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HOMOGENEOUS COUPLED CELL NETWORKS WITH $\mathbf{S}_3$-SYMMETRIC QUOTIENT

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Abstract. A coupled cell network represents dynamical systems (the coupled cell systems) that can be seen as a set of individual dynamical systems (the cells) with interactions between them. Every coupled cell system associated to a network, when restricted to a flow-invariant subspace defined by the equality of certain cell coordinates, corresponds to a coupled cell system associated to a smaller network, called *quotient network*.

In this paper we consider homogeneous networks admitting a $\mathbf{S_3}$-symmetric quotient network. We assume that a codimension-one synchrony-breaking bifurcation from a synchronous equilibrium occurs for that quotient network. We aim to investigate, for different networks admitting that $\mathbf{S_3}$-symmetric quotient, if the degeneracy condition leading to that bifurcation gives rise to branches of steady-state solutions outside the flow-invariant subspace associated with the quotient network. We illustrate that the existence of new solutions can be justified directly or not by the symmetry of the original network. The bifurcation analysis of a six-cell asymmetric network suggests that the existence of new solutions outside the flow-invariant subspace associated with the quotient is 'forced' by the symmetry of a five-cell quotient network.

1. **Introduction.** We follow the theory of coupled cell networks formalized by Golubitsky and Stewart, see [3] for a survey. Coupled cell networks represent dynamical systems (the coupled cell systems) that can be seen as a set of individual dynamical systems (the cells) that are interacting between them. In particular, a cell is an ODE (ordinary differential equation) or a system of ODEs. Schematically, the architecture of a coupled cell network can be represented by a graph whose nodes represent the cells and whose edges specify the couplings between them.

Networks with only one type of cell and one kind of edge are called *homogeneous* if an additional property of the network is satisfied: the number of edges directed to each cell is equal for all cells. This number is called the *valency* of the network.

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Figure 1 shows an example of an homogeneous three-cell network with valency 2. This network is $\mathbf{S}_3$-symmetric and it is the only $\mathbf{S}_3$-symmetric homogeneous network with valency 2.

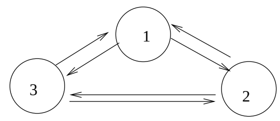

FIGURE 1. Three-cell $\mathbf{S}_3$-symmetric network.

Figure 2 shows two examples of homogeneous four-cell networks with valency 2.

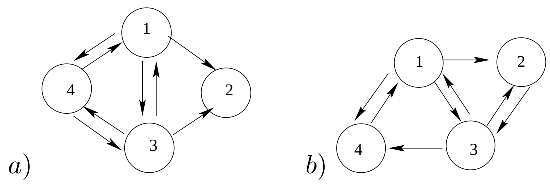

FIGURE 2. The valency 2 homogeneous four-cell networks admitting the $\mathbf{S}_3$-symmetric quotient of Figure 1.

In this paper we consider the special class of homogeneous networks. In this case, the identification of coupled cell systems is straightforward. The phase space of each cell is $\mathbf{R}^k$, where k can be any positive integer. We call k the *dimension of the internal dynamics* of a cell. The total phase space of the coupled cell systems is then $P = (\mathbf{R}^k)^n$ where n is the number of cells. We assume $n \geq 2$ since we do not consider networks with only one cell.

The coupled cell systems associated to such networks have the form $\dot{X} = F(X)$, where $X = (x_1, \ldots, x_n) \in P$ and the n coordinate functions of F are defined by the same function; this special structure is a result of the differential equations defining the time evolution of each cell being identical. For valency 2 homogeneous networks, the systems have the form

$$\dot{x}_i = f(x_i, \overline{x_j, x_l}) \tag{1}$$

where $f : \left(\mathbf{R}^k\right)^3 \to \mathbf{R}^k$ is smooth and j, l are the cells coupled to cell i. The bar over the second and third coordinates in (1) indicates that $f(u, v, w) = f(u, w, v)$, and reflects the fact of the couplings from cells j and l to cell i being of the same type.

