Abstract

Coupled cell systems are networks of dynamical systems (the cells), where the links between the cells are described through the network structure, the coupled cell network. Synchrony subspaces are spaces defined in terms of equalities of certain cell coordinates that are flow-invariant for all coupled cell systems associated with a given network structure. In this paper we show how to obtain the lattice of synchrony subspaces for a general network and present an algorithm that generates that lattice. We prove that this problem is reduced to get the lattice of synchrony subspaces for regular networks. For a regular network we obtain the lattice of synchrony subspaces based on the eigenvalue structure of the network adjacency matrix.

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1 Introduction

Consider a network architecture, that is, a finite set of nodes (the cells) linked together by a finite number of arrows. Assume the cells represent individual dynamics and the arrows the interactions between the individuals. We have then the coupled cell systems, that is, the dynamical systems consistent with that network structure. See for example the approach of Stewart, Golubitsky et al. [11, 8, 7] that we follow in this paper or Field [5]. The network structure imposes restrictions at the dynamics that can occur.

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for the associated coupled cell systems. In particular, it can force the existence of flow-invariant subspaces defined in terms of equalities of certain cell coordinates. These flow-invariant subspaces, called \textit{synchrony subspaces}, have a major impact at the dynamics of the coupled cell systems associated with the given network. See for example Aguiar \textit{et al.} [2]. Stewart, Golubitsky and co-workers [11, 8] describe these spaces using the network structure. Given a network and a space given by equalities of certain cell coordinates, we can define an equivalence relation on the set of nodes with two cells in the same class when the coordinates are equal. In [11, 8] it is proved that the space is of synchrony (for any coupled cell system associated with the network) if and only if this relation satisfies a set of conditions on the network – the relation is said to be \textit{balanced} (see Definition 2.4). Thus the synchrony subspaces associated with a network structure are in one-to-one correspondence with the balanced equivalence relations on the set of cells of the network. Moreover, by Stewart [10] (see also Aldis [3]) the set of all such balanced equivalence relations forms a complete lattice taking the relation of refinement; recall that a \textit{lattice} is a partially ordered set such that every pair of elements as a unique least upper bound or \textit{join}, and a unique greatest lower bound or \textit{meet}. Moreover, a \textit{complete lattice} \(X\) is a lattice where every subset \(Y \subseteq X\) has a unique least upper bound or join, and a unique greatest lower bound or meet. Using the one-to-one correspondence between balanced equivalence relations and synchrony subspaces, it follows that the set of synchrony subspaces associated with a network, taking the relation of inclusion \(\subseteq\), is also a complete lattice [10]. See Section 3.

The aim of this paper is to describe how to obtain the lattice of synchrony subspaces of a given a network. As we shall show, this reduces basically to the problem of how to obtain the lattice of synchrony subspaces of regular networks. Here, we say that a network is \textit{regular} if it has only one cell type and one arrow type and the number of arrows directed to each cell is constant. For a regular network we obtain the lattice of synchrony subspaces based on the eigenvalue structure of the network adjacency matrix (see Definition 2.1 (iv)) and we present an algorithm that generates the lattice. Our approach exploring the eigenvalue structure of the network adjacency matrices has been motivated by the work of Kamei [9] on the class of regular networks where the adjacency matrix has only simple eigenvalues.

We restrict our discussion in this section to the regular case. Basically, in our framework, the key point is that a synchrony subspace of a regular network is a space that is left invariant by the corresponding network adjacency matrix and is polydiagonal, that is, it is described by a set of equality conditions on the cell coordinates.

It is known that the subspaces invariant under a linear map can be described in terms of its eigenvectors and generalized eigenvectors. Moreover, the set of such invariant subspaces forms a complete lattice under the relation of inclusion and this lattice can be described using the irreducible invariant subspaces - the Jordan subspaces - the invariant subspaces having a unique eigenvector (up to multiplication by a scalar). In this lattice the join operation is the intersection and the meet operation is the sum.

In our work, there were two fundamental difficulties that had to be overcomed: (a) how to list the possible polydiagonal subspaces that contain (generalized) eigenvectors of the network adjacency matrix; (b) how to generalize the concept of irreducible (synchrony) subspace aiming to describe the lattice of synchrony subspaces through a (small) set of
irreducible synchrony subspaces.

