Coupled cell networks: minimality

Manuela A. D. Aguiar * and **Ana Paula S. Dias**[1]

[1] Centro de Matemática da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.

Non-isomorphic coupled cell networks with the same number of cells and with equivalent dynamics are said to be ODE-equivalent. Moreover, they are all related by a linear algebra condition involving their graph adjacency matrices. A network in an ODE-class is said to be minimal if it has a minimum number of edges among all the networks of the class. In this short paper we review the characterization of the minimal networks of an ODE-equivalence class and we present an example for the case of non-homogeneous networks.

Networks of nonlinear dynamical systems model many real world problems. See for example Stewart[4] and references therein. Mathematically, a *coupled cell network* can be idealised as a directed graph representing schematically a set of dynamical systems (the *cells*) that are coupled together, and the couplings (the *edges*) among them. Following the theory of Stewart etal [5] and Golubitsky etal [3], we associate to each coupled cell network a class of ordinary differential equations (ODEs) compatible with the structure of the network – the class of *coupled cell systems*.

Non-isomorphic coupled cell networks can have equivalent dynamics. Such networks are said to be *ODE-equivalent* [3]. Dias and Stewart [2] prove that up to re-numeration of cells, two coupled cell networks are ODE-equivalent if and only if the set of all corresponding linear vector fields coincide. Moreover, cells can be assumed to be one-dimensional for that purpose.

In [1] we compare ODE-equivalent coupled cell networks in terms of the number of edges. A network in an ODE-equivalence class is said to be *minimal* if it has a minimum number of edges.

Using the results in [2] on ODE and linear equivalence of networks, the problem of finding all the minimal networks in a given ODE-equivalence class can be posed in terms of the networks adjacency matrices. In [1] we describe the minimal subclass of a given coupled cell network and present an algorithm that computes it.

We start by addressing the minimality of homogeneous networks. That is, networks where all cells are isomorphic. Two cells are said to be *isomorphic* if there is a bijection between the two sets of edges directed to them that preserves the edge-type. For an ODE-class of an homogeneous coupled cell network, we can consider the lattice consisting of the vectors with integer entries in the real subspace generated by the adjacency matrices of any network in the class. The minimal networks are obtained by finding vectors (adjacency matrices) with shortest length in a cone of that lattice (whose rank grows with the number of edge-types). That is, the problem of finding all the minimal networks corresponds to find all the *minimal bases* (with non-negative integer entries) of the real vector space generated by the adjacency matrices of the networks in the ODE-class. The minimal bases define the adjacency matrices of the minimal networks. As a corollary, we obtain that an ODE-class containing an homogeneous coupled cell network with only one type of edges has a unique minimal network. However, in general, for ODE-classes containing networks with more than one edge-type there is more than one minimal network.

In [1] we show that the minimality of coupled cell networks basically reduces to the minimality of homogeneous networks. Figure 1 (a) shows an example of a non-homogeneous coupled cell network. Here, identical cells and identical edges are represented, respectively, by the same symbol. Multiple couplings of the same type between two cells are represented by just one arrow with the number of couplings attached to it. Note that cells $1, 2$ and $3, 4$ are isomorphic. So this network decomposes into the homogeneous networks (b) and (c). Applying the methods in [1], the networks (d), (e) are minimal networks in the ODE-classes of (b) and (c), respectively. Composing the networks (d) and (e) we obtain (f) which is a minimal network in the ODE-class of network (a).

* CMUP is supported by FCT through the programmes POCTI and POSI, with Portuguese and European Community structural funds.