For example, consider the homogeneous four-cell network *a*) in Figure 2. The associated coupled cell systems have the form:

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, \overline{x_3, x_4}) \\ \dot{x}_2 & = & f(x_2, \overline{x_1, x_3}) \\ \dot{x}_3 & = & f(x_3, \overline{x_1, x_4}) \\ \dot{x}_4 & = & f(x_4, \overline{x_1, x_3}) \end{array} \tag{2}$$

where $x_1,\ x_2,\ x_3,\ x_4 \in \mathbf{R}^k$ and $f : \left(\mathbf{R}^k\right)^3 \to \mathbf{R}^k$. It is straightforward to check that $\Delta_1 = \{X : x_2 = x_4\}$ is a flow-invariant subspace for (2). A subspace given by equality of some cell coordinates which is flow-invariant for every coupled cell system associated to a coupled cell network is called a *polysynchronous subspace.* Thus, the space Δ_1 is a polysynchronous subspace for (2).

It follows that the systems of differential equations (2) restricted to Δ_1 have the form

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, \overline{x_2, x_3}) \\ \dot{x}_2 & = & f(x_2, \overline{x_1, x_3}) \\ \dot{x}_3 & = & f(x_3, \overline{x_1, x_2}) \end{array} \tag{3}$$

and describe synchronous dynamics of (2) that satisfy

$$(x_1(t), x_2(t), x_3(t), x_2(t)).$$

Observe that the class of coupled cell systems (3) is the one associated to the three-cell network in Figure 1. Following [4], the network in Figure 1 is the three-cell *quotient network* associated with the four-cell network *a*) in Figure 2 and the polysynchronous subspace Δ_1. In [4] it is proved that in the context of coupled cell systems associated with networks where self-coupling and multiarrows are permitted, every coupled cell system associated with a network when restricted to a polysynchronous subspace corresponds to a coupled cell system associated to a smaller network called the *quotient network*.

For every homogeneous coupled cell system, the diagonal subspace

$$\triangle = \{(x, x, \ldots, x) : x \in \mathbf{R}^k\} \subset \left(\mathbf{R}^k\right)^n$$

is flow-invariant. Moreover, for any homogeneous network the class of *admissible vector fields* (those that respect the architecture of the network) restricted to $\triangle$ is the set of all vector fields on $\triangle$. Thus, it is reasonable to assume that there exists a synchronous equilibrium in $\triangle$, which we may assume, after a change of coordinates, is at the origin.

Let $F : \left(\mathbf{R}^k\right)^n \times \mathbf{R} \to \left(\mathbf{R}^k\right)^n$ be an admissible vector field depending on a real bifurcation parameter λ. Let $J = (dF)_{(0,0)}$ and $J^c = J|E^c$, where E^c denotes the center subspace.

Codimension-one bifurcations divide into *steady-state* (J^c has a zero eigenvalue) and *Hopf* bifurcation (J^c has purely imaginary eigenvalues). Each of these bifurcation types divide into *synchrony-preserving* ($E^c \subset \triangle$) and *synchrony-breaking* ($E^c \not\subset \triangle$).

For the remainder of this paper we focus on synchrony-breaking steady-state bifurcations from a synchronous equilibrium.

Consider a network G with phase space P. Suppose Δ_1 is a polysynchronous subspace of P. Then $J^c(\Delta_1) \subseteq \Delta_1$. Denote by G_1 the quotient network associated to Δ_1 and G. Assume a codimension-one steady-state bifurcation occurs for the coupled cell systems associated to G_1 (and so for G). We investigate if the degeneracy condition leading to this bifurcation gives rise to branches of steady-state solutions outside of Δ_1 for the coupled cell systems associated with G. We aim to compare the impact of such a synchrony-breaking steady-state bifurcation on the bifurcations with the same degeneracy condition that can occur for the different networks that admit G_1 as a quotient network.

In this paper we address this issue for the special case where the quotient network G_1 is the $\mathbf{S_3}$-symmetric network given in Figure 1. The networks in Figure 2 are the only four-cell homogeneous networks that quotient to that network, see Section 2 for details.

For simplicity, suppose $k = 1$ and so the cell phase-spaces are one-dimensional. In (3), suppose

$$f(0,0,\lambda) = 0 \quad \text{and} \quad f_u(0) - f_v(0) = 0.$$

In this case, codimension-one steady-state bifurcations for systems (3) lead to three nontrivial transcritical symmetry related branches, see [2, Ch 1]. When the eigenvalue $f_u(0) - f_v(0)$ is critical, we have that the critical space E^c for all coupled cell systems associated to networks in Figure 2 satisfy $E^c \subseteq \Delta_1$. Therefore, when we consider that degeneracy condition, besides the three solutions bifurcating branches in Δ_1, no new branches of steady-state solutions bifurcating from the trivial branch appear for the four-cell networks. See Section 2.

