Chaotic and deterministic switching in a two-person game

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Abstract

We study robust long-term complex behaviour in the Rock-Scissors-Paper game with two players, played using reinforcement learning. The complex behaviour is connected to the existence of a heteroclinic network for the dynamics. This network is made of three heteroclinic cycles consisting of nine equilibria and the trajectories connecting them. We provide analytical proof both for the existence of *chaotic switching* near the heteroclinic network and for the relative asymptotic stability of at least one cycle in the network, leading to behaviour ranging from almost deterministic actions to chaotic-like dynamics. Our results are obtained by making use of the symmetry of the original problem, a new approach in the context of learning.

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1 Introduction

Recent years have seen a thriving expansion within the subject of learning in games, both from the point of view of population dynamics in biology and from that of strategic thinking applied to economics. Two landmark references are Hofbauer and Sigmund [17] concerning the first viewpoint and Fudenberg and Levine [10] concerning the latter. See also Hofbauer and Sigmund [19] and Fudenberg and Levine [11] for more recent updates. Clearly, the two perspectives are not mutually exclusive and share common concerns such as asymptotic behaviour and convergence to equilibrium, learning rules or adaptation processes leading, or not, to equilibrium in the long-run.

Results have been achieved in many settings, both experimentally, numerically or analytically. Our results are analitycal but we refer to the papers by Roth and Erev [23, 7] and by Henrich *et al.* [15], as well as references in these papers, for an experimental treatment of learning. Numerical simulations are pervasive in the literature, out of necessity when models become too hard to solve. We refer to the work of Chawanya [5] and Sato *et al.* [25, 26] on this point.[2]

A central issue in learning is that of the learning procedure itself. This may consist, for instance, in simple imitation, in taking into account previous best-responses or in responding to some reinforcement received after an action. In the two latter cases, memory (or lack thereof) is also an issue: if there is memory loss then the effect of recent events is stronger than that of earlier ones; with perfect memory, all events affect the agent in the same way. See Hofbauer and Sigmund [19]. Since different games produce different outcomes with different learning processes, it has been an issue to decide which learning processes will eventually lead to equilibrium for each type of game. See Roth and Erev [23] for a comparison between behaviour in experiments and learning models.

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[2]The references to the literature are intended as sample references and are, by no means, comprehensive.

Most learning processes or models fit into, possibly a variation, of fictitious play: a process where players are engaged in playing a finite game repeatedly (this includes infinite repetition). The state of play is given by a probability vector, describing the mixed strategies of the agents. The dynamics of play are described by a dynamical system having the probability vectors describing the mixed strategies as state variables. One important question is that of finding out whether the dynamics will converge to a Nash equilibrium in the long-run. In the case of two players, one with 2 strategies and another with $n \geq 2$ strategies, Berger [2] showed that, both in discrete- and continuous-time, the dynamics approach equilibrium, thus solving the problem of asymptotic behaviour. However, Roth and Erev [23] do point out that, in view of experimental results, the "intermediate term predictions of dynamic learning models may be even more important than their asymptotic properties". In fact, transient dynamics can be rather different from asymptotic behaviour, as pointed out by Izquierdo *et al.* [20].

It is also well-known that asymptotic behaviour may not coincide with the Nash equilibrium of the finite game. In such cases, it is important to describe the asymptotic behaviour, which may range from periodic (see Shapley [28] and Sparrow *et al.* [29]) to chaotic (see Richards [22], Sato *et al.* [25, 26] and also Sparrow *et al.* [29]). Lack of convergence to equilibrium was observed in experiments by Feltovich [8]. Chaotic behaviour may arise or be described in different ways: Richards [22] and Sparrow *et al.* [29] address the existence of chaotic behaviour in the Shapley game [28], the former using a geometric argument and the latter by looking at the stable manifold of the periodic orbit (this is a result announced for a future paper). Sato *et al.* [25, 26] provide numerical evidence for the existence of complicated dynamics in a Rock-Scissors-Paper game with two players, arising from the existence of a heteroclinic network for the dynamics.

Our results add to the description of asymptotic behaviour when it does not converge to equilibrium. We do this by way of an example, even though the techniques we use may be applied to any game with analogous properties. We use Sato *et al.* [25, 26] and show, analytically, that in the Rock-Scissors-Paper (henceforth, RSP) game with two players there is *infinite switching*, leading to behaviour ranging from almost deterministic actions to chaotic-like dynamics. Each pair of choices, one for each player, among the possible actions of Rock, Scissors or Paper, is an equilibrium of a dynamical system that describes the dynamics of play. In this game there are 9 equilibria connected by trajectories and forming what is known as a heteroclinic network. After one choice of action, each player may make a certain number of choices at the next moment of play. The trajectories connecting the equilibria in the network reflect precisely these possible sequences of play. The existence of switching means that, given any possible sequence of play in the network, there are initial choices of action for each player such that the choices made throughout the game are exactly those described by the sequence. Thus, every possible sequence of actions may indeed take place in a game, including both simple sequences, involving a small number of equilibria, and sequences of play involving all equilibria chosen in random order.

This provides a distinct route to chaos from that considered by Richards [22] and Sparrow *et al.* [29] in a context that is equally simple. Furthermore, in our example there is coexistence of random (in which trajectories of play follow complex patterns) and almost deterministic (in which players alternate between two actions) behaviour. We will show that when a draw is penalized, players avoid sequences that involve draws, thus restricting the actions in a deterministic way. This is related to the stability of the cycles in the network.

In proving our results we make strong use of the symmetry of the problem thus opening a new way of dealing with the issue of asymptotic behaviour in this context of games. The symmetry allows us to reduce the study of the dynamics near a network involving 9 equilibria in a 4-dimensional space to that of the dynamics near a network involving 3 equilibria and a smaller number of trajectories connecting them.

The following section provides the preliminary results and notation required. In section

3, we describe the heteroclinic network in the RSP game, as well as the quotient network, with 3 equilibria, induced by symmetry. The properties of the networks are essential for the results that follow. Section 4 is devoted to the study of the dynamics of the reduced problem with 3 equilibria. We prove the existence of infinite switching near the quotient network, study the stability of the cycles that constitute the network, and the stability of the network as a whole. This is divided into subsections that lead to the proof of theorem 4.5. The last subsection of section 4 deals with the stability of the cycles in the quotient network and provides an explanation for a preference for one of the cycles of play, when the payoff for ties is negative for at least one player. In section 5, we extend the results obtained for the quotient network to the original network of the RSP game. Section 6 concludes.

2 Preliminary results and notation

Consider a system of differential equations

$$\dot{x} = f(x, \lambda), \tag{1}$$

with $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and f a smooth vector field.

Symmetry We introduce some background on group theory and equivariant dynamics needed throughout the paper. Other concepts and results not defined here can be found in Bredon [4], Chossat and Lauterbach [6], Golubitsky and Schaeffer [12] or Field [9].

Let Γ be a compact Lie group acting on $\mathbb{R}^n$. System (1) is *equivariant by* Γ or Γ-*symmetric* if it commutes with the action of Γ, that is

$$f(\gamma x, \lambda) = \gamma f(x, \lambda), \ \forall \gamma \in \Gamma \ \forall x \in \mathbb{R}^n.$$

Let G be a subgroup of Γ. The set of points $x \in \mathbb{R}^n$ that are kept invariant by the action of the elements in G is a subspace of $\mathbb{R}^n$, the *fixed-point subspace* of G

$$Fix(G) = \{x \in \mathbb{R}^n : \ \delta x = x, \ \forall \delta \in G\}.$$

Fixed-point subspaces possess the important property of being invariant by the flow of f, that is, the dynamics of a state in $Fix(G)$ remain in $Fix(G)$.

The *isotropy subgroup* of a point $x \in \mathbb{R}^n$, denoted by Γ_x, corresponds to the elements of Γ that fix x,

$$\Gamma_x = \{\gamma \in \Gamma : \ \gamma x = x\}.$$

The group Γ *acts freely* on a set $S \in \mathbb{R}^n$, if $\Gamma_x = \{Id\}$, for all $x \in S$, with Id the identity element. That is, the only element of Γ that can fix a point $x \in X$, different from the origin, is the identity element.

The Γ-*orbit* of a point $x \in \mathbb{R}^n$ is the set of images of x under the action of the group Γ

$$\Gamma(x) = \{\gamma x : \gamma \in \Gamma\}.$$

An analogous definition applies to the Γ-orbit of any flow invariant set. An important and straightforward consequence is that the elements in the Γ-orbit of an equilibrium of system (1) are also equilibria of the system. More generally, if S is a flow invariant set, then so are the sets γS, $\gamma \in \Gamma$, in its group orbit. Thus, the elements in the Γ-orbit of solution curves of system (1) are conjugated solution curves. Moreover, the elements in the Γ-orbit of a fixed-point subspace are fixed-point subspaces with conjugated dynamics.

Let S be a subset of $\mathbb{R}^n$. The set of all Γ-orbits of S, denoted by S/Γ, is called the *quotient space* or *orbit space*.

If S is a manifold and the group Γ acts freely on S then the orbit space S/Γ will again be a manifold. If S is a smooth finite-dimensional manifold, Γ is compact and Γ acts smoothly on S, then the orbit space S/Γ is a stratified manifold ([6]). For a definition of stratification and orbit-stratum see Definitions 4.10.(10-12) of Chossat and Lauterbach [6].

If S is invariant by the flow of f, then the flow of f restricts to a flow on S/Γ. By the Smooth Lifting Theorem (theorem 0.2) in Schwarz [27], when Γ is a compact Lie group and S is a smooth manifold, for each smooth Γ-equivariant vector field on S, there is a corresponding smooth strata-preserving vector field on S/Γ.

