

Regular Synchrony Lattices for Product Coupled Cell Networks

Manuela A. D. Aguiar¹ and Ana Paula S. Dias²

¹*Faculdade de Economia, Centro de Matemática, Universidade do Porto,
Rua Dr Roberto Frias, 4200-464 Porto, Portugal; maguiar@fep.up.pt*

²*Dep. Matemática, Centro de Matemática, Universidade do Porto,
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal; apdias@fc.up.pt*

November 13, 2014

Abstract

There are several ways of constructing (bigger) networks from smaller networks. We consider here the cartesian and the Kronecker (tensor) product networks. Our main aim is to determine a relation between the lattices of synchrony subspaces for a product network and the component networks of the product. In this sense, we show how to obtain the lattice of regular synchrony subspaces for a product network from the lattices of synchrony subspaces for the component networks. Specifically, we prove that a tensor of subspaces is of synchrony for the product network if and only if the subspaces involved in the tensor are synchrony subspaces for the component networks of the product. We also show that, in general, there are (irregular) synchrony subspaces for the product network that are not described by the synchrony subspaces for the component networks, concluding that, in general, it is not possible to obtain the all synchrony lattice for the product network from the corresponding lattices for the component networks. We also make the following remark concerning the fact that the cartesian and Kronecker products, as graph operations, are quite different, implying that the associated coupled cell systems have distinct structures. Although, the kinds of dynamics expected to occur are difficult to compare, we establish an inclusion relation between the lattices of synchrony subspaces for the cartesian and Kronecker products.

AMS classification scheme numbers: 34C15 37C10 06B23 05C76 05C50

Coupled cell systems (CCS) are dynamical systems where the full system can be seen as a set of interacting smaller dynamical systems (the cells). The motivation for interpreting certain dynamical systems as CCS comes from many real life applications where it is advantageous to interpret the evolution in time of the system as determined by the dynamics of individual units and the links between these units. Properties encoding information of the cells and the interactions can be given by a network – a graph where the nodes represent the cells and the arrows the interactions between the cells. The product of networks is a natural process of constructing real networks from smaller ones. One ultimate goal is to derive how far we can describe the dynamics of the CCS associated with a (large) product network based on the dynamics of the CCS consistent with the structure of the component networks of the product. We have focused this dynamics question on the synchrony subspaces (flow-invariant sub-

spaces defined by equalities of certain cell coordinates), being our main aim to determine a relation between the synchrony subspaces for a product network and the component networks of the product. We show how to obtain a special set of synchrony subspaces for a product network from the sets of synchrony subspaces for the component networks - the synchrony subspaces that are tensors of synchrony subspaces for the component networks of the product. Moreover, we show that, in general, there are synchrony subspaces for the product network that are not described by the synchrony subspaces for the component networks, concluding so that, in general, it is not possible to obtain the all set of synchrony subspaces for the product network from the corresponding sets for the component networks.

1 Introduction

Coupled cell systems (CCS) are dynamical systems

where the full system can be seen as a set of interacting smaller dynamical systems (the *cells*). The motivation for interpreting certain dynamical systems as CCS comes from many real life applications where it is advantageous to interpret the evolution in time of the system as determined by the dynamics of individual units and the links between these units. Properties encoding information of the cells and the interactions can be given by a network. More precisely, a *network architecture* is a graph where the nodes represent the cells and the arrows the interactions between the cells. Cells that have the same phase space are denoted by the same symbol. Interactions that, from the dynamics point of view, have the same role are denoted by the same arrow type. The CCS associated with a network structure are the differential equations where the involved vector fields respect that structure. Observe that a network is then a directed graph where nodes and arrows can be distinguished, say using equivalence relations (on the set of nodes and on the set of arrows). See for example the approach of Stewart, Golubitsky *et al.* [31, 19, 18] or Field [15].

The product of networks, that we consider here, is a natural process of constructing real networks from smaller ones. See for example Atay and Biyikoglu [13], Leskovec *et al.* [29]. From Graph Theory, it is known that there are several ways of forming from two digraphs a new digraph whose vertex set is the cartesian product of their vertex sets – these constructions are usually referred as ‘products’. Two of such are the *cartesian product* and the *Kronecker (tensor) product* and are the ones we consider here. See Figures 1 and 2. One ultimate goal is to derive how far we can describe the dynamics of the CCS associated with a (large) product network based on the dynamics of the CCS consistent with the structure of the component networks of the product.

It is known that the network structure imposes restrictions at the dynamics that can occur for the associated CCS. One important such restriction is the existence of *synchrony subspaces* – flow-invariant subspaces defined in terms of equalities of certain cell coordinates. One of the novelties introduced by the theory of CCS is the existence of flow-invariant subspaces (synchrony subspaces) that are independent of the network symmetries and of the particular symmetries of the associated CCS. By Stewart [30] (see also Aldis [1]) the set of all synchrony subspaces for a network forms a complete lattice taking the relation of inclusion. An algebraic and algorithmic description of the lattice of synchrony subspaces for a network is given by Aguiar and Dias [5]. See also Kamei and Cock [23].

Flow-invariant subspaces have a very important role

in the study of dynamical systems: in a high dimensional dynamical system we can restrict the study of the dynamics to the flow-invariant subspaces, which are lower dimensional; also, it is known that flow-invariant subspaces favour the existence of heteroclinic cycles and networks. There is a vast literature on heteroclinic cycles and networks in symmetric systems, induced by the lattice of fixed point subspaces, see for example, the review articles by Krupa [25] and Homburg and Sandstede [22]. The study of the existence of heteroclinic cycles and networks in CCS, induced by the lattice of synchrony subspaces, is new and of current interest, see, for example, Aguiar *et al.* [3], where it is shown how flow-invariant subspaces in CCS can support robust heteroclinic attractors, assuming asymmetric inputs and no global or local symmetries in the network, Ashwin and Postlethwaite [12] where there are presented two methods of realizing arbitrarily complex directed graphs as robust heteroclinic networks for flows generated by ordinary differential equations, and Field [16], where results are proven that enable realization of heteroclinic networks in coupled homogeneous and heterogeneous systems of identical cells. Another possible way of constructing CCS with dynamics supporting heteroclinic networks, that we intend to address in the near future, based on the results obtained here, is by making the product of CCS where it is known the existence of robust heteroclinic cycles. See Ashwin and Field [8] for a specific example with nine cells.

It is well known that the existence of heteroclinic networks can lead to the occurrence of complex dynamics. See, for example, Aguiar *et al.* [4], Aguiar *et al.* [6], Homburg and Knobloch [21] and Labouriau *et al.* [27], where, under some generic assumptions on an heteroclinic or homoclinic network, it is proven chaotic dynamics in the neighbourhood of the network, such as switching and suspended horseshoes. A mechanism for switching near a heteroclinic network is described in Kirk *et al.* [24]. See also Homburg and Sandstede [22] and references therein. For CCS displaying unexpected heteroclinic behaviour see, for example, the review in Ashwin and Karabacak [10] that examines some recent work on robust heteroclinic networks that can appear as attractors for CCS and Ashwin *et al.* [11] on heteroclinic cycles that appear in neural microcircuits modelled as coupled dynamical cells.

The identification of the synchrony subspaces that arise from the component networks of the product, that we consider in this work, is the first step to the construction of heteroclinic networks in product networks from smaller networks having heteroclinic behaviour. The construction of product heteroclinic networks in product coupled cell networks will provide a

general method, in the context of CCS, to construct and analyse products, as for example, of planar homoclinic and heteroclinic attractors as in Ashwin and Fieldl [9] and Agarwal *et al.* [2], in the context of symmetric systems. The main aim of the present work is, then, to establish a relation between the lattices of synchrony subspaces for a product network and the component networks of the product. See Aguiar and Ruan [7], for an analogous study for the join and coalescence network operations. Moreover, despite the definitions of cartesian product and Kronecker (tensor) product networks are quite different, we show that there is an inclusion relation between the lattices of synchrony subspaces for these products.