Many other networks (with more cells) can have the $\mathbf{S}_3$-symmetric quotient network of Figure 1.

It is known that for symmetric networks where the symmetry does not leave invariant the critical space of the coupled cell systems restricted to Δ_1, new solution branches outside Δ_1 can appear. Nevertheless, in [1] there are examples of symmetric networks where the symmetry leaves invariant the critical space of the coupled cell systems restricted to Δ_1 and new solution branches outside Δ_1 appear.

We can ask if the existence of bifurcating solution branches outside the polysynchronous subspace can also occur for asymmetric networks. We present an example in Section 3 which shows that additional solutions may indeed exist. However, the existence of new bifurcating branches for this example is associated with a symmetric quotient network.

2. Homogeneous four-cell networks with $\mathbf{S}_3$-symmetric quotient.

Given an homogeneous network and an equivalence relation on the set of cells, choose a coloring of the cells by setting cells in the same equivalence class with the same color. Consider the polydiagonal subspace Δ_1 of the total phase space given by setting as equal the cell coordinates of cells with the same color. Stewart *et al.* [6, Theorem 23] prove that the polydiagonal subspace Δ_1 is a flow-invariant space for every coupled cell system respecting the given network architecture if and only if a combinatorial property holds on the network: if we consider two cells of the same color, there is a color-preserving bijection between the corresponding sets of cells that are coupled to them.

In Proposition 1 below we enumerate the four-cell networks with valency 2 that admit the quotient network in Figure 1. This enumeration proceeds by constructing all networks of four cells and valency 2 that satisfy the combinatorial property when coloring the cells of the network with three colors. The methods of Aguiar *et al.* [1] simplify this procedure.

Proposition 1. *The valency 2 homogeneous four-cell networks admitting the $\mathbf{S}_3$-symmetric quotient network of Figure 1, are listed in Figure 2.*

In Table 1 we list the admissible coupled cell systems for the four-cell networks in Figure 2. Observe that for both networks, the space $\Delta_1 = \{X : x_2 = x_4\}$ is flow-invariant. Moreover, the coupled cell systems restricted to this polysynchronous space correspond to the three-cell network of Figure 1.

2.1. Codimension-one bifurcations.

Given an homogeneous n-cell network, let a_{ij} be the number of directed edges from cell j to cell i. The $n \times n$ matrix of

Four-Cell Network	**Symmetry**	**Equations**
$a)$	$\mathbf{Z}_2 =< (13) >$	$\dot{x}_1 = f(x_1, \overline{x_3, x_4})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_1, x_4})$ $\dot{x}_4 = f(x_4, \overline{x_1, x_3})$
$b)$	$\mathbf{Z}_2 =< (13)(24) >$	$\dot{x}_1 = f(x_1, \overline{x_3, x_4})$ $\dot{x}_2 = f(x_2, \overline{x_1, x_3})$ $\dot{x}_3 = f(x_3, \overline{x_2, x_1})$ $\dot{x}_4 = f(x_4, \overline{x_1, x_3})$

TABLE 1. Admissible systems for networks in Figure 2.

nonnegative integers $A = (a_{ij})$ is the *adjacency* matrix of the network. If the network has valency p then

$$a_{i1} + \cdots + a_{in} = p \quad \text{for} \quad i = 1, \ldots, n.$$

In Table 2 we list, up to an isomorphism, the adjacency matrix and the corresponding eigenvalues (E'vals) and eigenvectors (E'vectors) for each network in Figures 1 and 2.

	E'vals	**E'vectors**		**E'vals**	**E'vectors**
$A^{(3)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 -1 -1	$(1,1,1)$ $(1,-1,0)$ $(0,1,-1)$	$A_a^{(4)} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	2 0 -1 -1	$(1,1,1,1)$ $(0,1,0,0)$ $(1,0,-1,0)$ $(-1,1,0,1)$
			$A_b^{(4)} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	2 0 -1 -1	$(1,1,1,1)$ $(1,-1,-1,1)$ $(1,0,-1,0)$ $(1,-1,0,-1)$

TABLE 2. Adjacency matrices $A_a^{(4)}$, $A_b^{(4)}$ associated to the four-cell networks admitting the three-cell $\mathbf{S}_3$-symmetric quotient network in Figure 1 with adjacency matrix $A^{(3)}$, and their eigenvalues (E'vals) and eigenvectors (E'vectors). The first eigenvalue corresponds to the synchrony eigenvector $(1, 1, 1, 1)^T$.

