$

If the slope of this regression equation is positive then, when maturity approaches, out-of-the-money option implied volatilities increase relative to in-the-money option implied volatilities. Table 5 shows the results.

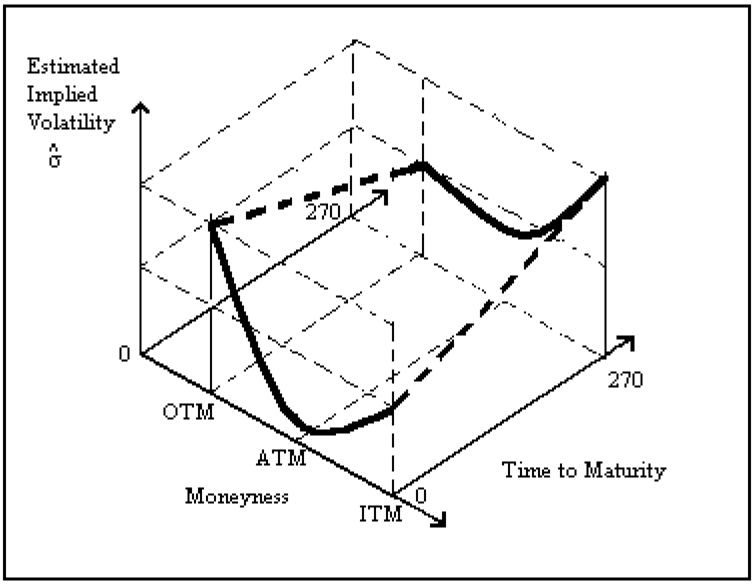
Table 5 - The smile skewness and time to maturity

	$U_{G,i} = \beta_0 + \beta_1 (T - t)_i$
β_0	-4.429069
t-ratio	-4.981124
P-value	0.000000
β_1	0.021606*
t-ratio	3.324410
P-value	0.000900
F–Stat.	14.162420*
R^2	0.025328

* - These results are statistically significant at a 95% confidence level.

As hypothesised, the maturity of the options tend to be associated with an increase of the exercise price bias for both in and out-of-the-money options, but out-of-the-money options tend to become more biased. Additionally, we also observe that when time to maturity is greater than 205, it is expected in-the-money implied volatilities will be greater than out-of-the-money implied volatilities. But for shorter time periods, implied volatilities of in-the-money options are lower than out-of-the-money options implied volatilities. Therefore, a possible explanation for previous empirical findings on the smile shape is time to maturity. In fact, the maturity approach changes the options smile asymmetry, converting a “wry grin” typical for longer term series into a “reverse grin” for nearly expiring options, with a more or less symmetric smile for middle term options. Figure 2 gives an idea of the possible shape for the smile in a 3 dimensional domain. Inside the same range of moneyness [0.94; 1.06] the bias changes according to time to maturity.

Figure 2 – The Smile Asymmetry along Time to Maturity



For longer-term options (T-t=270 days) the U-shape pattern of the smile shows implied volatilities of in-the-money options higher than implied volatilities of out-of-the-money options. For shorter-term options (T-t=4) the smile pattern reverses: implied volatilities of out-of-the-money options are higher than implied volatilities of in-the-money options.

Although the patterns found seem to differ from the pattern suggested by Hull and White [1988], some similarities may be observed. In fact as maturity approaches the smile becomes deeper with out-of-the-money implied volatilities higher than in-the-money implied volatilities (see Table 2). However, even for longer term options the smile still persists, but in-the-money options have higher implied volatilities.

5.2. Volatility and Smile Effect

According to Table 2, the smile is expected to increase with the decrease of the underlying stock price volatility. This should be a consequence of equation 14. As with previous tests, we examined whether relative and complete measures for the smile magnitude, are linearly related to the volatility of the underlying stock price. As a proxy for the underlying stock price, we selected the estimated implied volatility for the at-the-money series assuming the cubic B-spline fitting.

$$U_{out,i} = \beta_0 + \beta_1 \, \hat{\sigma}_{imp,i} (\text{at - the - money}) \qquad \textbf{Eq. 27}$$

$$U_{in,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 28}$$

$$U_{at,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 29}$$

The results did not show signs of heteroscedasticity and the figures are presented in Table 6.