Note that, although the lattice of synchrony subspaces, as a set, is a subset of the lattice of the invariant subspaces under the network adjacency matrix, it is not in general a sublattice - the meet operation is the same, the intersection of subspaces, but the join of two synchrony subspaces is not given in general by their sum. In fact, the join of two synchrony subspaces is given by their sum only when the sum is a polydiagonal subspace.

Concerning our first difficulty, we define the concept of minimal synchrony subspace associated to an eigenvector or Jordan chain - the intersection of all synchrony subspaces containing the eigenvector or Jordan chain (see Definition 5.8). We prove that the set of all minimal synchrony subspaces forms a sum-dense set for the lattice of synchrony subspaces (Theorem 5.11). That is, any synchrony subspace can be given by a sum of minimal synchrony subspaces. It follows then that, our first task - to list the possible polydiagonal subspaces containing (generalized) eigenvectors - can be reduced to the listing of the minimal synchrony subspaces.

Considering the second difficulty, we introduce the concept of sum-irreducible synchrony subspace (Definition 5.5) - it cannot be represented as a sum of proper synchrony subspaces. We prove then that every synchrony subspace associated with a network is a sum of sum-irreducible synchrony subspaces (Proposition 5.6). Joining the two results, as just mentioned, concerning the concepts of minimal and sum-irreducible synchrony subspaces, it follows that we can generate the lattice of synchrony subspaces associated with a regular network through the set of the minimal synchrony subspaces that are sum-irreducible. See Corollary 5.12 and Remark 5.13.

In Section 6 we present an algorithm that outputs for a regular network the set of minimal synchrony subspaces that are sum-irreducible and the lattice of synchrony subspaces (Algorithm 6.3). The algorithm contains four fundamental steps. Given a regular network, in the first two steps, it obtains a set of synchrony subspaces containing the minimal synchrony set for the lattice of the network synchrony subspaces. In the third step finds the irreducible sum-dense set for that lattice and in the fourth step generates the lattice.

We illustrate the implementation of the algorithm with five regular network examples: three where the network adjacency matrix is semi-simple (Examples 6.7, 6.8, 6.9) and two examples where the network adjacency matrix is not semi-simple (Examples 6.10 and 6.11). By inspection of these examples, it is clear that the lattice of synchrony subspaces for the last two examples (see Figures 10, 12) is smaller (and most of its elements are sum-irreducible) than for the semi-simple examples (Figures 5, 6). That reflects the fact that, in the non semi-simple cases, most of the irreducible invariant subspaces for the network adjacency matrices are Jordan spaces generated by Jordan chains.

The paper is organized in the following way. Section 2 introduces briefly a few concepts concerning networks and corresponding admissible vector fields - the coupled cell systems. It also recalls the concepts of balanced equivalence relation and synchrony subspace and the result establishing the one-to-one correspondence between the two concepts [8]. In section 3 we start by recalling a few basic concepts concerning complete lattices and the result in [10] proving that the set of all balanced equivalence relations of a network forms a complete lattice. The results in Section 4 relate the lattice of synchrony subspaces for nonhomogeneous networks and for nonregular homogeneous networks with the lattice.
of synchrony subspaces for their identical-edge subnetworks, a kind of regular networks. 
Thus, the most important question to be addressed is how to obtain the lattice of syn- 
chrony subspaces for regular networks. This is the issue addressed in Sections 5 and 6. 
In Section 5, we start by recalling the theory of invariant subspaces for linear maps; then 
we introduce the concepts of sum-irreducible and minimal synchrony subspace associated 
to an eigenvector or Jordan chain; finally, we prove in Proposition 5.6 that the lattice of 
synchrony subspaces associated with a network structure can be obtained using the sum 
operation of synchrony subspaces that are sum-irreducible. In Section 6 we present an 
algorithm (Algorithm 6.3) that calculates the set of sum-irreducible subspaces for a given 
regular network and generates the corresponding lattice of synchrony subspaces. We il-
ustrate the implementation of the algorithm with five network examples in Section 6.3. 
Finally, in Sections 7 and 8 we show how to optimize the process of obtaining the lattice 
of synchrony subspaces for general nonregular networks using Algorithm 6.3.