In Proposition 2, which is a generalization of [5, Proposition 3.1], we relate the eigenvalues of J and their associated eigenvectors to those of the $n \times n$ adjacency matrix A of the network.

Next we compute the eigenvalues of J for an homogeneous n-cell network. Observe that using tensor products the state space of a n-cell homogeneous network is $\mathbf{R}^{nk} = \mathbf{R}^k \otimes \mathbf{R}^n$, where $\mathbf{R}^k$ is the phase space of internal dynamics for each cell and n is the number of cells.

Let f be defined as in (1). Let $Q = (d_{x_i} f)_0$ be the linearized internal dynamics and let $R = (d_{x_j} f)_0 = (d_{x_l} f)_0$ be the linearized coupling. Note that Q and R are

$k \times k$ matrices. Using tensor product notation

$$J = Q \otimes I + R \otimes A \tag{4}$$

where I is the $n \times n$ identity matrix and A an $n \times n$ adjacency matrix. Denote the eigenvalues of A by $\mu_1, \ldots, \mu_n$ where μ_1 corresponds to the synchrony eigenvector $(1, \ldots, 1) \in \triangle$ and is equal to the valency of the network.

The proof of the following proposition is similar to the one given for Proposition 3.1 in [5].

Proposition 2. *The eigenvalues of J are the union of the eigenvalues of the n $k \times k$ matrices $Q + \mu_j R$, $j = 1, \ldots, n$, including algebraic multiplicity. The eigenvectors of J are the vectors $u \otimes w$, where $u \in \mathbf{C}^k$ is an eigenvector of Q and $w \in \mathbf{C}^n$ is an eigenvector of A.*

Remark 1. Let $J^{(4)}$ be the Jacobian associated to a four-cell network in Figure 2 and let $J^{(3)}$ be the Jacobian associated to the corresponding quotient three-cell network of Figure 1. Then the eigenvalues of $J^{(4)}$ include the eigenvalues of $J^{(3)}$. Recall (see Section 1), that the coupled cell systems associated to the quotient network are obtained by restricting the coupled cell systems associated to a four-cell network to the flow-invariant subspace

$$\Delta_1 = \{X : \; x_2 = x_4\}.$$

Note that Δ_1 is a flow-invariant subspace for coupled cell systems associated with the networks of Figure 2. Hence, Δ_1 is flow-invariant for $J^{(4)}$. That is,

$$J^{(4)}(\Delta_1) \subseteq \Delta_1$$

$\diamond$

2.2. **Steady-state bifurcations for the four-cell networks.** The eigenvalues of each of the four $k \times k$ matrices $Q + \mu_j R$ are generically simple. So the possible steady-state bifurcation types do not depend on k, and we assume $k = 1$. In this case the 4×4 matrix J has four eigenvalues $\gamma_j = Q + \mu_j R$, where Q and R are 1×1 matrices. Say, γ_1 corresponds to the synchrony eigenvector $(1, 1, 1, 1) \in \triangle$. From Table 2 and Proposition 2 it follows that two eigenvalues of J are equal, for instance, γ_3, γ_4, and are associated with two noncolinear eigenvectors. Therefore, only two types of synchrony-breaking codimension-one steady-state bifurcations can occur for the networks in Figure 2: simple eigenvalue ($\gamma_2 = 0$); and double eigenvalues ($\gamma_3 = \gamma_4 = 0$) with two eigenvectors.

Observe that -1 is an eigenvalue of the adjacency matrices associated with each of the networks in Figures 1 and 2 (see Table 2). The algebraic and geometric multiplicity of the eigenvalue -1 is always two. Hence, Proposition 2 implies that the Jacobian matrix J at the origin, associated to each of those networks, has $f_u(0) - f_v(0)$ as an eigenvalue with algebraic and geometric multiplicity equal to two. Thus, in each of the networks in Figures 1 and 2 the synchrony-breaking bifurcations can occur with double critical eigenvalue $f_u(0) - f_v(0)$, whose geometric multiplicity is two.