Framework of the paper

In Section 3, we present the definitions of cartesian and Kronecker product networks of identical-edge networks.

In Section 4, we establish an inclusion relation between the lattices of synchrony subspaces for the cartesian and Kronecker products by showing that any synchrony subspace for the cartesian product network is a synchrony subspace for the Kronecker product network. See Proposition 4.5.

Our main results are Theorem 5.10 and Theorem 6.5, in Section 5 and Section 6, respectively, where we prove that the collection of all regular synchrony subspaces for a (cartesian or Kronecker) product network is a lattice. By a *regular subspace* we mean a subspace of the product space given as a tensor product of subspaces of the phase spaces for the networks involved in the product. Moreover, we prove that the regular synchrony lattices for the two types of product networks coincide and are given by the tensor product of the synchrony lattices for the product network components.

We also show that, in general, the regular synchrony lattice and the synchrony lattice for a (cartesian or Kronecker) product network do not coincide.

2 Coupled Cell Networks

Following Stewart, Golubitsky *et al.* [31, 19, 18], a *coupled cell network* is a directed graph (digraph) whose nodes represent the cells and the directed arrows (or edges or arcs) the couplings. Note that we can have self loops, or multiarrows (parallel arcs) – arrows with the same head and tail cells. Also, equivalence relations on the set of nodes and on the set of arrows can be defined symbolizing the following: (a) Two nodes are in the same *cell equivalence class* if they represent individual dynamics with the same state space. (b)

Two arrows are in the same *arrow equivalence class* if they represent couplings of the same type. Here, the following consistency condition is assumed: if two arrows are of the same type then the corresponding head cells are in the same cell equivalence class and the same holds for the corresponding tail cells.

Definition 2.1 (i) Given a network, the *input set of a cell* of the network is the set of arrows directed to that cell.

(ii) Two cells of a network are said to be (*input*) *isomorphic* if there is an arrow-type preserving bijection between the corresponding input sets.

(iii) A *homogeneous* coupled cell network is a network in which all cells are (input) isomorphic.

(iv) A *regular* coupled cell network is a homogenous network with only one arrow type. For a regular network, the *valency* is the number of arrows of the input set of any cell and the *adjacency matrix* is the matrix where the (i, j) entry is the number of arrows from cell j to cell i , assuming the set of cells is $\{1, \dots, n\}$. If v is the valency of a regular network then the corresponding adjacency matrix has v constant row sum.

(v) For a general coupled cell network with set of cells $\{1, \dots, n\}$ and k arrow equivalence classes, we define k adjacency matrices, one for each arrow type, say $A_1, \dots, A_k \in M_{n \times n}(\mathbf{Z}_0^+)$, in the following way: the (i, j) entry of the matrix A_p is the number of arrows of type p from cell j to cell i .

(vi) A coupled cell network is said *injective* if its adjacency matrices are injective. \diamond

In Figure 1, we have from left to the right, a two-cell regular network with valency two, a three-cell regular network with valency two, and a six-cell homogeneous network with two arrow types. Note that for each of the three networks, all cells are input isomorphic. For example, for the six-cell network, every cell receives two solid arrows and two dashed arrows.

Coupled cell systems

Following [31, 19, 18], the connection between CCS and coupled cell networks is made in the following way: to each coupled cell c is associated a choice of *cell phase space* P_c which is assumed to be a finite-dimensional real vector space, say \mathbf{R}^k for some $k > 0$. If cells c and d are cell equivalent then it is required that $P_c = P_d$ and the two spaces are identified canonically. If $\mathcal{C} = \{1, \dots, n\}$ denotes the set of cells of the network, then the *total phase space* P of the coupled cell system is the direct product of the cell phase spaces, $\prod_{c \in \mathcal{C}} P_c$, and we employ the coordinate system $x = (x_c)_{c \in \mathcal{C}}$ on P . Given a network \mathcal{G} and a fixed choice of the total phase space

P , we describe now the CCS that correspond to the class of the systems of ordinary differential equations, $\dot{X} = F(X)$, $X \in P$, compatible with the structure of the network. The system associated with cell j has the form $\dot{x}_j = f_j(x_j; x_{i_1}, \dots, x_{i_m})$ where the first argument x_j in f_j represents the internal dynamics of the cell and each of the remaining variables x_{i_p} represents a coupling between cell i_p and cell j . Thus $x_j \in P_j, x_{i_p} \in P_{i_p}, p = 1, \dots, m$ and we assume $f_j : P_j \times P_{i_1} \times \dots \times P_{i_m} \rightarrow P_j$ is smooth. Moreover, identical couplings directed to cell j correspond to the invariance of f_j under permutation of the corresponding variables. Systems associated with (input) isomorphic cells are identical up to permutation of the variables accordingly to the input sets of the cells. The vector fields F are called \mathcal{G} -admissible.

If for example we consider the six-cell homogeneous network on the right of Figure 1, as all cells are input isomorphic, we have that CCS having structure consistent with this network must be of the following type: the system associated with cell ij has the form

$$\dot{x}_{ij} = f(x_{ij}; x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$$

where the first argument x_{ij} in f represents the internal dynamics of the cell and each of the remaining variables x_{i_p} represents a coupling between cell i_p and cell ij . All cells must have the same phase space, say \mathbf{R}^k for some $k > 0$, and so the total phase space is then $(\mathbf{R}^k)^6$. Moreover, $f : \mathbf{R}^k \times (\mathbf{R}^k)^4 \rightarrow \mathbf{R}^k$ is smooth, and invariant say, under the second and third coordinates (representing the couplings associated with solid arrows), and under the fourth and fifth coordinates (representing the couplings associated with dashed arrows). Note that the cell systems are given by the same function f , as all cells are input isomorphic.

3 Product networks

We define now the cartesian and the Kronecker (tensor) products of identical-edge networks. Following Golubitsky *et al.* [17], we consider the cartesian product of networks as a variant of the usual definition of cartesian product of digraphs where the arrows types from each component network involved in the product remain distinct in the product. We adopt the usual definition of Kronecker product of digraphs for the Kronecker product of networks.

3.1 Cartesian product network

Definition 3.1 For $i = 1, 2$, consider the network \mathcal{N}_i with set of cells (nodes) \mathcal{C}_i , set of arrows \mathcal{E}_i , and adjacency matrix A_i . Following [17], we define the *cartesian product* of \mathcal{N}_1 and \mathcal{N}_2 , denoted by $\mathcal{N}_1 \boxtimes \mathcal{N}_2$, as the

network with set of cells (nodes) the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ and with adjacency matrices $A_1 \otimes \text{Id}_{\#\mathcal{C}_2}, \text{Id}_{\#\mathcal{C}_1} \otimes A_2$. \diamond

Assume that \mathcal{N}_i has r_i cells, say $\mathcal{C}_i = \{1, \dots, r_i\}$, for $i = 1, 2$. Denote by ij the cell of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ corresponding to $(i, j) \in \mathcal{C}_1 \times \mathcal{C}_2$. It follows then that there is an arrow from cell ij to cell kl in $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if:

$$i = k \text{ and } (j, l) \in \mathcal{E}_2, \text{ or } j = l \text{ and } (i, k) \in \mathcal{E}_1. \quad (3.1)$$

Observe that the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ has two adjacency matrices $A_1 \otimes \text{Id}_{\#\mathcal{C}_2}, \text{Id}_{\#\mathcal{C}_1} \otimes A_2$ and so two arrow types: the arrows of the type corresponding to $A_1 \otimes \text{Id}_{\#\mathcal{C}_2}$ connect cell ij to cell kj when $(i, k) \in \mathcal{E}_1$; the arrows of the type corresponding to $\text{Id}_{\#\mathcal{C}_1} \otimes A_2$ connect cell ij to cell il when $(j, l) \in \mathcal{E}_2$.