Heteroclinic network Let p_i, $i = 1, \dots, r$ be saddle equilibria for the flow of f. By saddle equilibria we mean that the equilibria p_i have non-trivial stable and unstable manifolds, $W^s(p_i) \neq \{p_i\}$ and $W^u(p_i) \neq \{p_i\}$, $i = 1, \dots, r$.

A *heteroclinic connection* from p_i to p_j, denoted by $[p_i \to p_j]$, is a trajectory in $W^u(p_i) \cap W^s(p_j)$.

There is a *heteroclinic cycle* connecting the saddle equilibria p_i, $i = 1, \dots, r$ if there is a reordering of the equilibria such that there are heteroclinic connections $[p_i \to p_{i+1}]$, for $i = 1, \dots, r-1$, and $[p_r \to p_1]$.

A *heteroclinic network* is defined to be a connected union of heteroclinic cycles. It follows that, given any two equilibria in the network, there is a sequence of connections taking one to the other. We will also refer to the equilibria in the network as *nodes* of the network.

The existence of heteroclinic networks is a common phenomenon in problems where there exist invariant spaces. This can be a consequence of symmetry (see Krupa [21] or Field [9]) or of the formulation of the problem itself, as is the case of games or population dynamics (see Hofbauer [16] or Hofbauer and Sigmund [18]).

Switching Let Σ be a heteroclinic network for the flow of f. We loosely follow the set-up in Aguiar *et al.* [1].

We define a (finite) *path* on the network Σ as a sequence of connections (c_i), $i = 1, \dots, s$ in Σ such that $c_i = [p_i \to p_j]$ and $c_{i+1} = [p_j \to p_k]$, with p_i, p_j and p_k equilibria in Σ. An *infinite path* corresponds to an infinite sequence of connections (c_i), $i \in \mathbb{N}$. Note that, we consider $i \in \mathbb{N}$, and not $i \in \mathbb{Z}$, for an infinite path because our original problem is one of game theory and so there is a beginning of play.

Given a path on the network Σ, we say that there is a trajectory for the flow of f that *follows* that path, if for every neighbourhood V of the sequence of connections in Σ defining that path, there is a trajectory for the flow of f contained in V. That is to say, there is a trajectory for the flow of f as close as required to the sequence of connections in Σ defining the path.

We say there is finite (infinite) *switching* near a network if for every finite (infinite) path on the network there is a trajectory, near the network, for the flow of f that follows that path.

We refer to the type of switching thus described as *chaotic*.

3 A heteroclinic network in the Rock-Scissors-Paper game

We start by recalling the description of the Rock-Scissors-Paper game (see, for instance, Sato *et al.* [26]). Two agents X and Y have the option of playing one of three actions : 'rock' (R), 'scissors' (S) and 'paper' (P). An agent playing R (S, P) beats the other playing S (P, R, respectively).

Let $x_1, x_2, x_3 \geq 0$, with $x_1 + x_2 + x_3 = 1$, denote the probability of agent X playing the action R, S, or P, respectively. Analogously, for y_1, y_2 and y_3 and agent Y.

For each agent, the state space is a two-dimensional simplex, and the collective state space $\Delta = \Delta_X \times \Delta_Y$ is four-dimensional.

The normalized interaction matrices are

$$A = \begin{bmatrix} \frac{2}{3}\epsilon_x & 1-\frac{1}{3}\epsilon_x & -1-\frac{1}{3}\epsilon_x \\ -1-\frac{1}{3}\epsilon_x & \frac{2}{3}\epsilon_x & 1-\frac{1}{3}\epsilon_x \\ 1-\frac{1}{3}\epsilon_x & -1-\frac{1}{3}\epsilon_x & \frac{2}{3}\epsilon_x \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{2}{3}\epsilon_y & 1-\frac{1}{3}\epsilon_y & -1-\frac{1}{3}\epsilon_y \\ -1-\frac{1}{3}\epsilon_y & \frac{2}{3}\epsilon_y & 1-\frac{1}{3}\epsilon_y \\ 1-\frac{1}{3}\epsilon_y & -1-\frac{1}{3}\epsilon_y & \frac{2}{3}\epsilon_y \end{bmatrix},$$

where $\epsilon_x, \epsilon_y \in (-1.0, 1.0)$ are the rewards for ties. Unlike Sato *et al.* [26], we exclude the boundary of the interval. Note that, on the boundary, a tie is either as good as a win or as bad as a defeat.

We consider the case of perfect memory and equal rates of adaptation, and so the dynamics are given by the following equations (these are the replicator equations, extensively used in population dynamics)

$$\begin{aligned} \dot{x}_i &= x_i \left[(Ay)_i - x^T Ay\right] \\ \dot{y}_j &= y_j \left[(Bx)_j - y^T Bx\right] \end{aligned} \tag{2}$$

with $i, j = 1, 2, 3$. Reinforcement learning, for each player, is described by the terms in brackets.

Notice that the coordinate hyperplanes are invariant by the flow of (2).

System (2) is equivariant under the symmetry group Γ of order 3 generated by the action of

$$\sigma(x_1, x_2, x_3, y_1, y_2, y_3) = (x_3, x_1, x_2, y_3, y_1, y_2).$$

The intersection of $Fix(\Gamma) = \{(x, x, x, y, y, y); x, y \in \mathbb{R}\}$ with Δ corresponds to the Nash equilibrium $(x^*, y^*) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which is a saddle with 2-dimensional stable and unstable manifolds. Besides the Nash equilibrium there are nine equilibria, that correspond to the vertices of Δ, given by (x, y) where $x, y \in \{R, S, P\}$ with $R = (1, 0, 0)$, $S = (0, 1, 0)$ and $P = (0, 0, 1)$.

Note that the set of the nine equilibria can be partioned into Γ-orbits as follows

$$\begin{aligned} \Gamma((R, P)) &= \{(R, P), (S, R), (P, S)\} \equiv \xi_0 \\ \Gamma((R, S)) &= \{(R, S), (S, P), (P, R)\} \equiv \xi_1 \\ \Gamma((R, R)) &= \{(R, R), (S, S), (P, P)\} \equiv \xi_2 \end{aligned}$$

We have denoted by ξ_i, $i = 0, 1, 2$, respectively, the Γ-orbit of the equilibria (R, P), (R, S) and (R, R). Along ξ_0, agent Y wins over agent X, whereas along ξ_1 the opposite occurs. Along ξ_2 there is a draw in play.

Proposition 3.1. *There is a heteroclinic network Σ in Δ involving all the equilibria at the vertices of Δ.*

Proof. The existence of a heteroclinic network in the intersection of the invariant hyperplanes with Δ is highly likely. We use the standard technique of Guckenheimer and Holmes [13]. We must confirm that the eigenvalues of the equilibria at the vertices have the correct signs and that there are no equilibria on the one-dimensional edges joining the equilibria at the vertices.

The analysis of the eigenvalues and eigendirections is easier if we work on invariant spheres rather than on simplices. We thus make the coordinate change: $x_i = u_i^2$ and $y_i = v_i^2$, $i = 1, 2, 3$. In the new coordinates the system is given by

$$\begin{aligned} \dot{u}_i &= \frac{1}{2} u_i \left[(Av^2)_i - (u^2)^T Av^2\right] \\ \dot{v}_j &= \frac{1}{2} v_j \left[(Bu^2)_j - (v^2)^T Bu^2\right] \end{aligned} \tag{3}$$

with $i,j=1,2,3$, where $u^2=(u_1^2,u_2^2,u_3^2)^T$ and $v^2=(v_1^2,v_2^2,v_3^2)^T$, with $u_1^2+u_2^2+u_3^2=1$ and $v_1^2+v_2^2+v_3^2=1$ invariant by the flow.

The manifold $\Delta=\Delta_x\times\Delta_y$ becomes the *fundamental domain*

$$D=\{(u_1,u_2,u_3,v_1,v_2,v_3)\in(\mathbb{R}_0^+)^6:\ u_1^2+u_2^2+u_3^2=1, v_1^2+v_2^2+v_3^2=1\},$$

of the manifold given by the direct product of two 2-dimensional spheres.

There are many similarities between the geometry of the flow for the systems (2) and (3)(Krupa [21]). In fact, the coordinate change corresponds to a smooth conjugacy of the flows restricted, respectively, to Δ and D. In particular, trajectories on the edges of Δ joining the equilibria at the vertices of Δ are analogous to trajectories on the edges of D joining the corresponding equilibria. The sign of the eigenvalues of the linearization at the equilibria is preserved even though their magnitude is decreased by half.

Let $e_i, i=1,\dots,6$ denote the vectors of the canonical basis of $\mathbb{R}^6$. Table 1 contains the information about the eigenvalues and eigenvectors for the three Γ-orbits of equilibria, ξ_0, ξ_1 and ξ_2.

Equilibria	e_1	e_2	e_3	e_4	e_5	e_6
ξ_0	$1+\frac{1}{3}\epsilon_x$	$1>0$	$\frac{1+\epsilon_x}{2}>0$	$\frac{-1+\epsilon_y}{2}<0$	$-1<0$	$-1+\frac{1}{3}\epsilon_y$
ξ_1	$-1+\frac{1}{3}\epsilon_x$	$\frac{-1+\epsilon_x}{2}<0$	$-1<0$	$\frac{1+\epsilon_y}{2}>0$	$1+\frac{1}{3}\epsilon_y$	$1>0$
ξ_2	$-\frac{2}{3}\epsilon_x$	$\frac{-1-\epsilon_x}{2}<0$	$\frac{1-\epsilon_x}{2}>0$	$-\frac{2}{3}\epsilon_y$	$\frac{-1-\epsilon_y}{2}<0$	$\frac{1-\epsilon_y}{2}>0$

Table 1: Eigenvalues and eigenvectors for the Γ-orbits of equilibria of system (3). The vectors e_i are those of the canonical basis of $\mathbb{R}^6$ in a system of local coordinates at each point of the group orbit of each ξ_j, $j=0,1,2$.