2 Background

We recall briefly a few concepts concerning networks and coupled cell systems. Following 
Stewart, Golubitsky et al. [11, 8, 7], a network is a directed graph whose nodes represent 
the cells and the arrows (or edges) the couplings. Equivalence relations on the set of 
odes and on the set of arrows can be defined symbolizing the following:
(a) Two nodes are in the same cell equivalence class if they represent individual dynamics 
with the same state space.
(b) Two arrows are in the same arrow equivalence class if they represent couplings of the 
same type.
The following consistency condition is assumed: if two arrows are of the same type then 
the corresponding head cells are in the same cell equivalence class and the same holds for 
the corresponding tail cells.

Definition 2.1 (i) Given a network, the input set of a cell of the network is the set of 
arrows directed to that cell.
(ii) Two cells of a network are said to be (input) isomorphic if there is an arrow-type 
preserving bijection between the corresponding input sets.
(iii) A homogeneous coupled cell network is a network in which all cells are (input) iso-
morphic.
(iv) A regular coupled cell network is a homogenous network with only one arrow type. 
For a regular network, the valency is the number of arrows of the input set of any cell and 
the adjacency matrix is the matrix where the (i, j) entry is the number of arrows from cell 
j to cell i, assuming the set of cells is \{1, \ldots, n\}. If v is the valency of a regular network 
then the corresponding adjacency matrix has v constant row sum.
(v) For a general coupled cell network with set of cells \{1, \ldots, n\} and k arrow equivalence 
classes, we define k adjacency matrices, one for each arrow type, say \(A_1, \ldots, A_k\), in the 
following way: the (i, j) entry of the matrix \(A_p\) is the number of arrows of type \(p\) from 
cell \(j\) to cell \(i\).

Example 2.2 Figure 1 shows two examples of 5-cell regular networks.
Following [11, 8, 7], the connection between coupled cell systems and coupled cell networks is made in the following way: to each coupled cell $c$ is associated a choice of cell phase space $P_c$ which is assumed to be a finite-dimensional real vector space, say $\mathbb{R}^k$ for some $k > 0$. If cells $c$ and $d$ are cell equivalent then it is required that $P_c = P_d$ and the two spaces are identified canonically. If $\mathcal{C} = \{1, \ldots, n\}$ denotes the set of cells of the network, then the total phase space $P$ of the coupled cell system is the direct product of the cell phase spaces, $\prod_{c \in \mathcal{C}} P_c$, and we employ the coordinate system $x = (x_c)_{c \in \mathcal{C}}$ on $P$. Given a network $\mathcal{G}$ and a fixed choice of the total phase space $P$, we describe now the coupled cell systems that correspond to the class of the systems of ordinary differential equations, $\dot{X} = F(X)$, $X \in P$, compatible with the structure of the network. The system associated with cell $j$ has the form

$$\dot{x}_j = f_j(x_j; x_{i_1}, \ldots, x_{i_m})$$

where the first argument $x_j$ in $f_j$ represents the internal dynamics of the cell and each of the remaining variables $x_{i_p}$ represents a coupling between cell $i_p$ and cell $j$. Thus $x_j \in P_j$, $x_{i_p} \in P_{i_p}$, $p = 1, \ldots, m$ and we assume $f_j : P_j \times P_{i_1} \times \cdots \times P_{i_m} \to P_j$ is smooth. Moreover, identical couplings directed to cell $j$ correspond to the invariance of $f_j$ under permutation of the corresponding variables. Systems associated with (input) isomorphic cells are identical up to permutation of the variables accordingly to the input sets of the cells. The vector fields $F$ are called $\mathcal{G}$-admissible.

For homogeneous networks (where all cells are input isomorphic) we have only one type of systems, that is, $f_j \equiv f$ for all $j$. For valency $v$ regular networks, the cell systems have the form

$$\dot{x}_j = f(x_j; \overline{x_{i_1}}, \ldots, \overline{x_{i_v}})$$

where the overbar in $f$ indicates that $f$ is invariant under any permutation of the cell coordinates $x_{i_1}, \ldots, x_{i_v}$ (representing the cells with arrows of the same type directed to cell $j$).