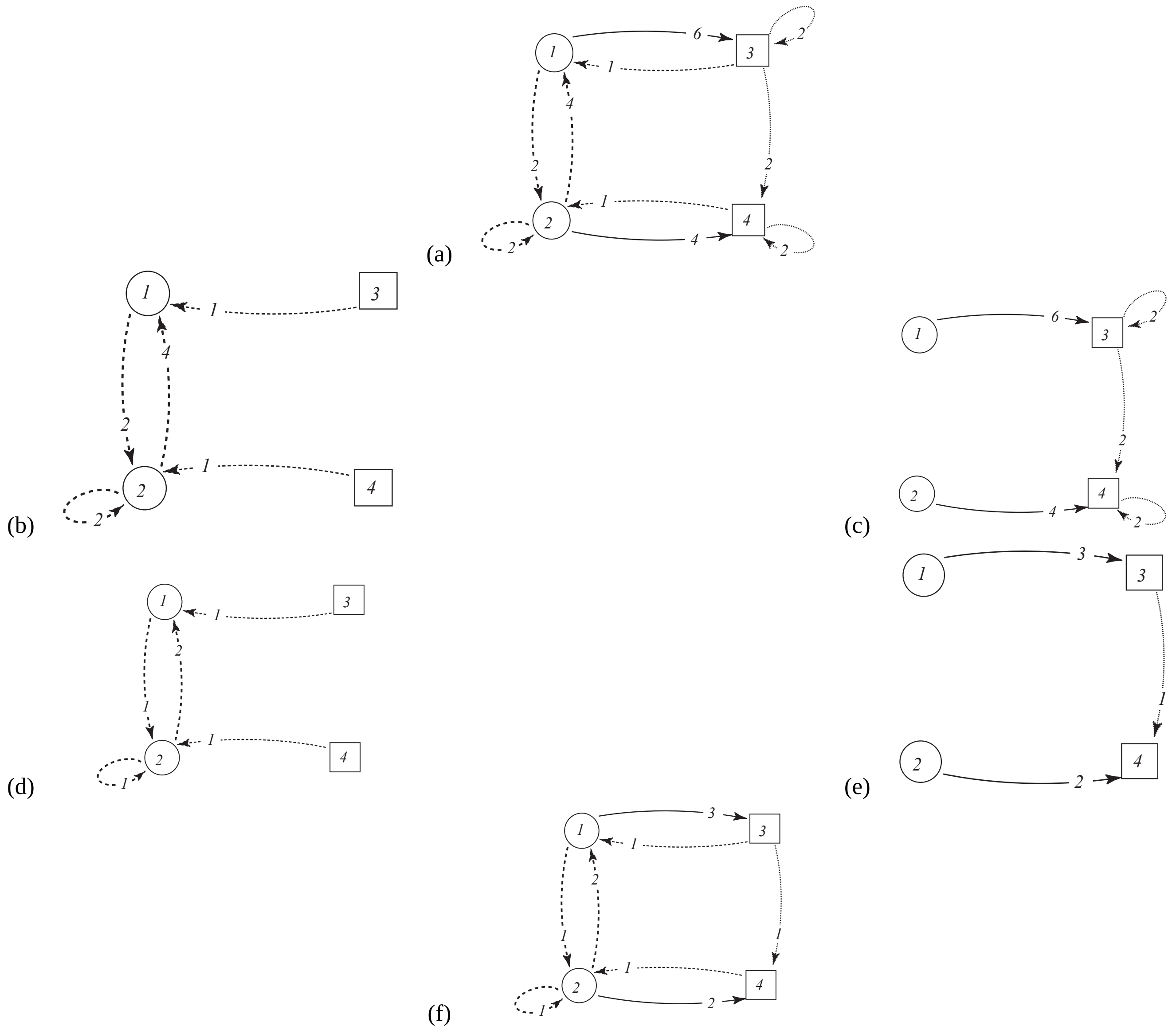

Fig. 1 (f) is a minimal network of the non-homogeneous network (a). Networks (d) and (e) are minimal networks of the homogeneous networks (b) and (c), respectively.

References

[1] M.A.D.Aguiar and A.P.S.Dias. Minimal Coupled Cell Networks. *Nonlinearity* **20**, 193-219 (2007).

[2] A.P.S. Dias and I. Stewart. Linear Equivalence and ODE-equivalence for Coupled Cell Networks, *Nonlinearity* **18**, 1003–1020 (2005).

[3] M. Golubitsky, I. Stewart and A. Török. Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows, *SIAM J. Appl. Dynam. Sys.* **4**(1), 78–100 (2005) .

[4] I. Stewart. Network opportunity, *Nature* **427**, 601-604 (2004) .

[5] I. Stewart, M. Golubitsky and M. Pivato. Symmetry groupoids and patterns of synchrony in coupled cell networks, *SIAM J. Appl. Dynam. Sys.* **2**, 609-646 (2003).