Tedious, but straightforward, computations show that there are no equilibria on the one-dimensional edges joining the equilibria at the vertices. The signs of the non-radial eigenvalues indicated in table 1 together with the Poincaré-Bendixson Theorem, applied on the two-dimensional invariant spaces, allow us to conclude for the existence of a heteroclinic network Σ involving the equilibria at the vertices. See figure 1, for an image of the connections in the network. □

The network Σ is the heteroclinic network numerically observed by Sato *et al.* [26, section 4.3.2]. Numerical simulations in [26] reveal interesting chaotic dynamics in the neighbourhood of the network Σ, namely the existence of chaotic switching.

As we mentioned in the proof of proposition 3.1, the dynamics of the flow of system (2) defined on Δ are conjugated to the dynamics of the flow of system (3) defined on D. Since the manifold D is smooth, it makes sense to use D in order to look for a suitable quotient space in which the flow is differentiable, so that we can study the local dynamics near the heteroclinic network Σ for the flow of system (3).

In order to provide an analytical proof of the complex behaviour observed in [26], we start by noting that the network Σ corresponds to the union of the following three heteroclinic

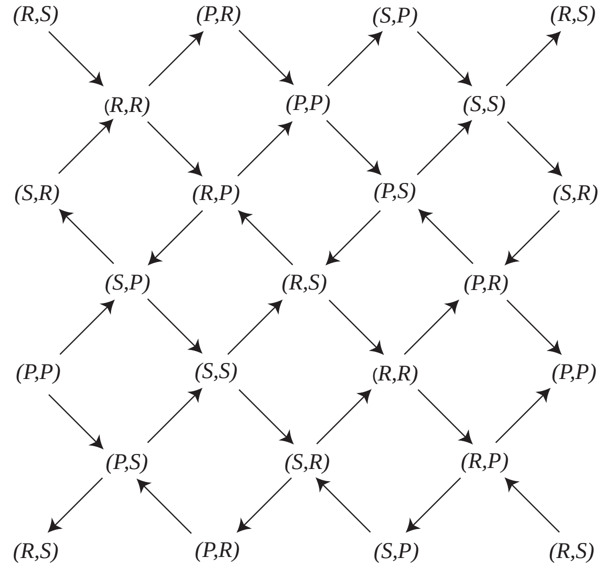

Figure 1: Connections in the network Σ.

cycles:

$$\begin{array}{ll} C_0: & (R,P) \to (S,P) \to (S,R) \to (P,R) \to (P,S) \to (R,S) \to (R,P) \\ C_1: & (R,S) \to (R,R) \to (P,R) \to (P,P) \to (S,P) \to (S,S) \to (R,S) \\ C_2: & (S,R) \to (R,R) \to (R,P) \to (P,P) \to (P,S) \to (S,S) \to (S,R). \end{array}$$

An equivalent description, which we shall not use, may be obtained from the union of the following two heteroclinic cycles:

$$\begin{array}{ll} C_3: & (R,S) \to (R,R) \to (R,P) \to (S,P) \to (S,S) \to (S,R) \to (P,R) \to (P,P) \to (P,S) \to (R,S) \\ C_4: & (S,R) \to (R,R) \to (P,R) \to (P,S) \to (S,S) \to (R,S) \to (R,P) \to (P,P) \to (S,P) \to (S,R). \end{array}$$

The heteroclinic cycles are invariant by the action of Γ, that is, $\Gamma(C_i) = C_i$, $i = 0, 1, 2$. Furthermore,

$$\begin{array}{l} \Gamma\left((R,P) \to (S,P) \to (S,R)\right) = C_0 \\ \Gamma\left((R,S) \to (R,R) \to (P,R)\right) = C_1 \\ \Gamma\left((S,R) \to (R,R) \to (R,P)\right) = C_2. \end{array}$$

3.1 Quotient heteroclinic network

The group Γ fixes the Nash equilibrium $(u^*, v^*) = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ in D and acts freely on $D - \{(u^*, v^*)\}$. Thus, the orbit space D/Γ is a stratified manifold with the two regular strata $(D - \{(u^*, v^*)\})/\Gamma$ and $\{(u^*, v^*)\}$. Since the flow of system (3) is Γ-equivariant, it respects the stratification. We work then on the orbit-stratum $(D - \{(u^*, v^*)\})/\Gamma$ and consider the restriction of the flow of system (3) to this manifold. By the Smooth Lifting Theorem (theorem 0.2) in Schwarz [27], we get a smooth flow in $(D - \{(u^*, v^*)\})/\Gamma$. The heteroclinic network Σ in D drops down to a heteroclinic network Σ_Γ in $(D - \{(u^*, v^*)\})/\Gamma$ which is the union of the following three heteroclinic cycles, as illustrated in figure 2,

$$\begin{array}{ll} C_0: & \xi_0 \to \xi_1 \to \xi_0 \\ C_1: & \xi_1 \to \xi_2 \to \xi_1 \\ C_2: & \xi_0 \to \xi_2 \to \xi_0. \end{array}$$

In what follows, we prove switching near the quotient network Σ_Γ, which is considerably simpler than Σ. By going from the quotient to the original space, we obtain the existence of

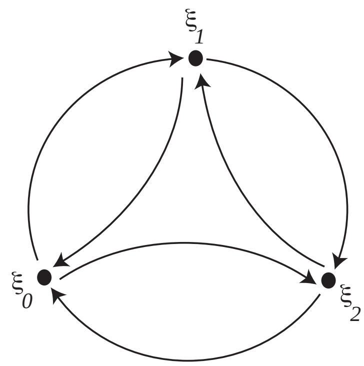

Figure 2: The heteroclinic network Σ_Γ.

switching near the heteroclinic network Σ. We also show that only the cycle C_0 is relatively asymptotically stable for $\epsilon_x + \epsilon_y < 0$, as was numerically observed by Sato *et al.* [26]. Additionally, we show that cycles C_1 and C_2 are also relatively asymptotically stable for parameter values satisfying $\epsilon_x + \epsilon_y > 0$. The region of stability is however much smaller in this case, which explains why the stability of these cycles was not observed numerically. Therefore the network is relatively asymptotically stable.

4 Dynamics near the quotient heteroclinic network

Consider the restriction of the system of differential equations (3) to the 4-dimensional manifold $(D - \{(u^*, v^*)\})/\Gamma$ in $\mathbb{R}^6$. In the restricted flow, consider the quotient heteroclinic network Σ_Γ consisting of the three (hyperbolic) saddle equilibria $\xi_k,\ k = 0, 1, 2$ and the three heteroclinic cycles $C_i,\ i = 0, 1, 2$.

Each saddle has 2-dimensional stable and unstable manifolds. We denote by e_{ij} the positive eigenvalues in the unstable direction, connecting equilibrium ξ_i to equilibrium ξ_j, and by $-c_{ij}$ the negative eigenvalues in the stable direction, connecting equilibrium ξ_j to equilibrium ξ_i.

In the next result, we provide sufficient conditions for the flow to be C^1 linearizable around each equilibrium $\xi_k,\ k = 0, 1, 2$. These are conditions on $\epsilon_x, \epsilon_y \in (-1, 1)$ obtained from the eigenvalues of the linear part of the flow in table 1.

Proposition 4.1. *The flow is C^1 linearizable around each equilibrium $\xi_k,\ k = 0, 1, 2$ provided all of the following inequalities hold*
(i) $\epsilon_x \neq \epsilon_y$;
(ii) $\epsilon_y \neq 2\epsilon_x + 1$;
(iii) $\epsilon_y \neq 2\epsilon_x - 1$;
(iv) $\epsilon_x \neq 2\epsilon_y - 1$;
(v) $\epsilon_x \neq 2\epsilon_y + 1$.

Proof. We use the C^1 extension by Ruelle [24] of Hartman's results [14]. Ruelle's sufficient condition for C^1 linearization is that

$$\mathrm{Re}(\lambda_i) \neq \mathrm{Re}(\lambda_j) + \mathrm{Re}(\lambda_k),\ \text{when } \mathrm{Re}(\lambda_j) < 0 < \mathrm{Re}(\lambda_k), \tag{4}$$

where Re denotes the real part of a number and λ_i is an eigenvalue of the linear part of the flow.

At ξ_0, the eigenvalues of the linear part of the flow are those in the first row of table 2.

	u_1	u_2	u_3	u_4
ξ_0	$e_{01} = 1$	$e_{02} = \frac{1+\epsilon_x}{2}$	$-c_{01} = -1$	$-c_{02} = -\frac{1-\epsilon_y}{2}$
ξ_1	$e_{12} = \frac{1+\epsilon_y}{2}$	$e_{10} = 1$	$-c_{12} = -\frac{1-\epsilon_x}{2}$	$-c_{10} = -1$
ξ_2	$e_{20} = \frac{1-\epsilon_y}{2}$	$e_{21} = \frac{1-\epsilon_x}{2}$	$-c_{20} = -\frac{1+\epsilon_x}{2}$	$-c_{21} = -\frac{1+\epsilon_y}{2}$

Table 2: For each saddle ξ_k ($k = 0, 1, 2$), the table provides the eigenvalues in the directions of the vectors in the first line. Note that these directions are only defined locally around each saddle in the fundamental domain D.