**Example 2.3** The coupled cell systems associated to the network on the left of Figure 1 satisfy

$$\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_4) \\
\dot{x}_2 &= f(x_2, \overline{x_1}, x_5) \\
\dot{x}_3 &= f(x_3, \overline{x_1}, x_5) \\
\dot{x}_4 &= f(x_4, \overline{x_1}, \overline{x_3}) \\
\dot{x}_5 &= f(x_5, \overline{x_1}, \overline{x_3})
\end{align*}$$
where \( f : (\mathbb{R}^k)^3 \to \mathbb{R}^k \) is smooth and invariant under permutation of the last two cell coordinates.

## 2.1 Balanced equivalence relations

We recall the definition of a balanced equivalence relation on the set of cells of a network. Balanced equivalence relations of a network play a crucial role when describing the synchrony spaces of a network.

**Definition 2.4 ([8])** Given a network \( G \), an equivalence relation \( \bowtie \) on the network set of cells \( C \) is balanced if for every \( c, d \in C \) with \( c \bowtie d \), there exists an isomorphism between the input sets, \( I(c) \) and \( I(d) \), of \( c \) and \( d \), respectively, say \( \beta : I(c) \to I(d) \), preserving the arrow equivalence relation and such that for all \( i \in I(c) \), the tail cells of \( i \) and \( \beta(i) \) are in the same \( \bowtie \) class.

**Example 2.5** Consider the 5-cell regular network on the left of Figure 1. The equivalence relation on the set of cells \( C = \{1, \ldots, 5\} \) with classes \( \{1, 2, 3\} \), \( \{4, 5\} \) is balanced.

**Remark 2.6** Given a network \( G \) with set of cells \( C \), we can define an equivalence relation \( \sim_I \) on \( C \) in the following way: given \( c, d \in C \), then \( c \sim_I d \) if and only if cells \( c \) and \( d \) are (input) isomorphic (recall Definition 2.1). Now observe that if \( \bowtie \) is a balanced equivalence relation on \( C \) then it refines \( \sim_I \).

## 2.2 Synchrony subspaces

**Definition 2.7** Given a network \( G \), an equivalence relation \( \bowtie \) on the network set of cells \( C \) refining the cell equivalence relation, and a choice of the total phase space \( P \), define the polydiagonal subspace

\[
\Delta_{\bowtie} = \{ x \in P : x_c = x_d \text{ whenever } c \bowtie d, \ \forall c, d \in C \}.
\]

The polydiagonal subspace \( \Delta_{\bowtie} \) of \( P \) is called a synchrony subspace if it is flow-invariant for all \( G \)-admissible vector fields on \( P \).

**Example 2.8** Consider the 5-cell regular network on the left of Figure 1 with set of cells \( C = \{1, \ldots, 5\} \). Taking \( \bowtie \) the equivalence relation on \( C \) with classes \( \{1, 2, 3\} \), \( \{4, 5\} \), then \( \Delta_{\bowtie} = \{ x \in P : x_1 = x_2 = x_3, x_4 = x_4 \} \). It follows easily from the network admissible equations (see Example 2.3) that \( \Delta_{\bowtie} \) is a synchrony subspace. Moreover, their restriction to \( \Delta_{\bowtie} \) is:

\[
\begin{align*}
\dot{x}_1 &= f(x_1, \overline{x_1}, \overline{x_4}) \\
\dot{x}_4 &= f(x_4, \overline{x_1}, \overline{x_1})
\end{align*}
\]

We present now the result of [8] relating balanced equivalence relations on the set of cells and the synchrony spaces of a network.
Theorem 2.9 ([8]) Given a network $\mathcal{G}$, an equivalence relation $\bowtie$ on the network set of cells $\mathcal{C}$ and a choice $P$ of the total phase space, then $\Delta_\bowtie$ is a synchrony subspace if and only if $\bowtie$ is balanced.

Proof See Golubitsky et al. [8, Theorem 4.3]. The proof of this result is divided into two steps. Check directly that $\bowtie$ being balanced is sufficient for $\Delta_\bowtie$ to be a synchrony subspace. The necessity is established by considering linear vector fields. \hfill \Box

From Theorem 2.9, it follows that synchrony subspaces are determined only by the network topology. Moreover, we have:

Corollary 2.10 Let $\mathcal{G}$ be a coupled cell network with set of cells $\mathcal{C}$ and let $\bowtie$ be an equivalence relation on $\mathcal{C}$. Then for any choice of the total phase space $P$, the subspace $\Delta_\bowtie$ is a synchrony subspace if and only if it is flow-invariant for all linear admissible vector fields choosing the cell phase spaces to be $\mathbb{R}$.