Remark 3.2 Definition 3.1 generalizes the definition of product network of [17] where it is defined the product of regular networks. \diamond

Example 3.3 In Figure 1, we show a 2-cell regular network \mathcal{N}_1 , a 3-cell regular network \mathcal{N}_2 and the homogeneous product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. \diamond

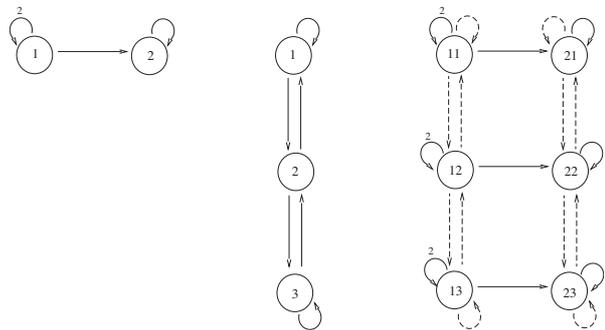


Figure 1: (From left to right) A 2-cell network \mathcal{N}_1 . A 3-cell network \mathcal{N}_2 . The cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$.

3.2 Kronecker (tensor) product network

Definition 3.4 For $i = 1, 2$, consider the network \mathcal{N}_i with set of cells (nodes) \mathcal{C}_i , set of edges \mathcal{E}_i , and adjacency matrix A_i . Following [32], we define the *Kronecker product* of \mathcal{N}_1 and \mathcal{N}_2 , denoted by $\mathcal{N}_1 \otimes \mathcal{N}_2$, as the network with set of cells (nodes) the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ and with adjacency matrix $A_1 \otimes A_2$. \diamond

As before, assume that \mathcal{N}_i has r_i cells, say $\mathcal{C}_i = \{1, \dots, r_i\}$, for $i = 1, 2$ and denote by ij the cell of $\mathcal{N}_1 \otimes \mathcal{N}_2$ corresponding to $(i, j) \in \mathcal{C}_1 \times \mathcal{C}_2$. It follows then that there is an arrow from cell ij to cell kl in $\mathcal{N}_1 \otimes \mathcal{N}_2$ if and only if:

$$(i, k) \in \mathcal{E}_1 \text{ and } (j, l) \in \mathcal{E}_2. \quad (3.2)$$

Example 3.5 In Figure 2, we show the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ of the 2-cell and 3-cell networks of Example 3.3. \diamond

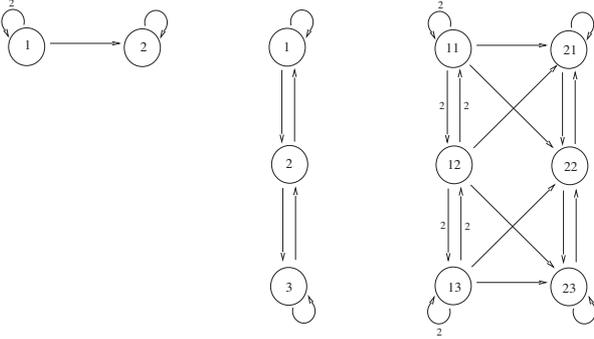


Figure 2: (From left to right) A 2-cell network \mathcal{N}_1 . A 3-cell network \mathcal{N}_2 . The Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$.

Remark 3.6 If u and w are eigenvectors of A_1 and A_2 associated to eigenvalues λ_1 and λ_2 , respectively, then $v = u \otimes w$ is an eigenvector of $A_1 \otimes A_2$ associated to the eigenvalue $\lambda_1 \lambda_2$. More generally, every eigenvector v of $A_1 \otimes A_2$ associated to an eigenvalue λ is given by a linear combination $v = \sum_i \alpha_i u_i \otimes w_i$, with u_i and w_i eigenvectors of A_1 and A_2 associated with eigenvalues λ_i^1 and λ_i^2 , respectively, such that $\lambda_i^1 \lambda_i^2 = \lambda$. \diamond

From Remark 3.6, we obtain the following result:

Proposition 3.7 Every generalized eigenspace $G_\mu^{A_1 \otimes A_2}$ of $A_1 \otimes A_2$ is given by

$$G_\mu^{A_1 \otimes A_2} = \oplus_{i,j} \left(G_{\lambda_i^{A_1}} \otimes G_{\beta_j^{A_2}} \right),$$

with $G_{\lambda_i^{A_1}}$ and $G_{\beta_j^{A_2}}$ generalized eigenspaces of A_1 and A_2 associated to eigenvalues λ_i and β_j , respectively, such that $\lambda_i \beta_j = \mu$.

4 Lattices of synchrony subspaces

In this section we start by recalling the definition of synchrony subspace for a network and some results

concerning the set of all synchrony subspaces for a network, which forms a lattice under the subspace inclusion operation. We then establish the relation between the lattices of synchrony subspaces for the cartesian and Kronecker product networks.

4.1 The lattice of synchrony subspaces for a coupled cell network

A network structure imposes the existence of certain flow-invariant subspaces for any coupled cell system associated with that structure. These subspaces are called synchrony subspaces. Following [31, 19]:

Definition 4.1 Given a network \mathcal{G} , a *synchrony subspace* Δ of the total phase space P is a subspace of P characterized by a set of cell coordinates equalities which is flow-invariant for all \mathcal{G} -admissible vector fields on P . \diamond

As remarked in [5, Section 2.2], it follows from the results in [31, 19] that, for any choice of the total phase space, a polydiagonal subspace is a synchrony subspace if and only if it is left invariant for *all linear* network admissible vector fields choosing the cell phase spaces to be \mathbf{R} . Equivalently, a polydiagonal subspace is a synchrony subspace if and only if it is left invariant by all the network adjacency matrices.

Let $V_{\mathcal{G}}$ be the set of synchrony subspaces for \mathcal{G} and note that the intersection of two synchrony subspaces for a network is again a synchrony subspace. As proved by Stewart [30] (see also [5, Section 3.3]), taking the partial order on $V_{\mathcal{G}}$ given by inclusion \subseteq of spaces, it follows that $V_{\mathcal{G}}$ is a lattice: every pair of synchrony subspaces has a *unique least upper bound or join* and a *unique greatest lower bound or meet*. The meet operation is the intersection and the join can be defined in terms of the meet. Moreover, the lattice is *complete* as every subset of $V_{\mathcal{G}}$ has a unique least upper bound or join, and a unique greatest lower bound or meet. (For details on lattices and completeness of lattices in general, see for example Davey and Priestley [14].)

Another example of a lattice, it is the set of the invariant subspaces under the network adjacency matrices, taking again the partial order given by inclusion of spaces. Now, the meet and the join operations are intersection and sum, respectively. Note that the lattice $V_{\mathcal{G}}$, as a set, it is a subset of the lattice of the invariant subspaces under the network adjacency matrices.

Example 4.2 The lattice of synchrony subspaces for the 3-cell network \mathcal{N}_2 at Figure 2 is formed by three spaces: two are trivial, corresponding to the full total

phase space P and the diagonal space $\{\mathbf{x} : x_1 = x_2 = x_3\}$, and the third one is $\{\mathbf{x} : x_1 = x_3\}$. \diamond

4.2 Relation between the lattices of synchrony subspaces for the cartesian and Kronecker products

Let \mathcal{N}_1 and \mathcal{N}_2 be networks with only one edge type. Consider the product networks $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and $\mathcal{N}_1 \otimes \mathcal{N}_2$, and the corresponding lattices of synchrony subspaces $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ and $V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$.