Assume $f_u(0) - f_v(0) = 0$. We show that for the coupled cell systems associated to the four-cell networks in Figure 2 these bifurcations correspond to the local steady-state synchrony-breaking bifurcations from the synchronous equilibrium that can occur for the coupled cell systems restricted to $\Delta_1 = \{X : x_2 = x_4\}$. That is, correspond to the bifurcations with double critical eigenvalue that can occur in the three-cell $\mathbf{S}_3$-symmetric quotient network.

Theorem 1. *Assume that the coupled cell systems defined by $f(u, \overline{v, w}, \lambda)$ associated to networks a) or b) of Figure 2 satisfy*

$$\begin{array}{ll} f(0,0,0,\lambda) \equiv 0, & f_u(0) - f_v(0) = 0, \\ f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0, & f_{uu}(0) - 2f_{uv}(0) - f_{vv}(0) + 2f_{vw}(0) \neq 0. \end{array}$$

Then there are three transcritical branches of unstable solutions bifurcating from the trivial solution of the form

$$(x(\lambda), x(\lambda), y(\lambda), x(\lambda), \lambda) \qquad (x(\lambda), y(\lambda), x(\lambda), y(\lambda), \lambda) \qquad (y(\lambda), x(\lambda), x(\lambda), x(\lambda), \lambda)$$

where $x(0) = 0 = y(0)$ and $x'(0) \neq 0 \neq y'(0)$.

Proof. A simple calculation shows that the critical space for the coupled cell systems associated to networks of Figure 2 is given by

$$E_c = \langle (-1, 0, 1, 0)^t, (-1, 1, 0, 1)^t \rangle \subseteq \Delta_1 \tag{5}$$

where the subspace $\Delta_1 = \{X \in \mathbf{R}^4 : \ x_2 = x_4\}$ is flow-invariant. Hence, the study of the bifurcations given by $f_u(0) - f_v(0) = 0$ that can occur in networks in Figure 2 reduces to the analysis of these bifurcations for the coupled cell systems restricted to Δ_1. The coupled cell systems restricted to Δ_1 have the form given in (3) and correspond to admissible systems associated to the $\mathbf{S}_3$-symmetric network in Figure 1. It is known that for this network the codimension-one bifurcations lead to three nontrivial transcritical symmetry related branches of unstable solutions [2, Ch 1] whose form is

$$(x(\lambda), x(\lambda), y(\lambda), \lambda) \qquad (x(\lambda), y(\lambda), x(\lambda), \lambda) \qquad (y(\lambda), x(\lambda), x(\lambda), \lambda)$$

with $x(0) = 0 = y(0)$ and $x'(0) \neq 0 \neq y'(0)$. The instability of these solutions associated with the directions in the space Δ_1 imply the instability of the solutions in the all space. □

Remark 2. Consider the networks of Figure 2. Observe that network *a*) is $\mathbf{Z}_2 =< (13) >$-symmetric, network *b*) is $\mathbf{Z}_2 =< (13)(24) >$-symmetric and both $\mathbf{Z}_2$-symmetries leave Δ_1 invariant. ◇

3. **Example.** Consider the asymmetric six-cell network in Figure 3.

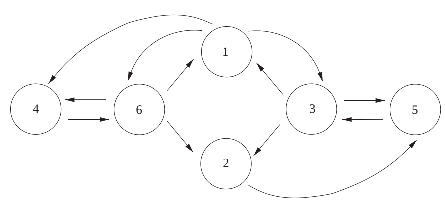

FIGURE 3. Homogeneous six-cell network with valency 2 and no symmetry.

As before, we assume that the phase space for each cell is one-dimensional. Observe that the subspace $\Delta_1 = \{X : x_1 = x_2,\ x_3 = x_4,\ x_5 = x_6\}$ is a polysynchronous subspace for the asymmetric six-cell network. The quotient network associated to Δ_1 is the $\mathbf{S}_3$-symmetric quotient network in Figure 1. Assume $f_u(0) - f_v(0) = 0$, then a steady-state synchrony-breaking bifurcation from the trivial equilibrium occurs for the coupled cell systems restricted to Δ_1 (and so associated to the $\mathbf{S}_3$-symmetric quotient network). Next we investigate the consequences of the above degeneracy condition for the steady-state synchrony-breaking bifurcations associated to the coupled cell systems corresponding to the six-cell network. In what follows, we conclude that, in addition to the three nontrivial transcritical branches of solutions in Δ_1, these bifurcations lead to new three transcritical branches of solutions outside Δ_1.