Remark 2.11 For $n$-cell homogeneous networks, the linear admissible vector fields, assuming the cell phase spaces to be $\mathbb{R}$, are generated by the identity map on $\mathbb{R}^n$ and the linear maps on $\mathbb{R}^n$ associated to the network adjacency matrices. It follows then that a polydiagonal $\Delta_\bowtie$ is a synchrony subspace if and only if the corresponding polydiagonal, assuming the cell phase spaces to be $\mathbb{R}$, is left invariant by the network adjacency matrices. \hfill \Diamond

3 Complete lattices

In this section we start by reviewing some basic facts about lattices and complete lattices. Details can be found, for example, at Davey and Priestley [4]. We then recall the results establishing that both the lattices of balanced equivalence relations and synchrony subspaces for a given network are complete lattices, taking the relation of refinement and inclusion $\subseteq$ of spaces, respectively.

3.1 Basic definitions

Given a partially ordered set $X$ with a binary relation $\geq$ and a subset $Y \subseteq X$, an element $x$ of $X$ is an upper bound of $Y$ if $x \geq y$ for all $y \in Y$. Further, an upper bound $x$ of $Y$ is said to be the least upper bound of $Y$ if every upper bound $x'$ of $Y$ satisfies $x' \geq x$. Dually, we define lower bound and greatest lower bound.

Now recall that a lattice is a partially ordered set $X$ such that every pair of elements $x, y \in X$ has a unique least upper bound or join, denoted by $x \vee y$, and a unique greatest lower bound or meet, denoted by $x \wedge y$.

A complete lattice is a lattice where every subset $Y \subseteq X$ has a unique least upper bound or join, and a unique greatest lower bound or meet. A complete lattice has a top (maximal) element, denoted $\top$, and a bottom (minimal) element, denoted $\bot$. Observe that every finite lattice is complete, see [4, Corollary 2.12].
A sublattice $M$ of a lattice $X$ is a subset such that

$$x \in M \text{ and } y \in M \implies x \lor y \in M \text{ and } x \land y \in M .$$

**Remark 3.1** Observe that a subset of a lattice $X$ may form a lattice according to the definition of lattice without being a sublattice of $X$. For example, it may happen that the greatest lower bound of $x$ and $y$ in the subset differs from the greatest lower bound $x \land y$ of $x$ and $y$ in $X$.

An element $x$ in a lattice $L$ is \textit{join-irreducible} if

(i) $x \neq 0$ (in case $L$ has a zero);
(ii) $x = a \lor b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

A \textit{meet-irreducible} element is defined dually. See for example Davey [4, Definition 8.7].

A subset $S$ of a lattice $L$ is called \textit{join-dense} in $L$ if for every element $a \in L$, there exists a subset $A$ of $S$ such that $a = \lor A$. The dual of join-dense is \textit{meet-dense}. See for example [4, Definition 2.34].

### 3.2 Complete lattice of balanced equivalence relations

Let $\mathcal{G}$ be a network with set of cells $\mathcal{C}$. Denote by $M_\mathcal{G}$ the set of equivalence relations on $\mathcal{C}$ and take the relation of refinement on $M_\mathcal{G}$: given two equivalence relations, $\rhd_i$ and $\rhd_j$, we say that $\rhd_i$ refines $\rhd_j$, and we write $\rhd_i \prec \rhd_j$, when for all $c \in \mathcal{C}$ we have

$$[c]_i \subseteq [c]_j .$$

Here, $[c]_l$ denotes the $\rhd_l$-equivalence class of cell $c$, for $l = i, j$. It follows that $M_\mathcal{G}$ is a complete lattice with the meet and join operations on the set $M_\mathcal{G}$ defined in the following way:

**Meet operation:** we have that $\rhd_k = \rhd_i \land \rhd_j$ where if $c, d \in \mathcal{C}$ then $c \rhd_k d$ if and only if $c \rhd_i d$ and $c \rhd_j d$.

**Join operation:** we have that $\rhd_k = \rhd_i \lor \rhd_j$ where if $c, d \in \mathcal{C}$ then $c \rhd_k d$ if and only if there exists a finite chain $c = c_q, \ldots, c_s = d$ such that for all $t$ with $q \leq t \leq s - 1$ either $c_t \rhd_i c_{t+1}$ or $c_t \rhd_j c_{t+1}$.