Assume the networks \mathcal{N}_1 and \mathcal{N}_2 have r_1 and r_2 cells, respectively. Let A_1 and A_2 be the adjacency matrices of \mathcal{N}_1 and \mathcal{N}_2 , respectively. It follows then that the adjacency matrices of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the $r_1 r_2 \times r_1 r_2$ -matrices $A_1 \otimes \text{Id}_{r_2}$ and $\text{Id}_{r_1} \otimes A_2$, and $A_1 \otimes A_2$ is the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. We have the following relation between the matrices $A_1 \otimes \text{Id}_{r_2}$, $\text{Id}_{r_1} \otimes A_2$ and $A_1 \otimes A_2$:

Remark 4.3 The adjacency matrices $A_1 \otimes \text{Id}_{r_2}$ and $\text{Id}_{r_1} \otimes A_2$ of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ satisfy

$$(A_1 \otimes \text{Id}_{r_2}) \circ (\text{Id}_{r_1} \otimes A_2) = (\text{Id}_{r_1} \otimes A_2) \circ (A_1 \otimes \text{Id}_{r_2}) = A_1 \otimes A_2.$$

\diamond

Lemma 4.4 A subspace S of $\mathbf{R}^{r_1} \otimes \mathbf{R}^{r_2}$ which is invariant under both $A_1 \otimes \text{Id}_{r_2}$ and $\text{Id}_{r_1} \otimes A_2$ it is also invariant under $A_1 \otimes A_2$.

Proof If a subspace S is invariant under $A_1 \otimes \text{Id}_{r_2}$ and $\text{Id}_{r_1} \otimes A_2$ then, from Remark 4.3, we have

$$\begin{aligned} (A_1 \otimes A_2)(S) &= (A_1 \otimes \text{Id}_{r_2}) \circ (\text{Id}_{r_1} \otimes A_2)(S) \\ &\subseteq (A_1 \otimes \text{Id}_{r_2})(S) \subseteq S. \end{aligned}$$

Thus, S is $A_1 \otimes A_2$ -invariant. \square

Proposition 4.5 Given two networks \mathcal{N}_1 and \mathcal{N}_2 with only one edge type, we have the following inclusion relation between the lattices of synchrony subspaces for their product networks:

$$V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}.$$

Proof For $i = 1, 2$, denote by A_i the adjacency matrix of \mathcal{N}_i . Then the adjacency matrices of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the matrices $A_1 \otimes \text{Id}_{r_2}$ and $\text{Id}_{r_1} \otimes A_2$, and $A_1 \otimes A_2$ is the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. Using Lemma 4.4 and the fact that a synchrony subspace for a coupled cell network is a polydiagonal subspace that is left invariant under the adjacency matrices of the network, we obtain $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$. \square

The following example illustrates that, in general, the inclusion in Proposition 4.5 is strict.

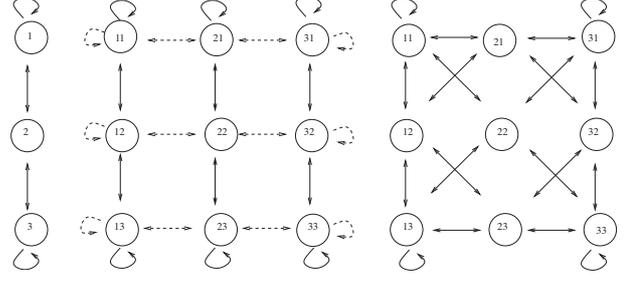


Figure 3: (From left to right) A 3-cell network \mathcal{N} . The cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$. The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

Example 4.6 Consider the network \mathcal{N} and the product networks $\mathcal{N} \boxtimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{N}$, in Figure 3. Take the subspace $S = \{\mathbf{x} : x_{11} = x_{33}, x_{12} = x_{23}, x_{21} = x_{32}\}$. Note that S is a synchrony subspace for $\mathcal{N} \otimes \mathcal{N}$ but it is not a synchrony subspace for $\mathcal{N} \boxtimes \mathcal{N}$. Thus the network \mathcal{N} is an example where $V_{\mathcal{N} \boxtimes \mathcal{N}} \subsetneq V_{\mathcal{N} \otimes \mathcal{N}}$. \diamond

5 The lattice of synchrony subspaces for the Kronecker product

In this section, we address the problem of relating the lattice of synchrony subspaces for a Kronecker product network with the lattices of synchrony subspaces for the component coupled cell networks of the product. We start by reviewing certain results concerning the lattice of invariant subspaces for the Kronecker product of matrices, which relate with the lattices of invariant subspaces for each component matrix in the product. Motivated by these results, we then prove similar results for the lattice of synchrony subspaces for the Kronecker product of networks.

5.1 The lattice of invariant subspaces for the Kronecker product of matrices

Let P_1, P_2 be two real vector spaces and consider the linear subspace $P_1 \otimes P_2$. As in Kubrusly [26], we define regular subspaces of $P_1 \otimes P_2$ in the following way.

Definition 5.1 A subspace of the tensor product $P_1 \otimes P_2$ is *regular* if it is of the form $S_1 \otimes S_2$, with S_i a subspace of P_i , $i = 1, 2$. Otherwise, it is *irregular*. \diamond

From the results in Lemma 1 of Kubrusly [26], we have:

Lemma 5.2 For $i = 1, 2$, let A_i be a linear operator on a linear space P_i , and S_i a subspace of P_i . We have:

1. If S_i is an invariant subspace for A_i , $i = 1, 2$, then $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$.
2. If $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$, then, for $i = 1$ or $i = 2$, S_i is an invariant subspace for A_i .
3. If $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$ and if $S_i \not\subseteq \ker(A_i)$, for $i = 1, 2$, then S_i is an invariant subspace for A_i . Particular case:
 - (a) If $S = S_1 \otimes S_2$ is nonzero and invariant for $A_1 \otimes A_2$ and if A_i , for $i = 1, 2$, is injective, then S_i is an invariant subspace for A_i .

According to Lemma 5.2, we present next an example to illustrate that if the matrices A_1 and A_2 are not both injective, then we can have $S = S_1 \otimes S_2$ invariant for $A_1 \otimes A_2$ with only S_1 or S_2 invariant for A_1 or A_2 , respectively.

Example 5.3 Consider the linear matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Taking $u_1 = (1, 1)$ and $u_2 = (1, -1)$, we have that the generalized eigenspaces of A_1 are $G_2^{A_1} = \langle u_1 \rangle$ and $G_0^{A_1} = \langle u_2 \rangle$. Now, if $v_1 = (1, 1, 1)$, $v_2 = (1, -1, 0)$, $v_3 = (0, 0, 1)$, then the generalized eigenspaces of A_2 are $G_2^{A_2} = \langle v_1 \rangle$ and $G_0^{A_2} = \langle v_2, v_3 \rangle$. Thus, by Proposition 3.7, the generalized eigenspaces of $A_1 \otimes A_2$ are $G_4^{A_1 \otimes A_2} = \langle u_1 \otimes v_1 \rangle$ and $G_0^{A_1 \otimes A_2} = \langle u_1 \otimes v_2, u_1 \otimes v_3, u_2 \otimes v_1, u_2 \otimes v_2, u_2 \otimes v_3 \rangle$. Note that the linear operators A_1 , A_2 (and $A_1 \otimes A_2$) are not injective. As $u_2 \otimes v_1$ and $u_2 \otimes v_3$ are eigenvectors of $A_1 \otimes A_2$ associated with the eigenvalue 0, it follows that their sum is also an eigenvector of $A_1 \otimes A_2$. Thus the subspace $\langle u_2 \rangle \otimes \langle v_1 + v_3 \rangle$ is invariant for $A_1 \otimes A_2$, where the subspace $\langle u_2 \rangle$ is invariant for A_1 but the subspace $\langle v_1 + v_3 \rangle$ is not invariant for A_2 . \diamond