We have the following possibilities to verify (4)

$$\begin{array}{lcl}
-1 < 0 < 1 & : & 0 \neq \frac{1+\epsilon_x}{2};\ \ 0 \neq \frac{-1+\epsilon_y}{2} \\
-1 < 0 < \frac{1+\epsilon_x}{2} & : & \frac{-1+\epsilon_x}{2} \neq 1;\ \ \frac{-1+\epsilon_x}{2} \neq \frac{-1+\epsilon_y}{2} \Leftrightarrow \epsilon_x \neq \epsilon_y \\
\frac{-1+\epsilon_y}{2} < 0 < \frac{1+\epsilon_x}{2} & : & \epsilon_x + \epsilon_y \neq -2;\ \ \epsilon_x + \epsilon_y \neq 2 \\
\frac{-1+\epsilon_y}{2} < 0 < 1 & : & \epsilon_x \neq \epsilon_y;\ \ \frac{1+\epsilon_y}{2} \neq -1.
\end{array}$$

Given the restriction that $\epsilon_x, \epsilon_y \in (-1, 1)$, the only binding condition is $\epsilon_x \neq \epsilon_y$.

At ξ_1, we obtain the same restriction.

At ξ_2, we obtain the remaining four restrictions. □

Remark: The necessary and sufficient conditions for C^1 linearization of Hartman's [14] (Theorem 12.1 applied to differential equations) show that linearization is not possible for subsets of points on the lines described by the restrictions above. These restrictions are a set of measure zero in parameter space, and place no serious constraint on the analysis that follows.

From now on assume that ϵ_x and ϵ_y are such that the flow is C^1 linearizable. In a neighbourhood of each equilibrium ξ_k ($k = 0, 1, 2$), we choose coordinates for which the flow is linear, with the equilibrium at the origin and such that the local stable and unstable manifolds are coordinate planes. In the system of local coordinates at each ξ_k, the flow is induced by a differential equation of the form below. The directions defined by the first two coordinates are expanding and the remaining two contracting,

$$\begin{aligned}
\dot{x}_1 &= e_{k,k+1} x_1 \\
\dot{x}_2 &= e_{k,k+2} x_2 \\
\dot{x}_3 &= -c_{k,k+1} x_3 \\
\dot{x}_4 &= -c_{k,k+2} x_4 \pmod 3,
\end{aligned}$$

where e_{kj} and $-c_{kj}$ are the eigenvalues as above. The flow near ξ_k is then given by

$$\mathcal{F}_t(u_1, u_2, u_3, u_4) = (u_1 e^{e_{k,k+1} t}, u_2 e^{e_{k,k+2} t}, u_3 e^{-c_{k,k+1} t}, u_4 e^{-c_{k,k+2} t}) \pmod 3, \tag{5}$$

where, as stated above, we have chosen the u_1 and u_2 axes to correspond to the expanding directions and the u_3 and u_4 axes to correspond to the contracting directions.

In table 2, we present the eigenvalues in each local coordinate system. These are obtained from the original system of differential equations, see table 1.

4.1 Set-up for the dynamics

The dynamics both near each cycle and near the network may be described using the maps we characterize in this section. This is done following Brannath [3].

Consider a neighbourhood of each saddle where the flow can be linearized and define a cross-section for each of the four connections in this neighbourhood. The cross-sections are chosen to be transversal to the connection and the flow. Rescaling the coordinates near each saddle ξ_k $(k = 0, 1, 2)$, the cross-sections may be given by

$$\begin{aligned}
\Sigma^{\text{out}}_{k,k+1} &= \{(1, u_2, u_3, u_4) :\ 0 < u_2, u_3, u_4 < \alpha_k\} \\
\Sigma^{\text{out}}_{k,k+2} &= \{(u_1, 1, u_3, u_4) :\ 0 < u_1, u_3, u_4 < \alpha_k\} \\
\Sigma^{\text{in}}_{k,k+1} &= \{(u_1, u_2, 1, u_4) :\ 0 < u_1, u_2, u_4 < \alpha_k\} \\
\Sigma^{\text{in}}_{k,k+2} &= \{(u_1, u_2, u_3, 1) :\ 0 < u_1, u_2, u_3 < \alpha_k\} \pmod 3,
\end{aligned}$$

where $0 < \alpha_k < 1$ is a positive number, small enough to guarantee transversality of the flow near each saddle. The points in $\Sigma^{\text{out}}_{k,j}$ follow the connection from saddle ξ_k to saddle ξ_j. Analogously, the points in $\Sigma^{\text{in}}_{k,j}$ come from a neighbourhood of saddle ξ_j and are taken close to saddle ξ_k. A 2-dimensional representation of these sections is given in figure 3.

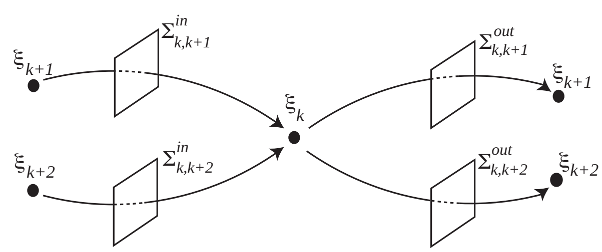

Figure 3: Two-dimensional representation of the cross-sections.

We define the maps $\Psi_{i,k,j}$ from a subset of points in $\Sigma^{\text{in}}_{k,i}$ to $\Sigma^{\text{out}}_{k,j}$ by the following rules, using the linearized flow:

$$\Psi_{k+1,k,k+1}(u_1, u_2, u_4) = \left(u_2 u_1^{-\frac{e_{k,k+2}}{e_{k,k+1}}}, u_1^{\frac{c_{k,k+1}}{e_{k,k+1}}}, u_4 u_1^{\frac{c_{k,k+2}}{e_{k,k+1}}} \right),$$

$$\Psi_{k+2,k,k+1}(u_1, u_2, u_3) = \left(u_2 u_1^{-\frac{e_{k,k+2}}{e_{k,k+1}}}, u_3 u_1^{\frac{c_{k,k+1}}{e_{k,k+1}}}, u_1^{\frac{c_{k,k+2}}{e_{k,k+1}}} \right),$$

$$\Psi_{k+1,k,k+2}(u_1, u_2, u_4) = \left(u_1 u_2^{-\frac{e_{k,k+1}}{e_{k,k+2}}}, u_2^{\frac{c_{k,k+1}}{e_{k,k+2}}}, u_4 u_2^{\frac{c_{k,k+2}}{e_{k,k+2}}} \right),$$

and

$$\Psi_{k+2,k,k+2}(u_1, u_2, u_3) = \left(u_1 u_2^{-\frac{e_{k,k+1}}{e_{k,k+2}}}, u_3 u_2^{\frac{c_{k,k+1}}{e_{k,k+2}}}, u_2^{\frac{c_{k,k+2}}{e_{k,k+2}}} \right).$$

From points in a cross-section $\Sigma^{\text{out}}_{k,i}$ to $\Sigma^{\text{in}}_{i,k}$, we define the maps $\Phi_{k,i}$ taking points along the connection from saddle ξ_k to saddle ξ_i in a flow-box fashion as follows

$$\Phi_{k,k+1}(u_2, u_3, u_4) = \left(u_2 G^1_{k,k+1}(u_2, u_3, u_4), u_3 G^2_{k,k+1}(u_2, u_3, u_4), u_4 G^3_{k,k+1}(u_2, u_3, u_4)\right)$$

and

$$\Phi_{k,k+2}(u_1, u_3, u_4) = \left(u_1 G^1_{k,k+2}(u_2, u_3, u_4), u_3 G^2_{k,k+2}(u_2, u_3, u_4), u_4 G^3_{k,k+2}(u_2, u_3, u_4)\right)$$

where $G^j_{k,k+i}$, $j = 1, 2, 3$, $i = 1, 2$, are continuous functions satisfying $c < G^j_{k,k+i} < C$ for some constants $c, C > 0$.

We define the maps $\Omega_{i,k,j,l}$, from $\Sigma^{\text{in}}_{k,i}$ to $\Sigma^{\text{in}}_{l,j}$, through the neighbourhoods of saddles ξ_k and ξ_j, as follows (see figure 4)

$$\Omega_{i,k,j,l} = \Phi_{j,l} \circ \Psi_{k,j,l} \circ \Phi_{k,j} \circ \Psi_{i,k,j}$$

taking points through the following sequence of cross-sections

$$\Sigma^{\text{in}}_{k,i} \longrightarrow \Sigma^{\text{out}}_{k,j} \longrightarrow \Sigma^{\text{in}}_{j,k} \longrightarrow \Sigma^{\text{out}}_{j,l} \longrightarrow \Sigma^{\text{in}}_{l,j}.$$

Notice that the maps $\Omega_{i,k,i,k}$ are first-return maps from $\Sigma^{\text{in}}_{k,i}$ to itself.

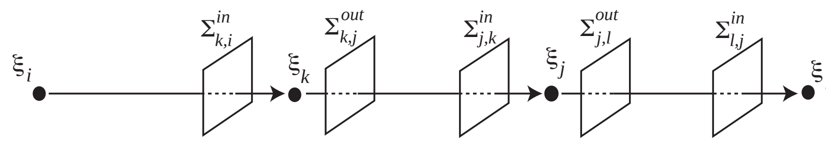

Figure 4: Representation of the cross-sections in the definition of $\Omega_{i,k,j,l}$.

By the Implicit Function Theorem, each map $\Phi_{k,k+i}$ is a diffeomorphism between neighbourhoods of $(0, 0, 0)$. We may then approximate the maps Φ by the identity, simplifying further calculations. From now on, we consider $\Omega_{i,k,j,l} = \Psi_{k,j,l} \circ \Psi_{i,k,j}$.