Now denote by $\Lambda_\mathcal{G}$ the set of balanced equivalence relations of a network $\mathcal{G}$ on the set of cells $\mathcal{C}$ and again note that $\Lambda_\mathcal{G}$ has a partially ordered structure, using the relation of refinement $\prec$ as defined above. Stewart [10] proves that the set $\Lambda_\mathcal{G}$ of balanced equivalence relations of a (locally finite) network forms a complete lattice. See also Aldis [3, Chapter 4].

However, the lattice of balanced equivalence relations $\Lambda_\mathcal{G}$ is not in general a sublattice of the lattice of equivalence relations $M_\mathcal{G}$. Although the join operation on equivalence relations restricts to give the join operation on balanced equivalence relations that does not happen in general with the meet operation. See Stewart [10, Example 5.5].

**Remark 3.2** Consider the lattice $\Lambda_\mathcal{G}$ of balanced equivalence relations for a network $\mathcal{G}$.

(i) If $\mathcal{G}$ is homogeneous then the top element is the balanced equivalence relation with only one class, and the bottom element corresponds to the equivalence relation where
each class is formed by a unique cell.

(ii) If $G$ is nonhomogeneous, again the bottom element corresponds to the equivalence relation where each class is formed by a unique cell. However, the top element does not have to correspond to the input-equivalence relation. Following the notation of [10], the top element corresponds to a unique coarsest balanced equivalence relation, where $\sqsupset\sqsubset_1$ is coarser than $\sqsupset\sqsubset_2$ if and only if $\sqsupset\sqsubset_2$ is finer than $\sqsupset\sqsubset_1$. This relation refines $\sim_I$ but it is not $\sim_I$ if the input relation $\sim_I$ is not balanced. Take as an example, the chain of three identical cells $1 \leftarrow 2 \leftarrow 3$ where $\sim_I$ has two classes, $\{1, 2\}$ and $\{3\}$, and it is not balanced. Aldis [1] presents an algorithm to compute the coarsest balanced equivalence relation of a given network $G$ in polynomial time (in the number of cells plus the number of edges of $G$).

3.3 Complete lattice of synchrony subspaces

Let $V_G^P$ the set of synchrony subspaces for $G$ assuming the total phase space is $P$. By Theorem 2.9 there is a one-to-one correspondence between the elements of $\Lambda_G$ and $V_G^P$. Moreover, an equivalence relation $\sqsupset\sqsubset$ on the network set of cells is balanced if and only if the associated polydiagonal $\Delta_{\sqsupset\sqsubset} \subseteq P$ is left invariant by all linear network admissible vector fields. Also, for that purpose, we can take the cell phase spaces to be $\mathbb{R}$ and so the total phase space is $P = \mathbb{R}^n$ (if $n$ is the number of cells) and check invariance of $\Delta_{\sqsupset\sqsubset} \subseteq \mathbb{R}^n$ by all the linear admissible vector fields on $\mathbb{R}^n$ (see Corollary 2.10). From now on we denote by $V_G$ the set of synchrony subspaces for $G$. Note that $V_G$ is a lattice taking the partial order on $V_G$ given by inclusion $\subseteq$ of spaces.

The map $\delta : \Lambda_G \to V_G$ defined by $\delta(\sqsupset\sqsubset) = \Delta_{\sqsupset\sqsubset}$ for $\sqsupset\sqsubset \in \Lambda_G$ is a lattice anti-isomorphism, that is, an isomorphism that reverses order, hence interchanges meet and join, see Stewart [10]. In particular, we have

$$\Delta_{\sqsupset\sqsubset_1 \lor \sqsupset\sqsubset_2} = \Delta_{\sqsupset\sqsubset_1} \cap \Delta_{\sqsupset\sqsubset_2}.$$  

Moreover, since $\Lambda_G$ is a complete lattice, it follows that $V_G$ is also a complete lattice.