Let $A^{(6)}$ and $J^{(6)}$ be respectively the adjacency and the Jacobian matrices associated to the six-cell asymmetric network. A simple calculation shows that -1 is an eigenvalue of $A^{(6)}$ with algebraic multiplicity three and geometric multiplicity two. Hence, by Proposition 2 it follows that $f_u(0) - f_v(0)$ is an eigenvalue of $J^{(6)}$ with algebraic and geometric multiplicity three and two, respectively. Assume $f_u(0) - f_v(0) = 0$. Let E_6^c and E_3^c denote the critical eigenspace associated to the critical eigenvalue $f_u(0) - f_v(0) = 0$ for the six-cell network and to the three-cell quotient network coupled cell systems, respectively. Observe that the geometric multiplicity of the zero eigenvalue is the same as the one for the coupled cell systems associated for the $\mathbf{S}_3$-symmetric quotient network. However, the algebraic multiplicity increased by one unity. Thus,

$$\dim E_6^c = \dim E_3^c + 1.$$

Hence, in addition to the three nontrivial transcritical bifurcating branches of solutions identified for the $\mathbf{S}_3$-symmetric quotient network coupled cell systems, it is reasonable to expect other branches of solutions outside Δ_1 to bifurcate from the trivial solution.

We describe briefly the generic local codimension-one steady-state synchrony-breaking bifurcation that can occur for the coupled cell systems associated to the six-cell network when $f_u(0)-f_v(0) = 0$. Observe that the subspace $\Delta_2 = \{X : x_1 = x_2\}$ is a flow-invariant subspace for the six-cell network coupled cell systems. The five-cell quotient associated to the polysynchronous subspace Δ_2, given in Figure 4, is $\mathbf{Z}_2 =< (36)(45) >$-symmetric. Let $A^{(5)}$ and $J^{(5)}$ be respectively the adjacency and

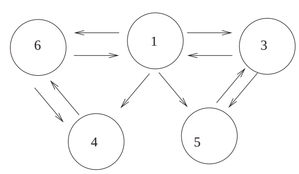

Figure 4. Five-cell symmetric quotient network associated to the six-cell network in Figure 3.

the Jacobian matrices associated to the five-cell symmetric network. A simple calculation using $A^{(5)}$ and the Proposition 2 shows that $f_u(0) - f_v(0)$ is an eigenvalue of $J^{(5)}$ with algebraic and geometric multiplicity being respectively three and two. Under the assumption $f_u(0) - f_v(0) = 0$ it follows that the $E_6^c \subseteq \Delta_2$. Thus, to find the bifurcating branches of equilibria for the six-cell network it is sufficient to analyze the bifurcations that can occur in the associated five-cell symmetric quotient network. Those local codimension-one steady-state bifurcations are classified in Aguiar *et al.* [1] obtaining six nontrivial transcritical bifurcating branches of solutions: three of them are the well-known branches of solutions associated to the $\mathbf{S}_3$-symmetric network; the other three transcritical bifurcating branches of solutions are outside Δ_1.

We remark that although the six-cell network in Figure 3 has no symmetry, the bifurcation analysis suggests that the symmetry of the five-cell quotient network in Figure 4 is the structure that forces the existence of new bifurcating branches of solutions outside Δ_1.

4. **Final remarks.** As it was pointed out, there are many networks admitting the $\mathbf{S_3}$-symmetric quotient network of Figure 1. Although the coupled cell systems associated to those networks are different, each admits a polysynchronous subspace such that the restriction to that subspace is the same – the coupled cell systems associated to the $\mathbf{S_3}$-symmetric network.

A degeneracy condition leading to a steady-state bifurcation for the quotient coupled cell systems can lead to new bifurcating branches for the original coupled cell systems. The examples presented here illustrate that the existence of new solutions can be justified directly or not by the symmetry of the original network.

The six-cell network of the example in Section 3 is asymmetric and presents new bifurcating branches. Yet, the bifurcation analysis in Aguiar *et al.* [1] highlights the existence for this network of a symmetric five-cell quotient network 'forcing' the existence of new solutions.

Therefore, although the symmetry is not sufficient to the appearance of new solution branches, it is reasonable to think that the existence of new solutions is a consequence of the existence of a symmetric quotient network.

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