As above, for $i = 1, 2$, let A_i be a linear operator on a linear space P_i , and S_i a subspace of P_i . Let $\text{Lat}(A_1) \otimes \text{Lat}(A_2)$ denote the collection of all nonzero regular subspaces $S_1 \otimes S_2$, where each S_i is an A_i -invariant subspace of P_i , and let $\text{RLat}(A_1 \otimes A_2)$ denote the collection of all regular invariant subspaces of $P_1 \otimes P_2$ for $A_1 \otimes A_2$. From the results in Theorem 1 of Kubrusly [26], we have:

Theorem 5.4 $\text{RLat}(A_1 \otimes A_2)$ is a lattice. If each A_i is injective, then

$$\text{RLat}(A_1 \otimes A_2) \setminus \{0\} = \text{Lat}(A_1) \otimes \text{Lat}(A_2) \subseteq \text{Lat}(A_1 \otimes A_2) \setminus \{0\},$$

and the inclusion may be proper even though every A_i is injective.

Theorem 5.4 states that for injective matrices, A_1 and A_2 , the lattice of regular invariant subspaces for $A_1 \otimes A_2$ is given by the product of the invariant subspaces for A_1 with the invariant subspaces for A_2 . Nevertheless, as the theorem states and as the following example illustrates, there can be other invariant subspaces for $A_1 \otimes A_2$, the irregular ones.

Example 5.5 Consider the injective linear operators on \mathbf{R}^2 and \mathbf{R}^3 , respectively, given by:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The generalized eigenspaces for A_1 are $G_2^{A_1} = \langle u_1 \rangle = \langle (1, 1) \rangle$ and $G_1^{A_1} = \langle u_2 \rangle = \langle (0, 1) \rangle$. The generalized eigenspaces for A_2 are $G_2^{A_2} = \langle w_1 \rangle = \langle (1, 1, 1) \rangle$, $G_{-1}^{A_2} = \langle w_2 \rangle = \langle (1, -2, 1) \rangle$ and $G_1^{A_2} = \langle w_3 \rangle = \langle (1, 0, -1) \rangle$. Consider now the Kronecker product matrix

$$A_1 \otimes A_2 = \begin{bmatrix} 2A_2 & 0A_2 \\ 1A_2 & 1A_2 \end{bmatrix}.$$

According to Proposition 3.7, we have the following generalized eigenspaces for $A \otimes B$,

$$\begin{aligned} G_4^{A_1 \otimes A_2} &= G_2^{A_1} \otimes G_2^{A_2} = \langle u_1 \otimes w_1 \rangle, \\ G_2^{A_1 \otimes A_2} &= \left(G_2^{A_1} \otimes G_1^{A_2} \right) \oplus \left(G_1^{A_1} \otimes G_2^{A_2} \right) \\ &= \langle u_1 \otimes w_3, u_2 \otimes w_1 \rangle, \\ G_{-2}^{A_1 \otimes A_2} &= G_2^{A_1} \otimes G_{-1}^{A_2} = \langle u_1 \otimes w_2 \rangle, \\ G_{-1}^{A_1 \otimes A_2} &= G_1^{A_1} \otimes G_{-1}^{A_2} = \langle u_2 \otimes w_2 \rangle, \\ G_1^{A_1 \otimes A_2} &= G_1^{A_1} \otimes G_1^{A_2} = \langle u_2 \otimes w_3 \rangle. \end{aligned}$$

We have $\text{RLat}(A_1 \otimes A_2) \setminus \{0\} \subsetneq \text{Lat}(A_1 \otimes A_2) \setminus \{0\}$ since, for example, the subspace $\langle u_1 \otimes w_1, u_2 \otimes w_1, u_2 \otimes w_2, u_2 \otimes w_3 \rangle$ is $A_1 \otimes A_2$ -invariant and irregular. \diamond

5.2 The lattice of synchrony subspaces for the Kronecker product of networks

Motivated by the results in Section 5.1, we prove in this section how to get the lattice of regular synchrony subspaces for the Kronecker product of networks from the lattices of synchrony subspaces for the component networks of the product. We also show that, in

general, the same is not possible for the all lattice of synchrony subspaces for the Kronecker product network.

As before, let P_1, P_2 be two real vector spaces, say \mathbf{R}^n and \mathbf{R}^m , and consider P_1, P_2 be two real vector spaces, say \mathbf{R}^n and \mathbf{R}^m . We start by recalling that a synchrony subspace for an identical-edge n -cell network is a polydiagonal subspace that is left invariant by the $n \times n$ network adjacency matrix. (See Definition 4.1 and the discussion that follows that definition.)

Lemma 5.6 *A regular subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is polydiagonal if and only if each S_i is a polydiagonal subspace of P_i , $i = 1, 2$.*

Proof Let $S = S_1 \otimes S_2$ be a regular subspace of $P_1 \otimes P_2$ where both S_1 and S_2 are polydiagonal subspaces of P_1 and P_2 , respectively. Trivially, we have that $S_1 \otimes P_1$ and $P_1 \otimes S_2$ are polydiagonal subspaces of $P_1 \otimes P_2$. As $S = S_1 \otimes S_2 = S_1 \otimes P_2 \cap P_1 \otimes S_2$, it follows that S is a polydiagonal subspace of $P_1 \otimes P_2$.

Assume now that $S = S_1 \otimes S_2$ is a regular subspace of $P_1 \otimes P_2$ where for example S_1 is not a polydiagonal subspace of P_1 . Recall that $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$. We prove that then S cannot be a polydiagonal subspace of $P_1 \otimes P_2$. Assume that S_1 has dimension r and S_2 has dimension s . Take the polydiagonal subspace P_{S_1} of P_1 given by the intersection of all polydiagonal subspaces of P_1 containing S_1 . If P_{S_1} has dimension $n - i$ we have that all the vectors of P_{S_1} and of S_1 satisfy i independent equalities. Moreover, as S_1 is not a polydiagonal, that is, S_1 is strictly contained in P_{S_1} , we have that $r < n - i$. Take also the polydiagonal subspace P_{S_2} of P_2 given by the intersection of all polydiagonal subspaces of P_2 containing S_2 . If P_{S_2} has dimension $m - j$ we have that all the vectors of P_{S_2} and of S_2 satisfy j independent equalities, and $s \leq m - j$. The polydiagonal $P_{S_1} \otimes P_{S_2}$ has dimension $(n - i)(m - j)$ and all the vectors in $P_{S_1} \otimes P_{S_2}$ and in $S_1 \otimes S_2$ satisfy $jn + i(m - j)$ independent equalities. In fact, as $S = S_1 \otimes S_2$, and we know that the maximum number of independent equalities satisfied by all the vectors of S_1 is i and of S_2 is j , then the maximum number of independent equalities satisfied by all the vectors of $S_1 \otimes S_2$ is $jn + i(m - j)$. Since $S_1 \otimes S_2$ has dimension $rs < (n - i)(m - j)$, we conclude that $S = S_1 \otimes S_2$ cannot be a polydiagonal subspace. \square

Lemma 5.7 *For $i = 1, 2$, let \mathcal{N}_i be an identical-edge coupled cell network. Assume \mathcal{N}_1 has n cells and \mathcal{N}_2 has m cells. Take P_1, P_2 to be the real vector spaces \mathbf{R}^n and \mathbf{R}^m , respectively. For $i = 1, 2$, let S_i a subspace of P_i . We have:*

1. *If S_i is a synchrony subspace for the network \mathcal{N}_i , for $i = 1, 2$, then the subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$.*
2. *If $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, then at least one of the S_i is a synchrony subspace for \mathcal{N}_i .*
3. *If \mathcal{N}_1 and \mathcal{N}_2 are injective networks and $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, then S_1 and S_2 are synchrony subspaces for \mathcal{N}_1 and \mathcal{N}_2 , respectively.*

Proof If S_i , $i = 1, 2$, is a synchrony subspace for \mathcal{N}_i with adjacency matrix A_i , by Lemma 5.2, the regular subspace $S = S_1 \otimes S_2$ is invariant under $A_1 \otimes A_2$, the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$; by Lemma 5.6, as S_1, S_2 are polydiagonal subspaces, $S_1 \otimes S_2$ is also a polydiagonal subspace. Thus $S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, proving statement 1.