Table 6 - The smile magnitude and volatility

	$U_{out,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{ATM})$	$U_{in,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{ATM})$	$U_{at,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{ATM})$
β_0	0.007968	0.008256	0.004840
t-ratio	9.872587	9.836481	9.922156
P-value	0.000000	0.000000	0.000000
β_1	0.006885*	0.005472*	-0.001344
t-ratio	3.218233	2.427515	-1.081689
P-value	0.001300	0.015400	0.279900
F–Stat.	10.357020*	5.892828*	1.170051
R^2	0.011402	0.008209	0.002142

* - These results are statistically significant at a 95% confidence level.

The results show that when reduced measures of the smile magnitude are used, we find a statistically significant positive relation between the underlying stock volatility and the smile. This means that when volatility increases, the exercise price bias tends to increase, for both in and out-of-the-money options. This is the opposite of what we should expect from equation 14 when the simulations given in Table 2 are considered. However, we did not find a statistically significant difference between the slopes of regression equations 27 and 28, with a significance level of 5%. Therefore, apparently, we can not expect a significant asymmetry when comparing the exercise price bias of in and out-of-the-money options, as we observed for the maturity effect.

In order to confirm this suspicion, we used the symmetry test, as previously, but adapted to the volatility effect on the smile.

$$U_{G,i} = \beta_0 + \beta_1 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 30}$$

If the slope of this regression equation is positive, then when volatility increases, in-the-money option implied volatilities increase relative to the out-of-the-money option implied volatilities. Table 7 shows the results.

Table 7 - The smile skewness and volatility

	$U_{G,i} = \beta_0 + \beta_1 \hat{\sigma}_{imp,i}(ATM)$
β_0	-6.077170
t-ratio	-6.935726
P-value	0.000000
β_1	11.385000*
t-ratio	5.865909
P-value	0.000000
F–Stat.	27.502390*
R^2	0.048039

* - These results are statistically significant at a 95% confidence level.

The result, slightly unexpected, shows a significant effect of underlying stock volatility on the increase of the exercise price bias. For high volatilities, the relative difference between in-the-money and out-of-the-money estimated implied volatilities tends to be higher than for lower volatilities. The results also enable us to conclude that $\hat{U}_G$ (the expected asymmetry measure for a given volatility) only turns positive when volatility reaches 53.4% which is a quite high value. Therefore, it seems that in most of the cases of our database, out-of-the-money estimated implied volatilities are higher than in-the-money estimated implied volatilities. These findings are also a possible explanation for previous inconsistencies in implied volatility analysis. The findings show an opposite effect to the expected bias given by equation 14 and considering the data of Table 2. The relative difference between in and out-of-the-money options decreases as volatility increases for all of the time-to-maturities observed.

5.3. Time to Maturity, Volatility and Smile Effect

As a result of the conclusions obtained in sections 5.1 and 5.2 we hypothesised that both variables (time to maturity and the volatility of the underlying stock) could explain the smile in a multiple linear regression model:

$$U_{out,i} = \beta_0 + \beta_1 (T - t)_i + \beta_2 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 31}$$

$$U_{in,i} = \beta_0 + \beta_1 (T - t)_i + \beta_2 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 32}$$

$$U_{at,i} = \beta_0 + \beta_1 (T - t)_i + \beta_2 \ \hat{\sigma}_{imp,i}(\text{at - the - money}) \qquad \textbf{Eq. 33}$$

The results did not show signs of heteroscedasticity and the figures are presented in Table 8.