Next, we characterize the set of points that are taken from $\Sigma^{\text{in}}_{k,i}$ to $\Sigma^{\text{out}}_{k,j}$ for each $i, j, k = \{0, 1, 2\}$ and $k \neq i, j$, that is, points for which $\Psi_{i,k,j}$ is well-defined.

So as not to make notation too cumbersome, we provide detail for $\Psi_{k+1,k,k+1}$, all other cases being similar. Consider the unit cube containing the cross-section $\Sigma^{\text{in}}_{k,k+1}$

$$\mathcal{Q} = \{(u_1, u_2, 1, u_4) : \ 0 < u_1, u_2, u_4 < 1\} \equiv \{(u_1, u_2, u_4) : \ 0 < u_1, u_2, u_4 < 1\}.$$

Similarly, there is a unit cube, $\mathcal{Q}'$ containing the cross-section $\Sigma^{\text{out}}_{k,k+1}$. Analogously to Brannath [3], we may view $\Psi_{i,k,j}$ as a map from $\mathcal{Q}$ to $\mathcal{Q}'$. Denote by $C^{k+1,k,k+1}$ the set of points in $\mathcal{Q}$ that are taken by $\Psi_{k+1,k,k+1}$ into $\mathcal{Q}'$. Since this assumes that there is a neighbourhood of $(0, 0, 1, 0)$, containing $\mathcal{Q}$, where the flow is linear and transverse to $\mathcal{Q}$, which may not be the case, the domain of definition of $\Psi_{k+1,k,k+1}$ is obtained by intersecting $C^{k+1,k,k+1}$ with an open neighbourhood of the origin. We therefore focus on the study of the sets $C^{i,k,j}$ describing the domain of definition of $\Psi_{i,k,j}$. We have

$$\Psi_{k+1,k,k+1}(u_1, u_2, u_4) = (u_2 u_1^{-\frac{e_{k,k+2}}{e_{k,k+1}}}, u_1^{\frac{c_{k,k+1}}{e_{k,k+1}}}, u_4 u_1^{\frac{c_{k,k+2}}{e_{k,k+1}}}) = (\bar{u}_2, \bar{u}_3, \bar{u}_4),$$

with $-\frac{e_{k,k+2}}{e_{k,k+1}} < 0$, $\frac{c_{k,k+1}}{e_{k,k+1}} > 0$ and $\frac{c_{k,k+2}}{e_{k,k+1}} > 0$. So, $\Psi_{k+1,k,k+1}(\mathcal{Q}) \subset \mathcal{Q}'$ if and only if $u_2 < u_1^{\frac{e_{k,k+2}}{e_{k,k+1}}}$. We obtain

$$C^{k+1,k,k+1} = \{(u_1, u_2, 1, u_4) \in \mathcal{Q} : u_2 < u_1^{\frac{e_{k,k+2}}{e_{k,k+1}}}\}.$$

Geometric representations of these points can be found in figure 5 for the two cases $\frac{e_{k,k+2}}{e_{k,k+1}}$ less or greater than one. We note that the complement of $C^{i,k,j}$ in $\mathcal{Q}$ is $C^{i,k,l}$ with $l \neq j$. This is consistent with the fact that the dynamics are conservative in the original set-up in Sato *et al.* [26].

Below, we describe the sets $C^{i,k,j}$ for all maps Ψ, obtained using the eigenvalues in table 2:

$$C^{i,0,1} = \{(u_1, u_2, v_i) \in \mathcal{Q} : \ u_2 < u_1^{\frac{1+\epsilon_x}{2}}\}$$

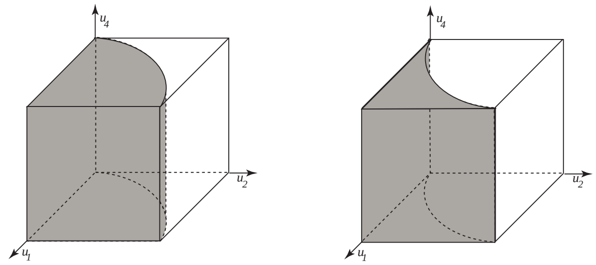

Figure 5: The left-hand side picture refers to the case $\frac{e_{k,k+2}}{e_{k,k+1}} < 1$ and the right-hand side picture to $\frac{e_{k,k+2}}{e_{k,k+1}} > 1$. In each case, the set $C^{k+1,k,k+1}$ consists of the points in the shaded region. The remaining points are those in $C^{k+1,k,k+2}$.

where $i = 1, 2$, $v_1 = u_4$, $v_2 = u_3$ and $C^{i,0,2} = \mathcal{Q} \backslash C^{i,0,1}$;

$$C^{i,1,0} = \{(u_1, u_2, v_i) \in \mathcal{Q} : \; u_2 > u_1^{\frac{2}{1+\epsilon_y}}\}$$

where $i = 0, 2$, $v_0 = u_3$, $v_2 = u_4$ and $C^{i,1,2} = \mathcal{Q} \backslash C^{i,1,0}$;

$$C^{i,2,0} = \{(u_1, u_2, v_i) \in \mathcal{Q} : \; u_2 < u_1^{\frac{1-\epsilon_x}{1-\epsilon_y}}\}$$

where $i = 0, 1$, $v_0 = u_4$, $v_1 = u_3$ and $C^{i,2,1} = \mathcal{Q} \backslash C^{i,2,0}$.

The domain of definition for the maps $\Omega_{i,k,j,l} = \Psi_{k,j,l} \circ \Psi_{i,k,j}$ is obtained from the sets above and is described by

$$\{(u_1, u_2, v) \in C^{i,k,j} : \; \Psi_{i,k,j}(u_1, u_2, v) \subset C^{k,j,l}\},$$

where $v = u_3$ or $v = u_4$, depending on i, k and j.

4.2 Switching at the nodes

As in Aguiar *et al.* [1], we say there is *switching at a node* ξ if, for any neighbourhood of a point in a connection leading to node ξ of a network, there exist trajectories starting in that neighbourhood that follow along all the possible connections forward from ξ.

Theorem 4.2. *There is switching at every node of the network* Σ_Γ.

Proof. We prove switching at a generic node ξ_k. Consider a connection $[\xi_i \to \xi_k]$. Let $p = (0, 0, 0)$ in $\Sigma^{\text{in}}_{k,i}$ be the point corresponding to the intersection of the connection $[\xi_i \to \xi_k]$ with $\Sigma^{\text{in}}_{k,i}$. Let U_p be a neighbourhood of p and set $V = U_p \cap \Sigma^{\text{in}}_{k,i}$. For any neighbourhood U_p the set V contains points in $C^{i,k,j}$ and points in the complement of $C^{i,k,j}$ in $\Sigma^{\text{in}}_{k,i}$. Points in $C^{i,k,j}$ follow the connection $[\xi_k \to \xi_j]$ and points in the complement follow the connection $[\xi_k \to \xi_i]$ from ξ_k thus proving switching at node ξ_k. $\square$

4.3 Switching along the connections

We say there is switching along a connection $[\xi_k \to \xi_j]$ if, for any neighbourhood of a point in a connection leading to node ξ_k, there exist trajectories starting in that neighbourhood that follow along the connection $[\xi_k \to \xi_j]$ and then along all the possible connections forward from ξ_j.

Note that switching at the nodes of the network does not guarantee switching along the connections.

This subsection establishes switching along every connection of the quotient heteroclinic network Σ_Γ. We present a detailed proof for the case of the connection $[\xi_1 \to \xi_0]$. The proof for the remaining connections is analogous.

We shall abuse notation, so as not to make it cumbersome, and refer to $C^{i,k,j}$ when what we mean is its intersection with the appropriate cross-section. Also, we shall use the cross-sections $\Sigma^{\text{out}}_{i,j}$ and $\Sigma^{\text{in}}_{i,j}$ when we are, in fact, calculating in the corresponding cubes $\mathcal{Q}'$ and $\mathcal{Q}$ (as in the case in the definition of $F^{i,k,j}$ below).

Consider the connection $[\xi_1 \to \xi_0]$. Points in $\Sigma^{\text{in}}_{0,1}$ are going to be sent to both $\Sigma^{\text{out}}_{0,1}$ and $\Sigma^{\text{out}}_{0,2}$, as we saw in theorem 4.2. We show in theorem 4.4 that the set, $C^{1,0,1}$, of points going into $\Sigma^{\text{out}}_{0,1}$ and the set, $C^{1,0,2}$, of points going into $\Sigma^{\text{out}}_{0,2}$ include points that come from both $\Sigma^{\text{in}}_{1,0}$ and $\Sigma^{\text{in}}_{1,2}$, thus establishing switching along the connection $[\xi_1 \to \xi_0]$. See figure 6.

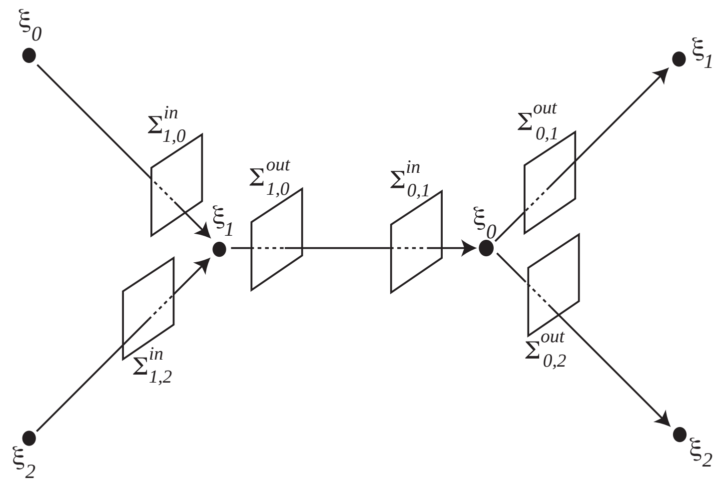

Figure 6: Behaviour along, before and after the connection $[\xi_1 \to \xi_0]$.