Assume now that $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$. Thus S is a polydiagonal subspace of $P_1 \otimes P_2$ that is left invariant under $A_1 \otimes A_2$, the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. By Lemma 5.6, we have that both S_1 and S_2 are polydiagonal subspaces. By Lemma 5.2 we have that at least one of the S_i is invariant under A_i , and so it is a synchrony subspace for \mathcal{N}_i , proving statement 2.

If, additionally, A_1 and A_2 are injective then, by Lemma 5.2, both S_1 and S_2 are invariant under A_1 and A_2 , respectively, and so are synchrony subspaces for \mathcal{N}_1 and \mathcal{N}_2 , respectively, proving statement 3. \square

The following lemma will be used in the proof of Proposition 5.9.

Lemma 5.8 *Let A be the adjacency matrix of an n -cell network. There is no polydiagonal subspace of \mathbf{R}^n contained in $\ker(A)$.*

Proof If a polydiagonal subspace P was contained in $\ker(A)$ then, in particular, the $(1, 1, \dots, 1)$ vector would belong to $\ker(A)$. Since A is a nonnegative matrix, that would only be possible if A was the zero matrix, which is not the case. \square

The following proposition generalizes the result of statement 3 of Lemma 5.7 for any identical-edge network, injective or not.

Proposition 5.9 *If S is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$ with $S = S_1 \otimes S_2$ then S_i is a synchrony subspace for \mathcal{N}_i , $i = 1, 2$.*

Proof Let A_i denote the adjacency matrix of \mathcal{N}_i , for $i = 1, 2$. Since $S = S_1 \otimes S_2$ is a synchrony subspace

for $\mathcal{N}_1 \otimes \mathcal{N}_2$, it is a polydiagonal subspace that is invariant for $A_1 \otimes A_2$. As $S = S_1 \otimes S_2$ is a polydiagonal subspace, from Lemma 5.6, we have that both S_1 and S_2 are polydiagonal subspaces. To prove that S_1 and S_2 are synchrony subspaces it remains to prove that they are invariant subspaces for A_1 and A_2 , respectively. If $\mathcal{N}_1 \otimes \mathcal{N}_2$ is an injective network that follows from statement 3 of Lemma 5.7. If $\mathcal{N}_1 \otimes \mathcal{N}_2$ is not an injective network, we have from Lemma 5.2, that at least one of the S_i , for $i = 1$ or $i = 2$, is an invariant subspace for N_i . But, in the situation where S_1 and S_2 are polydiagonal subspaces we cannot have that one of the S_i , for $i = 1$ or $i = 2$, is not an invariant subspace for A_i . In fact, if one of the S_i s was not invariant for A_i , then it would have to be contained at $\ker(A_i)$ by Lemma 5.2. But, by Lemma 5.8, that is impossible since S_i is a polydiagonal subspace. \square

As above, for $i = 1, 2$, let \mathcal{N}_i be a network with adjacency matrix A_i . Thus A_i is a linear operator on the linear space P_i , say $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$, if \mathcal{N}_1 has n cells and \mathcal{N}_2 has m cells. We generalize now the notation given at Section 5.1 for $\text{Lat}(A_1) \otimes \text{Lat}(A_2)$ and $\text{RLat}(A_1 \otimes A_2)$. Let $V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$ denote the collection of all regular subspaces $S_1 \otimes S_2$, where each S_i is a synchrony subspace for \mathcal{N}_i . Let $\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denote the collection of all regular synchrony subspaces of $P_1 \otimes P_2$ for $\mathcal{N}_1 \otimes \mathcal{N}_2$ and recall that $V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denotes the lattice of synchrony subspaces for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$. From the above results, we have:

Theorem 5.10 *$\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is a lattice and*

$$\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$$

and the inclusion may be proper even though every \mathcal{N}_i is injective.

Proof By the statement 2 of Lemma 5.7 and Proposition 5.9, a regular subspace $S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a synchrony subspace for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ if and only if both S_1 and S_2 are synchrony subspaces for the networks \mathcal{N}_1 and \mathcal{N}_2 , respectively. We have then that $\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$. Moreover, $\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is a lattice since it is the tensor of the two lattices $V_{\mathcal{N}_1}$ and $V_{\mathcal{N}_2}$. (See Corollary 2.7 of Grätzer *et al.* [20].) \square

We finish this section by showing examples where the inclusion $V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is proper, for both cases where the networks are injective or not. Moreover, our examples also show the existence of irregular and sum-irreducible synchrony subspaces for the Kronecker product – that is, there are synchrony subspaces that are not a sum of regular synchrony subspaces for the Kronecker product. The conclusion is

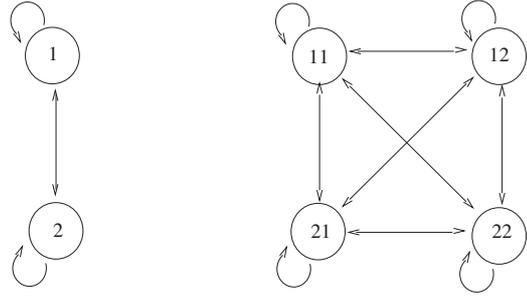


Figure 4: (Left) The network \mathcal{N} . (Right) The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

that, although, taking in advance the knowledge about the synchrony lattices for the component networks involved in a product, the synchrony lattice for the Kronecker product network is not, in general, completely determined by those lattices. That goes in the same lines as it is for the lattices of invariant subspaces under tensor product of linear operators, as seen in Section 5.1.

Example 5.11 Consider the network \mathcal{N} and the Kronecker product network $\mathcal{N} \otimes \mathcal{N}$ in Figure 4. The adjacency matrix of \mathcal{N} is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note that the networks are not injective. The generalized eigenspaces of A are $G_2^A = \langle (1, 1) \rangle$ and $G_0^A = \langle (1, -1) \rangle$. Thus, the generalized eigenspaces of $A \otimes A$ are $G_4^{A \otimes A} = \langle (1, 1) \otimes (1, 1) \rangle \cong \langle (1, 1, 1, 1) \rangle$ and $G_0^{A \otimes A} = \langle (1, 1) \otimes (1, -1), (1, -1) \otimes (1, 1), (1, -1) \otimes (1, -1) \rangle$. Note that $G_0^{A \otimes A} \cong \langle (1, -1, 1, -1), (1, 1, -1, -1), (1, -1, -1, 1) \rangle$. We have that $S = \{\mathbf{x} : x_{11} = x_{12} = x_{21}\} = \langle (1, 1, 1, 1), (1, 1, 1, -3) \rangle$ is an irregular synchrony subspace that cannot be given as a sum of regular synchrony subspaces. \diamond

Example 5.12 Consider the injective network \mathcal{N} in Figure 5 with adjacency matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

This is network 26 of Leite [28]. The generalized eigenspaces of A are $G_2 = \langle (1, 1, 1) \rangle$, $G_1 = \langle (0, 1, 0) \rangle$ and $G_{-1} = \langle (-2, 1, 4) \rangle$. The only non-trivial synchrony subspace for this network is $\{\mathbf{x} : x_1 = x_3\}$.