Table 8 - The smile magnitude and volatility

	$U_{out,i} = \beta_0 + \beta_1 (T)_i + \beta_2 \ \hat{\sigma}_i(ATM)$	$U_{in,i} = \beta_0 + \beta_1 (T)_i + \beta_2 \ \hat{\sigma}_i(ATM)$	$U_{at,i} = \beta_0 + \beta_1 (T)_i + \beta_2 \ \hat{\sigma}_i(ATM)$
β_0	0.013337	0.010135	0.007422
t-ratio	11.544910	9.342311	10.881190
P-value	0.000000	0.000000	0.000000
β_1	-4.53E-05*	-1.87E-05*	-2.50E-05*
t-ratio	-7.410475	-2.555405	-6.464311
P-value	0.000000	0.010800	0.000000
β_2	0.007274*	0.006416*	-0.000278
t-ratio	2.648737	2.320740	-0.248587
P-value	0.008200	0.020600	0.803800
F–Stat.	40.831090*	7.725074*	30.976340*
R^2	0.083443	0.021268	0.102240

* - These results are statistically significant at a 95% confidence level.

The results do not differ significantly from what we were expecting, implying that for both in and out-of-the-money options, the maturity and the volatility effect do not offset each other. They both seem to contribute to explaining the smile effect. It is also interesting to observe that, as we would expect, out-of-the-money options seem to be more sensitive to changes in volatility and in time to maturity than in-the-money options.

6. Conclusions

The “smile effect” is a result of an empirical observation of the options’ implied volatility with the same expiration date, across different exercise prices. It typically describes a U-shape form showing high implied volatility patterns for in and out-of-the-money options and low volatility figures for at-the-money options. We may find empirical evidence for this exercise bias in previous research. Some literature presents theoretical reasons for such a bias as Hull and White [1987], Taylor and Xu [1994] or Heynen [1994]. Reasons were stochastic volatility, trader’s behaviour, transaction costs, and the effect of dividends on pricing American options. But the most recent literature seems to conclude that the sophistication of financial modelling for option pricing is not enough to remove the “smile”.

We selected the Hull and White [1998] option pricing model, that assumes volatility is a stochastic variable following a mean reversion pattern, to search for theoretical relations that should exist between the smile shape, time-to-maturity and volatility.

We started by extracting a sample of call options quoted on LIFFE on nine heavily traded stocks requiring at least 5 exercise prices for each maturity. Then, we fitted B-spline curves to the existent observed implied volatilities to estimate unobserved option implied volatilities on 3 specific moneyness degrees.

This paper refines the empirical analysis of the smile, trying to establish statistically significant links between some variables which impact on option valuation and the shape of the smile. We tried to analyse empirically how implied volatility smiles vary with the approach of expiration as well as with changes in volatility.

We concluded that as time to maturity reduces, the magnitude of the smile increases. Both the absolute and the relative difference between implied volatility of in and at-the-money options or out and at-the-money options rise as expiration approaches. Further, this maturity bias seems to be more evident for out-of-the-money options than for in-the-money options, showing an asymmetric pattern. As a first conclusion, we found empirical support for the theoretical developments reported in the literature, where options seem to die smiling, however, differently, according to the moneyness. When observing longer time to maturity options, we found in-the-money implied volatilities larger than out-of-the-money implied volatilities. But for shorter time periods, implied volatilities of in-the-money options become smaller than out-of-the-money options implied volatilities. In fact, the maturity approach changes the options smile asymmetry, converting a “wry grin” typical for longer term series into “reverse grins” for expiring options, with a symmetric smile for middle term options. This conclusion differs from what could be expected when Hull and White [1998] model is considered, especially when longer-term options are considered.

Then we checked for possible evidence on the volatility effect. In the literature we found developments for a bias equation which shows that smiles should be volatility dependent, decreasing when volatilities increase. Our results, in contrast, show that, when reduced measures of the smile magnitude are used, we find a statistically significant positive relation between the underlying stock volatility and the smile. When volatility increases, the exercise price bias tend to rise, both for in and out-of-the-money options. As observed with the maturity effect, we also recognise an asymmetric behaviour for in and out-of-the-money options. For high volatilities, in-the-money options tend to evidence higher estimated implied volatilities than for lower volatilities.

Finally we found that when time and volatility are combined we increase the explanatory power of the smile bias.

This paper confirms previous findings on the maturity and volatility impact on the exercise price bias, also called the smile effect. However, additionally, we show empirical evidence for an asymmetric behaviour. Time to maturity and volatility seem to have a distinct influence on the smile effect when in and out-of-the-money options are observe.

We believe that in describing the smile and its association with other variables, we contribute to the refinement of future option pricing models.

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