Define

$$F^{i,k,j} = \{X \in \Sigma^{\text{out}}_{k,j} : \ \exists \bar{X} \in \Sigma^{\text{in}}_{k,i} : \quad X = \Psi_{i,k,j}(\bar{X})\} = \Psi_{i,k,j}\left(C^{i,k,j}\right),$$

the set of points in $\Sigma^{\text{out}}_{k,j}$ whose trajectory comes from $\Sigma^{\text{in}}_{k,i}$.

Switching along a connection $[\xi_k \to \xi_j]$ requires that $F^{i,k,j} \cap C^{k,j,l} \neq \emptyset$ for $i \neq j$ and $k \neq l$. In the next proposition we provide a description of the sets $F^{i,k,j}$.

Proposition 4.3. *The sets $F^{i,k,j}$ for the network Σ_Γ are as follows*

$$F^{0,1,j} = \{(v_j, u_3, u_4) \in \Sigma^{out}_{1,0} : \ u_3 < u_4^{\frac{1-\epsilon_x}{2}}\}$$

where $j = 0, 2$, $v_0 = u_1$, $v_2 = u_2$ and $F^{2,1,j} = \Sigma^{out}_{1,j} \backslash F^{0,1,j}$;

$$F^{1,0,j} = \{(v_j, u_3, u_4) \in \Sigma^{out}_{0,2} : \ u_4 < u_3^{\frac{1-\epsilon_y}{2}}\}$$

where $j = 1, 2$, $v_1 = u_2$, $v_2 = u_1$ and $F^{2,0,j} = \Sigma^{out}_{0,j} \backslash F^{1,0,j}$;

$$F^{0,2,j} = \{(v_j, u_3, u_4) \in \Sigma^{out}_{2,1} : \ u_4 < u_3^{\frac{1+\epsilon_y}{1+\epsilon_x}}\}$$

where $j = 0, 1$, $v_0 = u_2$, $v_1 = u_1$ and $F^{1,2,j} = \Sigma^{out}_{2,j} \backslash F^{0,2,j}$.

Proof. We provide a detailed proof for $F^{0,1,0}$. The other sets are obtained in an analogous way.

By definition, $F^{0,1,0}$ is the image of $C^{0,1,0}$ by $\Psi_{0,1,0}$. From section 4.1, we know that

$$C^{0,1,0} = \{(u_1, u_2, u_3) \in \mathcal{Q} : \ u_2 > u_1^{\frac{2}{1+\epsilon_y}}\}$$

and, using table 2, we have

$$\begin{aligned}\Psi_{0,1,0}(u_1,u_2,u_3) &= (u_1 u_2^{-\frac{e_{1,2}}{e_{1,0}}}, u_3 u_2^{\frac{c_{1,2}}{e_{1,0}}}, u_2^{\frac{c_{1,0}}{e_{1,0}}}) = \\ &= (u_1 u_2^{-\frac{1+\epsilon_y}{2}}, u_3 u_2^{\frac{1-\epsilon_x}{2}}, u_2) = (\bar{u}_1,\bar{u}_3,\bar{u}_4) \in \Sigma^{\text{out}}_{1,0}.\end{aligned}$$

In order to provide conditions for $(\bar{u}_1,\bar{u}_3,\bar{u}_4)$ to be in $F^{0,1,0}$, we calculate the image of the boundary of $C^{0,1,0}$ as follows

- when $u_2 = u_1^{\frac{2}{1+\epsilon_y}}$, we have

$$\Psi_{0,1,0}(u_1,u_2,u_3) = (1, u_3 u_1^{\frac{1-\epsilon_x}{2}\frac{2}{1+\epsilon_y}}, u_2) = (\bar{u}_1,\bar{u}_3,\bar{u}_4).$$

  This is satisfied if and only if $\bar{u}_1 = 1$, $\bar{u}_4 = u_1^{\frac{2}{1+\epsilon_y}} = u_2$ and $\bar{u}_3 = u_3 \bar{u}_4^{\frac{1-\epsilon_x}{2}}$, with $u_3 \in [0,1]$. Hence, $\bar{u}_3 \leq \bar{u}_4^{\frac{1-\epsilon_x}{2}}$.

- when $u_2 = 1$, we have

$$\Psi_{0,1,0}(u_1,u_2,u_3) = (u_1,u_3,1) = (\bar{u}_1,\bar{u}_3,\bar{u}_4),$$

  which occurs when $\bar{u}_4 = 1$.

- when $u_1 = 0$, we obtain

$$\Psi_{0,1,0}(u_1,u_2,u_3) = (0, u_3 u_2^{\frac{1-\epsilon_x}{2}}, u_2) = (\bar{u}_1,\bar{u}_3,\bar{u}_4).$$

  This is satisfied for $\bar{u}_1 = 0$, $\bar{u}_3 = u_3 \bar{u}_4^{\frac{1-\epsilon_x}{2}}$, with $u_3 \in [0,1]$, and $\bar{u}_4 \in [0,1]$.

Therefore, $F^{0,1,0} = \{(u_1,u_3,u_4) \in \Sigma^{\text{out}}_{1,0} :\ u_3 < u_4^{\frac{1-\epsilon_x}{2}}\}$. ☐

Remark: We point out that proposition 4.3 does not require any assumption on $\Phi_{k,j}$.

Theorem 4.4. *There is switching along every connection of the network Σ_Γ.*

Proof. The proof consists in showing that $F^{i,k,j}$ intersects $C^{k,j,l}$.

Similarly to what was done in the proof of proposition 4.3, we prove the result for $F^{0,1,0}$ and the sets $C^{1,0,1}$ and $C^{1,0,2}$. The remaining cases are analogous.

Since $\Phi_{1,0}$ is the identity, we can describe $F^{0,1,0}$ in $\Sigma^{\text{in}}_{0,1}$ by changing coordinates from (u_1,u_3,u_4) in $\Sigma^{\text{out}}_{1,0}$ to (u_1,u_2,u_4) in $\Sigma^{\text{in}}_{0,1}$. We then have

$$F^{0,1,0} = \{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} :\ u_2 < u_4^{\frac{1-\epsilon_x}{2}}\}.$$

From section 4.1, we know that

$$C^{1,0,1} = \{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} :\ u_2 < u_1^{\frac{1+\epsilon_x}{2}}\}$$

and

$$C^{1,0,2} = \{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} :\ u_2 > u_1^{\frac{1+\epsilon_x}{2}}\}.$$

The intersections $F^{0,1,0} \cap C^{1,0,1}$ and $F^{0,1,0} \cap C^{1,0,2}$ and their complements are pictured in figure 7. These are

$$\begin{aligned}F^{0,1,0} \cap C^{1,0,1} &= \{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} : u_2 < \min\{u_1^{\frac{1+\epsilon_x}{2}}, u_4^{\frac{1-\epsilon_x}{2}}\}\}, \\ F^{0,1,0} \cap C^{1,0,2} &= \{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} : u_1^{\frac{1+\epsilon_x}{2}} < u_2 < u_4^{\frac{1-\epsilon_x}{2}}\}.\end{aligned}$$

Since, for $F^{0,1,0} \cap C^{1,0,2}$ to make sense, we are implicitly assuming $u_1^{\frac{1+\epsilon_x}{2}} < u_4^{\frac{1-\epsilon_x}{2}}$, $F^{0,1,0} \cap C^{1,0,1}$ then becomes $\{(u_1,u_2,u_4) \in \Sigma^{\text{in}}_{0,1} : u_2 < u_1^{\frac{1+\epsilon_x}{2}}\}$. ☐

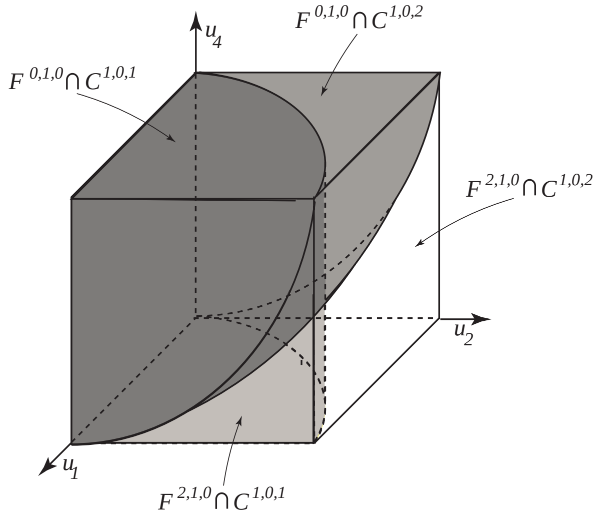

Figure 7: Intersections of $F^{i,1,0}$ with $C^{1,0,l}$, $i = 0, 2$ and $l = 1, 2$, inside $\Sigma^{\text{in}}_{0,1}$.

4.4 Switching near the network

If theorem 4.4 can be iterated a finite number of times then it induces finite switching near the network Σ_Γ. For that to happen, the F-sets in theorem 4.4 are not allowed to, for instance, be contained in just one C-set after a finite number of iterates, but they have to intersect all of them. If we can prove that the process may be continued forever then we get infinite switching.

Theorem 4.5. *There is infinite switching near the network Σ_Γ.*

Proof. The computations in the proof of theorem 4.4 show that the intersection of $F^{i,k,j}$ with $C^{k,j,l}$ is the intersection of open neighbourhoods of the origin inside $\Sigma^{\text{out}}_{k,j} \simeq \Sigma^{\text{in}}_{j,k}$.