Take the Kronecker product $\mathcal{N} \otimes \mathcal{N}$, see Figure 5, and the two-dimensional synchrony subspace for $\mathcal{N} \otimes$

\mathcal{N} given by $S = \{\mathbf{x} : x_{11} = x_{12} = x_{13} = x_{21} = x_{23} = x_{31} = x_{32} = x_{33}\}$. Note that

$$\begin{aligned} S &= \langle (1, 1, 1, 1, 1, 1, 1, 1), (0, 0, 0, 0, 1, 0, 0, 0) \rangle \\ &\equiv \langle (1, 1, 1) \otimes (1, 1, 1), (0, 1, 0) \otimes (0, 1, 0) \rangle \end{aligned}$$

where $(1, 1, 1) \otimes (1, 1, 1)$ and $(0, 1, 0) \otimes (0, 1, 0)$ are eigenvectors of $A \otimes A$. Now S is a synchrony subspace for the Kronecker product which is sum-irreducible, as it is not the sum of proper synchrony subspaces for $\mathcal{N} \otimes \mathcal{N}$. Moreover, S is not a regular synchrony subspace. Thus, this is another example, where the Kronecker product has synchrony subspaces that cannot be obtained using the synchrony subspaces for the component networks of the product. \diamond

6 The lattice of synchrony subspaces for the cartesian product network

In this section, we show that the results presented in Section 5.2 for the Kronecker product of networks are also valid for the cartesian product of networks.

As before, let \mathcal{N}_1 and \mathcal{N}_2 be two identical-edge networks, say with n and m cells, respectively, and take $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$. In what follows, we adopt the following notation: for $i = 1, 2$, denote by \mathcal{N}_i^d the disconnected subnetwork of \mathcal{N}_i with no edges. Thus the network $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ is the subnetwork of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ ignoring the edges corresponding to the network \mathcal{N}_2 – it is formed by r_2 copies of the network \mathcal{N}_1 . Similarly, ignoring the edges corresponding to the network \mathcal{N}_1 we obtain r_1 copies of the network \mathcal{N}_2 , that correspond to the network $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.

Proposition 6.1 *A polydiagonal S is a synchrony subspace for the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if S is a synchrony subspace for the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.*

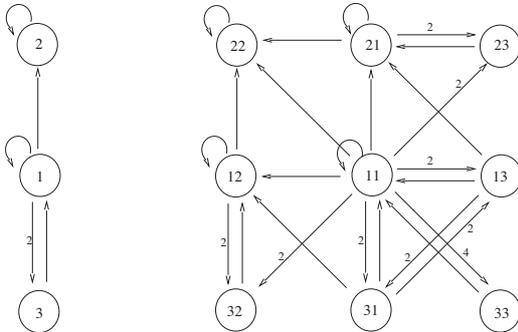


Figure 5: (Left) The network \mathcal{N} . (Right) The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

Proof Observe that the set of edges of the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the disjoint union of the two sets of edges of the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Moreover, the types of edges of each of these subnetworks are distinct. Equivalently, the adjacency matrices of the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the adjacency matrices of the two subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Now recall that a polydiagonal subspace S is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if it is left invariant under its adjacency matrices. \square

Proposition 6.2 *The lattice of synchrony subspaces for the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the set-wise intersection of the lattices of synchrony subspaces for the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.*

Proof Follows trivially from Proposition 6.1. \square

Lemma 6.3 *If S_i is a synchrony space for the network \mathcal{N}_i , for $i = 1, 2$, then the subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a regular synchrony space for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$.*

Proof Let S_1 be a synchrony subspace for \mathcal{N}_1 . Then $S_1 \otimes \mathbf{R}^{r_2}$ can be seen as r_2 distinct copies of the synchrony subspace S_1 for \mathcal{N}_1 – each one adapted to the coordinates of each of the r_2 copies of the network \mathcal{N}_1 in $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. Thus, trivially, $S_1 \otimes \mathbf{R}^{r_2}$ is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. Moreover, the adjacency matrix of the network $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$ leaves $S_1 \otimes \mathbf{R}^{r_2}$ invariant. Thus $S_1 \otimes \mathbf{R}^{r_2}$ is also a synchrony subspace for $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Similarly, we prove that if S_2 is a synchrony subspace for \mathcal{N}_2 , then $\mathbf{R}^{r_1} \otimes S_2$ is a synchrony subspace for both networks $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$ and $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. By Proposition 6.1, $S_1 \otimes \mathbf{R}^{r_2}$ and $\mathbf{R}^{r_1} \otimes S_2$ are both synchrony subspaces for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. Thus $S_1 \otimes \mathbf{R}^{r_2} \cap \mathbf{R}^{r_1} \otimes S_2 = S_1 \otimes S_2$ is a synchrony for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. \square

Proposition 6.4 *If S is a regular synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ with $S = S_1 \otimes S_2$ then S_i is a synchrony subspace for \mathcal{N}_i , $i = 1, 2$.*

Proof If $S = S_1 \otimes S_2$ is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$, then it is a synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, by the lattice inclusion $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ proved in Proposition 4.5. The result then follows by Proposition 5.9. \square

Let $\text{RV}_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ denote the collection of all regular synchrony subspaces for $P_1 \otimes P_2$ for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and recall that $\text{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denotes the lattice of regular synchrony subspaces for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ and $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ denotes the lattice of synchrony subspaces for the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. From the above results, we have:

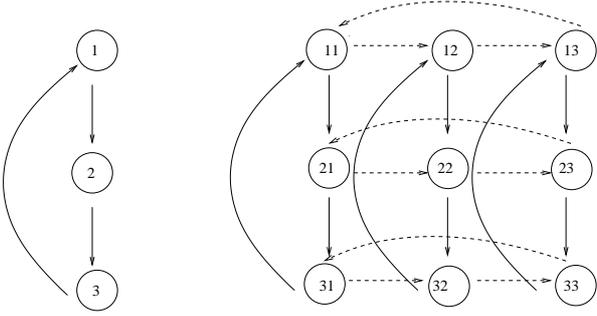


Figure 6: (Left) The 3-cell unidirectional ring network \mathcal{N} . (Right) The cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$.

Theorem 6.5 $RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ is a lattice and

$$RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = RV_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$$

and the inclusion may be proper even though every \mathcal{N}_i is injective.

Proof Note that, by Lemma 6.3 and Proposition 6.4, a regular subspace $S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a synchrony subspace for the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if both S_1 and S_2 are synchrony subspaces for the networks \mathcal{N}_1 and \mathcal{N}_2 , respectively. We have then that $RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$. By Theorem 5.10, it follows that $RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = RV_{\mathcal{N}_1 \otimes \mathcal{N}_2}$, and thus, it is a lattice. \square

We finish by showing an example where the inclusion $RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ is proper.

Example 6.6 Considering the cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$ of Figure 6, we show that $RV_{\mathcal{N} \boxtimes \mathcal{N}} \subsetneq V_{\mathcal{N} \boxtimes \mathcal{N}}$ by pointing out an irregular synchrony space in $V_{\mathcal{N} \boxtimes \mathcal{N}}$. For example, the polydiagonal in \mathbf{R}^9 defined by the equalities $x_{11} = x_{22} = x_{33}$, $x_{12} = x_{23} = x_{31}$, $x_{13} = x_{21} = x_{32}$ is an irregular synchrony subspace for the product network $\mathcal{N} \boxtimes \mathcal{N}$ since it is not of the form $S_1 \otimes S_2$ with S_i a synchrony subspace for the network \mathcal{N} . \diamond

7 Conclusion

In this work we have considered coupled cell networks constructed as the product of smaller coupled cell networks, using two types of product: the cartesian and the Kronecker product. It is natural to realize that a network constructed as product of two networks, besides being a graph with more nodes and arrows, can have additional properties that are intrinsic to the product operation. One important goal is to derive how far we can describe the dynamics of the CCS associated with a (large) product network based on the

dynamics of the CCS consistent with the structure of the component networks of the product.