In fact, the exponents $\frac{c_{k,k+i}}{e_{k,k+j}}$, with $i, j \in \{1, 2\}$ in the definition (see subsection 4.1) of the maps $\Psi_{k+i,k,k+j}$, $k \in \{0, 1, 2\}$, are positive, and so the second and third coordinates of the image tend to zero when approaching the origin. The exponents $-\frac{e_{k,k+2}}{e_{k,k+1}}$ and $-\frac{e_{k,k+1}}{e_{k,k+2}}$ are negative but, taking into account the domain of definition of the maps (the sets $C^{i,k,j}$), it is easy to verify that the first coordinate of the image also tends to zero when approaching the origin. This guarantees that we can iterate theorem 4.4 an infinite number of times.

Moreover, the boundaries of $F^{i,k,j}$ and $C^{k,j,l}$ intersect transversally inside $\Sigma^{\text{in}}_{j,k}$. Since the sets $C^{i,k,j}$ are described by conditions involving only the first two coordinates u_1 and u_2, we will analize the behaviour in these two coordinates. From now on, we will then be considering implicitly that we are working on planes $u_3 = k$ or $u_4 = k$, for constant k.

As we will see, the maps $\Psi_{k+i,k,k+j}$, $k \in \{0, 1, 2\}$ are expanding in the first coordinate. We show that neighbourhoods of the origin in $C^{i,k,j}$ are sent to horizontal strips through the whole of the $\Sigma^{\text{out}}_{k,j}$ that accumulate on the horizontal axis. This proves that the F-sets intersect all of the C-sets.

If we parametrize $C^{1,0,1}$ by

$$(u_1, u_2, u_4) \quad \text{with} \quad 0 \leq u_2 \leq u_1^{\frac{1+\epsilon_x}{2}},$$

then each vertical segment $u_1 = \overline{u_1}$ of length $\overline{u_1}^{\frac{1+\epsilon_x}{2}}$ is transformed into the horizontal segment $u_2 = \overline{u_1}$ with length 1. So, small vertical segments near the origin are streched and transformed into transverse segments.

Thus, the image of $C^{1,0,1}$ is the whole of $\Sigma^{\text{out}}_{0,1}$.

All the segments $u_1 = \alpha$ intersect transversally the curves in the parametrizations of the sets $C^{0,1,0}$ and $C^{0,1,2}$ in $\Sigma^{\text{in}}_{1,0}$.

If we parametrize $C^{1,0,2}$ by

$$(u_1, u_2, u_4) \text{ with } 0 \leq u_1 \leq u_2^{\frac{2}{1+\epsilon_x}},$$

then each horizontal segment $u_2 = \overline{u_2}$ of length $\overline{u_2}^{\frac{2}{1+\epsilon_x}}$ is transformed into the horizontal segment $u_2 = \overline{u_2}^{\frac{2}{1+\epsilon_x}}$ with length 1. So, small horizontal segments near the origin are stretched and transformed into parallel segments.

Thus, the image of $C^{1,0,2}$ is the whole of $\Sigma^{\text{out}}_{0,2}$.

All these segments intersect transversally the curves in the parametrizations of the sets $C^{0,2,0}$ and $C^{0,2,1}$ in $\Sigma^{\text{in}}_{2,0}$.

We get analogous results for the remaining sets and maps.

$\square$

Note that the switching we have just established using the Poincaré maps translates trivially into switching in the flow. Using the symmetry, we obtain switching near the heteroclinic network Σ of the dynamics of the Rock-Scissors-Paper game (see section 5 below).

4.5 Stability of the cycles and of the network

Analysing the conditions that define the sets $C^{i,k,j}$ we conclude that the region of points, in both $\Sigma^{\text{in}}_{0,1}$ and $\Sigma^{\text{in}}_{0,2}$, whose trajectory follows the connection $[\xi_0 \to \xi_1]$ is significantly bigger than that of points whose trajectory follows the connection $[\xi_0 \to \xi_2]$. Analogously, for $\Sigma^{\text{in}}_{1,0}$ and $\Sigma^{\text{in}}_{1,2}$ and the connections $[\xi_1 \to \xi_0]$ and $[\xi_1 \to \xi_2]$, respectively. This suggests that, together with the existence of infinite switching, there is a preference for one particular cycle, namely C_0. This implies some stability property for the cycle C_0.

We thus address the issue of stability for the cycles C_k $(k = 0, 1, 2)$ connecting equilibria ξ_k and ξ_{k+1} (mod 3) on the network. We use the notions of *relative asymptotic stability* and *essential asymptotic stability* used by Brannath [3] and which we include here for completion. Consider a flow on a compact metric space X.

Definition 4.6 (Definition 1.1 in [3]). *Given any subset N of X, a closed invariant subset A of $\bar{N}$ ($\bar{N}$ the closure of N) is said to be "stable, relatively to the set N", or "stable in N", if for every neighbourhood U of A there is a neighbourhood V of A, such that*

$$\forall\, x \in V \cap N : \quad x(t) \in U \quad \forall\, t \geq 0.$$

Let us call A "attracting" for $M \subset X$ if for every $x \in M$ the ω-limit $\omega(x)$ is a subset of A. Then A is said to be "asymptotically stable, relatively to N", or "asymptotically stable in N", if it is stable in N and there is a neighbourhood V of A such that A is attracting for $V \cap N$.

Definition 4.7 (Definition 1.2 in [3]). *A closed subset A of X, $X \in \mathbb{R}^n$, is "essentially asymptotically stable" if it is asymptotically stable relative to a set N which satisfies*

$$\lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(A) \cap N)}{\mu(B_\varepsilon(A))} = 1,$$

where $B_\varepsilon(A) = \{x \in X : dist(x, A) < \varepsilon\}$ and μ is the Lebesgue measure.

Note that the second notion is stronger than, and therefore implies, the first.

We have the following result on the stability of the cycles.

Theorem 4.8. *The relative asymptotic stability of the cycles C_0, C_1 and C_2 depends on the sign of $\epsilon_x + \epsilon_y$ as follows:*

- *if $\epsilon_x + \epsilon_y < 0$, then only the cycle C_0 is relatively asymptotically stable.*
- *if $\epsilon_x + \epsilon_y > 0$, then only the cycles C_1 and C_2 are relatively asymptotically stable.*

Proof. We prove results concerning the stability of cycle C_0 in detail. The statements concerning the other two cycles follow in an analogous way.

We analyze the flow in a neighbourhood of the cycle C_0, using the return map

$$\begin{array}{rcl} \Omega_{1,0,1,0} : \Sigma^{\text{in}}_{0,1} & \to & \Sigma^{\text{in}}_{0,1} \\ (u_1, u_2, u_4) & \mapsto & (u_2 u_1^{-\frac{e_{02}e_{10}+e_{12}c_{01}}{e_{01}e_{10}}}, u_4 u_1^{\frac{c_{02}e_{10}+c_{01}c_{12}}{e_{01}e_{10}}}, u_1^{\frac{c_{01}c_{10}}{e_{01}e_{10}}}). \end{array}$$

Given the values for e_{ij} and c_{ij} in table 2, we have

$$\Omega_{1,0,1,0}(u_1, u_2, u_4) = (u_2 u_1^{-\frac{2+\epsilon_x+\epsilon_y}{2}}, u_4 u_1^{\frac{2-\epsilon_x-\epsilon_y}{2}}, u_1).$$

This is contracting if and only if, using the Euclidean norm $||.||$,

$$||\Omega_{1,0,1,0}(u_1, u_2, u_4)|| < ||(u_1, u_2, u_4)||.$$

The following inequalities guarantee contractiveness:

$$\begin{cases} u_2 < u_1^{\frac{4+\epsilon_x+\epsilon_y}{2}} \\ u_4 < u_2 u_1^{\frac{-2+\epsilon_x+\epsilon_y}{2}} \\ u_1 < u_4 \end{cases}. \tag{6}$$

Let A^* be the set of points in the domain of $\Omega_{1,0,1,0}$ satisfying inequalities (6). The domain of $\Omega_{1,0,1,0}$ is

$$D_{\Omega_{1,0,1,0}} = \{(u_1, u_2, u_4) \in \Sigma^{\text{in}}_{0,1} :\ u_2 < u_1^{\frac{2+\epsilon_x+\epsilon_y}{2}}.$$

From the last two inequalities of (6), we have

$$u_2 > u_1^{\frac{4-\epsilon_x-\epsilon_y}{2}}.$$

The above together with the first inequality of (6) gives

$$u_1^{\frac{4-\epsilon_x-\epsilon_y}{2}} < u_1^{\frac{4+\epsilon_x+\epsilon_y}{2}}$$

which holds provided $\epsilon_x + \epsilon_y < 0$. Thus C_0 is attracting, since it is a fixed point for the return map. This implies that C_0 is relatively asymptotically stable with respect to N, if $\epsilon_x + \epsilon_y < 0$, for

$$N = \{\mathcal{F}_t(u_1, u_2, 1, u_4) :\ (u_1, u_2, u_4) \in A^*;\ t > 0\},$$

where $\mathcal{F}_t$ is as in (5).

When $\epsilon_x + \epsilon_y > 0$, we look at the eigenvalues of the Jacobian matrix of $\Omega_{1,0,1,0}$. This matrix is

$$J_{\Omega_{1,0,1,0}} = \begin{bmatrix} u_2 \frac{-2-\epsilon_x-\epsilon_y}{2} u_1^{\frac{-4-\epsilon_x-\epsilon_y}{2}} & u_1^{\frac{-2-\epsilon_x-\epsilon_y}{2}} & 0 \\ u_4 \frac{2-\epsilon_x-\epsilon_y}{2} u_1^{\frac{-\epsilon_x-\epsilon_y}{2}} & 0 & u_1^{\frac{2-\epsilon_x-\epsilon_y}{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues are such that their product is

$$Det = u_1^{-\epsilon_x-\epsilon_y}.$$

Since $0 < u_1 < 1$, we have $u_1^{-\epsilon_x-\epsilon_y} > 1$ when $\epsilon_x + \epsilon_y > 0$. Therefore, C_0 is unstable thus showing that for $\epsilon_x+\epsilon_y > 0$, at most the cycle C_1 and C_2 are relatively asymptotically stable.