In this paper we have focused this dynamics question on the lattice of synchrony subspaces. In the analysis of how far it is possible to describe the set of synchrony subspaces for a product network from the sets of synchrony subspaces for the component networks, our answer is complete when we restrict to regular synchrony subspaces, that is, synchrony subspaces that are given by the tensor product of spaces. We show in Theorems 5.10 and 6.5 that, for both kinds of products considered here, the set of regular synchrony subspaces for the product is a lattice and it is the tensor product of the lattices of the synchrony subspaces for the component networks of the product. That is, a synchrony subspace for the product (cartesian or Kronecker) is regular if and only if it is the tensor of synchrony subspaces for the component networks of the product. Moreover, we also show that, in general, there are synchrony subspaces for the product that are not possible to describe using only the synchrony subspaces for the component networks.

In a future work, we intend to use our results relating the lattice of synchrony subspaces for a product coupled cell network with those of its component networks in the construction of heteroclinic networks in CCS associated with product networks whose component networks are known to support heteroclinic behaviour in the associated dynamics. We observe that it follows, in particular, from our results that if we take a coupled cell system consistent with a product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ (or $\mathcal{N}_1 \boxtimes \mathcal{N}_2$), where it is known that at least one of the networks \mathcal{N}_i can support heteroclinic cycles, then that behaviour will occur for the product coupled system (for appropriated choices of the product vector fields). This follows from the definition itself of synchrony subspace and the fact that the dynamics of CCS associated with \mathcal{N}_i have to occur at the product CCS.

Acknowledgments

The authors were partially funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the projects PTDC/MAT/100055/2008 and PEst-C/MAT/UI0144/2013.

References

- [1] J. W. Aldis. A polynomial time algorithm to determine maximal balanced equivalence relations, *J. Bifur. Chaos Appl. Sci. Engrg.* **18** (2) (2008) 407–427.

- [2] N. Agarwal, A. Rodrigues and M.J. Field. Dynamics near the Product of Planar Heteroclinic Attractors, *Dynamical Systems: An International Journal* **26** (4) (2011) 447–481.
- [3] M. Aguiar, P. Ashwin, A. Dias, and M. Field. Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation, *J. Nonlinear Sci.* **21** (2) (2011) 271–323.
- [4] M. Aguiar, S. Castro and I. Labouriau. Dynamics near a Heteroclinic network, *Nonlinearity* **18** (2) (2005) 391–414.
- [5] M.A.D. Aguiar and A.P.S. Dias. The Lattice of Synchrony Subspaces of a Coupled Cell Network: Characterization and Computation Algorithm, *J. Nonlinear Sci.* **24** (6) (2014) 949–996.
- [6] M.A.D. Aguiar, I.S. Labouriau and A.A.P. Rodrigues. Switching near a network of rotating nodes, *Dynamical Systems: An Int. Journal* **25** (1) (2010) 75–95.
- [7] M.A.D. Aguiar and H. Ruan. Evolution of synchrony under combination of coupled cell networks, *Nonlinearity* **25** (2012) 3155–3187.
- [8] P. Ashwin and M. Field. Heteroclinic networks in coupled cell systems, *Arch. Ration. Mech. Anal.* **148** (2) (1999) 107–143.
- [9] P. Ashwin and M. Field. Product dynamics for homoclinic attractors, *Proc. Royal Soc., Ser. A* **461** (2005) 155–177.
- [10] P. Ashwin and O. Karabacak. Robust Heteroclinic Behaviour, Synchronization, and Ratcheting of Coupled Oscillators, *Dynamics, Games AND Science II* **2** (2011) 125–140.
- [11] P. Ashwin, O. Karabacak and T. Nowotny. Criteria for robustness of heteroclinic cycles in neural microcircuits, *J. Math. Neurosci.* **1** (1) (2011).
- [12] P. Ashwin and C. Postlethwaite. On designing heteroclinic networks from graphs, *Phys. D* **265** (2013) 26–39.
- [13] F.M. Atay and T. Biyikoglu. Graph operations and synchronization of complex networks, *Physical Review E* **72** 016217, 2005.
- [14] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.
- [15] M. Field. Combinatorial dynamics, *Dyn. Syst.* **19** (3) (2004) 217–243.
- [16] M. Field. Heteroclinic networks in homogeneous and heterogeneous identical cell systems, preprint (2014) <http://math.rice.edu/~mjf8/het/realization.pdf>
- [17] M. Golubitsky and R. Lauterbach. Bifurcations from synchrony in homogeneous networks: linear theory, *SIAM J. Appl. Dynam. Syst.* **8** (1) (2009) 40–75.
- [18] M. Golubitsky and I. Stewart. Nonlinear dynamics of networks: the groupoid formalism, *Bull. Amer. Math. Soc.* **43** (3) (2006) 305–364.
- [19] M. Golubitsky, I. Stewart and A. Török. Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows, *SIAM J. Appl. Dynam. Sys.* **4** (1) (2005) 78–100.
- [20] G. Grätzer, H. Lakser and R. Quackenbush. The structure of tensor products of semilattices with zero, *Trans. Amer. Math. Soc.* **267** (2) (1981) 503–515.
- [21] A.J. Homburg and J. Knobloch. Switching homoclinic networks, *Dynamical Systems* **25** (3) (2010) 351–358.
- [22] A.J. Homburg and B. Sandstede. Homoclinic and heteroclinic bifurcations in vector fields, *In H. Broer, F. Takens and B. Hasselblatt (Eds.), Handbook of dynamical systems* **3** Amsterdam: North-Holland (2010) 379–524.
- [23] H. Kamei and P. J. A. Cock. Computation of Balanced Equivalence Relations and Their Lattice for a Coupled Cell Network, *SIAM J. Appl. Dynam. Sys.* **12** (1) (2013) 352–382.
- [24] V. Kirk, E. Lane, C. M. Postlethwaite, A. M. Rucklidge, and M. Silber. A mechanism for switching near a heteroclinic network, *Dynamical Systems: An International Journal* **25** (3) (2010) 323–349.
- [25] M. Krupa. Robust heteroclinic cycles (review article), *J. of Nonl. Sci.* **7** (1997) 129–176.
- [26] C.S. Kubrusly. Regular lattices of tensor products, *Linear Algebra and its Applications* **438** (2013) 428–435.
- [27] I.S. Labouriau, A.A.P. Rodrigues and M.A.D. Aguiar. Chaotic double cycling, *Dynamical Systems: An Int. Journal* **26** (2) (2011) 199–233.
- [28] M.C.A. Leite and M. Golubitsky. Homogeneous three-cell networks, *Nonlinearity* **19** (2006) 2313–2363.
- [29] J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani. Kronecker graphs: an approach to modeling networks, *J. Mach. Learn. Res.* **11** (2010) 985–1042.
- [30] I. Stewart. The lattice of balanced equivalence relations of a coupled cell network, *Math. Proc. Cambridge Philos. Soc.* **143** (1) (2007) 165–183.
- [31] I. Stewart, M. Golubitsky and M. Pivato. Symmetry groupoids and patterns of synchrony in coupled cell networks, *SIAM J. Appl. Dynam. Sys.* **2** (2003) 609–646.
- [32] P. M. Weichsel. The Kronecker product of graphs, *Proc. Amer. Math. Soc.* **13** (1962) 47–52.