Concerning the cycle C_1, contractiveness occurs for $\epsilon_x + \epsilon_y > 0$ as can be seen by looking at the return map

$$\begin{aligned} \Omega_{2,1,2,1} : \Sigma^{\text{in}}_{1,2} &\rightarrow \Sigma^{\text{in}}_{1,2} \\ (u_1, u_2, u_4) &\mapsto (u_2 u_1^{-\frac{3-\epsilon_y}{1+\epsilon_y}}, u_4 u_1^{\frac{3+\epsilon_x}{1+\epsilon_y}}, u_1), \end{aligned}$$

and the following inequalities

$$\begin{cases} u_2 < u_1^{\frac{4}{1+\epsilon_y}} \\ u_4 < u_2 u_1^{-\frac{3+\epsilon_x}{1+\epsilon_y}} \\ u_1 < u_4 \end{cases}.$$

The set N for which C_1 is relatively asymptotically stable is defined in an analogous way to that of the cycle C_0.

When $\epsilon_x + \epsilon_y < 0$, the cycle C_1 is not relatively asymptotically stable since, for the Jacobian matrix of $\Omega_{2,1,2,1}$, we have that the determinant is

$$Det = u_1^{\frac{\epsilon_x+\epsilon_y}{1+\epsilon_y}}.$$

This is greater than one when $\epsilon_x + \epsilon_y < 0$.

As for the stability of the cycle C_2, we use the return map

$$\begin{aligned} \Omega_{0,2,0,2} : \Sigma^{\text{in}}_{2,0} &\rightarrow \Sigma^{\text{in}}_{2,0} \\ (u_1, u_2, u_4) &\mapsto (u_2 u_1^{-\frac{3-\epsilon_x}{1-\epsilon_y}}, u_4 u_1^{\frac{3+\epsilon_y}{1-\epsilon_y}}, u_1), \end{aligned}$$

and the inequalities

$$\begin{cases} u_2 < u_1^{\frac{4-\epsilon_x-\epsilon_y}{1-\epsilon_y}} \\ u_4 < u_2 u_1^{-\frac{3+\epsilon_y}{1-\epsilon_y}} \\ u_1 < u_4 \end{cases}$$

to show that C_2 is relatively asymptotically stable with respect to a set N defined as analogously when $\epsilon_x + \epsilon_y > 0$.

If $\epsilon_x + \epsilon_y < 0$, the eigenvalues of the Jacobian matrix are such that the determinant satisfies

$$Det = u_1^{\frac{\epsilon_x+\epsilon_y}{1-\epsilon_y}}.$$

This is greater than one when $\epsilon_x + \epsilon_y < 0$, finishing the proof of the theorem. □

Theorem 4.9. *None of the cycles C_i, $i = 0, 1, 2$, is essentially asymptotically stable.*

Proof. That C_0 is not stable when $\epsilon_x + \epsilon_y > 0$ and that C_1 and C_2 are not stable when $\epsilon_x + \epsilon_y < 0$ follows from the previous proof.

Otherwise, the sets N defined in the previous proof are such that, for $i = 0, 1, 2$,

$$\lim_{\varepsilon \to 0} \frac{\mu(N \cap B_\varepsilon(C_i))}{\mu(B_\varepsilon(C_i))} < 1.$$

□

We note also that it is not strange that numerical simulations have spotted the preference, reflected in relative asymptotic stability, for cycle C_0 but not for cycles C_1 or C_2. In fact, the domain of the return maps near the cycles C_1 and C_2 are as follows

$$D_{\Omega_{2,1,2,1}} = \{(u_1, u_2, u_4) \in \Sigma^{\text{in}}_{1,2} : \ u_2 < u_1^{\frac{3-\epsilon_y}{1+\epsilon_y}}\}$$

and

$$D_{\Omega_{0,2,0,2}} = \{(u_1, u_2, u_4) \in \Sigma^{\rm in}_{2,0} : \; u_2 < u_1^{\frac{3-\epsilon_x}{1-\epsilon_y}}\}.$$

The domain $D_{\Omega_{1,0,1,0}}$, when $\epsilon_x + \epsilon_y < 0$, has bigger volume than these two domains when $\epsilon_x + \epsilon_y > 0$. See figure 8.

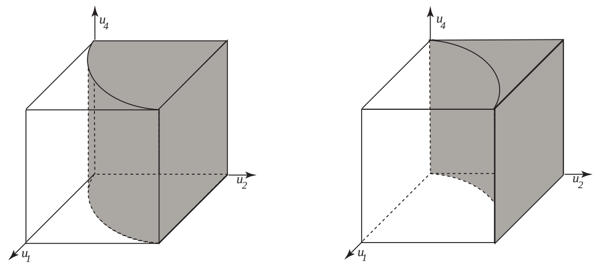

Figure 8: The figure on the left shows $D_{\Omega_{1,0,1,0}}$ when $\epsilon_x + \epsilon_y < 0$. On the right, we have a representation of $D_{\Omega_{2,1,2,1}}$ and $D_{\Omega_{0,2,0,2}}$ when $\epsilon_x + \epsilon_y > 0$.

We have the following result concerning the stability of the network.

Theorem 4.10. *The network Σ_Γ is relatively asymptotically stable.*

Proof. The dynamics described in the previous subsections show that there is a flow-invariant neighbourhood of the network. The stability results obtained in this subsection guarantee that, depending on the sign of $\epsilon_x + \epsilon_y$, there is a subset N of the flow-invariant neighbourhood satisfying the conditions of definition 4.6, but not of definition 4.7, such that trajectories, starting in N, are attracted to at least one of the cycles of the network. $\square$

5 Dynamics near the heteroclinic network in the RSP game

In this section, we show how the switching near the quocient heteroclinic network Σ_Γ can be lifted to the original network Σ of the RSP game. We use lemma 12 in Aguiar *et al.* [1]. We restate here, in the context of the present problem, both lemma 12 and its hypothesis for completion.

Assume that

♠ The finite group Γ acts orthogonally on a manifold M, with a subgroup G acting freely on M.
f is a Γ-equivariant vector field on M with a Γ-invariant network of equilibria Σ.
For any two nodes n_1, n_2 in Σ there is at most one trajectory connecting n_1 to n_2 in Σ.
The only element of G that fixes a node in Σ is the identity.

In the game of RSP, G and Γ are the same. The manifold M is $D - \{(u^*, v^*)\}$. The vector field f is given by equations (3).

Lemma 5.1. *[Lemma 12 in [1]] Let f be a vector field on M with a network of equilibria Σ satisfying ♠ and let $\Sigma_\Gamma = \Sigma/\Gamma$ be the quotient network on M/Γ for the quotient vector field. Then any two paths on Σ that coincide in one node and that drop down to the same path on Σ_Γ are the same.*

As an immediate consequence of lemma 5.1, we obtain switching along the connections. Furthermore, we have

Theorem 5.2. *There is infinite switching near the network Σ.*

Proof. The lifted images of the sets $C^{i,k,j}$ and those of the maps $\Psi_{k+i,k,k+j}$ satisfy the properties in the proof of proposition 4.5. Thus, there is switching near the original network since Σ is the group orbit of Σ_Γ. □

The existence of infinite switching guarantees that all possible sequences of play are realized by some trajectory. We then have, depending on initial conditions, trajectories that follow very simple paths near the network and trajectories following random-like sequences. Thus, depending on the initial choice of action, we will observe from very simple to extremely complex sequences of play. In particular, similar initial actions may lead to very distinct sequences of play.

The stability results obtained in subsection 4.5 are preserved under the symmetry. Therefore, the stability results extend trivially to the cycles in the original network Σ, through the group orbit of the cycles in the quotient network. These results support the observation made by Sato *et al.* [26] concerning the fact that for $\epsilon_x + \epsilon_y < 0$ agents seem to play according to the connections describing cycle C_0. Recall that the cycle C_0 connects equilibria for which agent Y wins over agent X, to equilibria where the reverse happens. Thus for $\epsilon_x + \epsilon_y < 0$, that is, when at least one agent is more penalized than the other is rewarded for a tie, ties are avoided. When $\epsilon_x + \epsilon_y > 0$ ties are more rewarded for at least one agent than penalized for the other. In this case, the relative asymptotic stability of cycles C_1 and C_2 shows that agents play for ties.

6 Concluding remarks

We describe asymptotic behaviour in a simple two-person learning game, where there is no convergence to the Nash equilibrium. We show that the asymptotic behaviour is determined by the existence of a heteroclinic network for the dynamics and chaotic switching near this network.

The presence of switching means that every path on the network is followed by a trajectory for the dynamics of play. The paths may be as simple as cycles (closed loops of strategies) or chaotic-like (following a random sequence of strategies), thus showing that in a game as simple as the Rock-Scissors-Paper game, players' strategies induce a variety of actions, ranging from almost deterministic to chaotic-like actions. Even though all sequences of actions are possible, they are not equally likely. Depending on initial conditions and the reward or penalty for ties, there is a preference for a particular sequence involving actions leading to, or avoiding, ties.

We make strong use of symmetry to obtain our results. Our techniques may be applied to any game with similar characteristics.

The generalization to more than two players is out of the scope of this paper and will appear elsewhere. The case of two players and more than three actions seems to be harder to tackle and to require a different mathematical approach.

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