**Remark 3.3** Given a linear $G$-admissible map $A : \mathbb{R}^n \to \mathbb{R}^n$, the set of $A$-invariant subspaces is a lattice (considering the partial order $\subseteq$) with the meet operation corresponding to the intersection and the join operation given by the sum. The sum corresponds to the subspace generated by the union. The top element is $\mathbb{R}^n$ and the bottom element is $\{0\}$. Moreover, that lattice is either finite or uncountably infinite. See for example Gohberg et al. [6, Proposition 2.5.4].

**Remark 3.4** (i) The lattice of synchrony subspaces is not in general a sublattice of the lattice of the $A$-invariant subspaces. (Recall Remark 3.1.) The meet operation is the same, the intersection of subspaces, but the join of two synchrony subspaces may not be given by their sum. The join of two synchrony subspaces is given by their sum only when this is a polydiagonal subspace. Note that the sum of two synchrony subspaces is always $A$-invariant but it may not be a polydiagonal subspace.

(ii) Apparently, there is no general form for the join operation in the lattice of synchrony subspaces. Nevertheless, the join can be defined in terms of the meet. The join of two synchrony subspaces $V_1$ and $V_2$ is the meet of all the elements in the lattice $V_G$ that are greater than or equal to both $V_1$ and $V_2$.  

\[\diamond\]
Remark 3.5 (i) Kamei [9] describes the lattice of balanced equivalence relations (and so the lattice of synchrony spaces) for regular networks using the eigenvalue structure of the network adjacency matrix for the cases where the adjacency matrix has only simple eigenvalues. We observe that by Corollary 2.10, the results of [9] for regular networks are valid for $k$-dimensional internal dynamics.

(ii) Recalling Remark 3.2, if $G$ is an homogeneous network, then the top element of the lattice $\Lambda G$ of balanced equivalence relations (the balanced equivalence relation with only one class) corresponds to the full synchronous polydiagonal space. The bottom element (the equivalence relation where each class is formed by a unique cell) corresponds to the total asynchronous polydiagonal space.

4 Description of the lattice of synchrony subspaces of a network

The question we address in this section is the description of the lattice of synchrony spaces for a network $G$. As we will see, this can be done in terms of the lattice of synchrony subspaces for the identical-edge subnetworks of $G$ that we define below.

Let $G$ be a coupled cell network with set of cells $C$ and set of arrows $E$. Assume $\sim C$ is an equivalence relation on $C$ where each $\sim C$-class represents a cell type. Also let $\sim E$ be the equivalence relation on $E$ where each $\sim E$-class determines an edge type and denote by $E_i, \ldots, E_l$ the $\sim E$-equivalence classes. Thus $E = \bigcup_j E_j$ where $j$ runs through the set \{1, \ldots, l\} and $\bigcup$ denotes disjoint union. We can write $G = (C, E, \sim C, \sim E)$.

Let $I_1, \ldots, I_k \subset C$ be the $\sim I$-equivalence classes of the cells $C$ in $G$. For $i = 1, \ldots, k$, denote by $E^{I_i}$ the subset of $E$ of the edges that are directed to cells in $I_i$ and $G^{I_i} = (C, E^{I_i}, \sim C, \sim E)$ where two edges $e_1, e_2 \in E^{I_i}$ are equivalent if they are equivalent as edges in $E$ of the network $G$. Thus, each network $G^{I_i}$ is a subnetwork of $G$.

Let $r_i$ be the number of edge types in $G^{I_i}$. For each edge type $E_{i_1}, \ldots, E_{i_{r_i}}$ with $i_1, \ldots, i_{r_i} \in \{1, \ldots, l\}$, consider the subnetwork of $G^{I_i}$ given by $G^{I_i}_{E_{i_j}} = (C, E^{I_i} \cap E_{i_j}, \sim C, \sim E)$. Thus, each $G^{I_i}_{E_{i_j}}$ is the subnetwork of $G^{I_i}$ with the edges of type $E_{i_j}$ directed to the cells in $I_i$.

Recall that $\Lambda G$ denotes the set of balanced equivalence relations for $G$. By Remark 2.6 any $\preceq \in \Lambda G$ refines $\sim I$. Denote by $\Lambda^{I_i}_{G}$ the set of balanced equivalence relations for $G^{I_i}$ that also refine $\sim I$. Finally, let $\Lambda^{E_{i_j}}_{G_{I_i}}$ be the set of balanced equivalence relations for $G^{I_i}_{E_{i_j}}$ (refining $\sim I$). We have that each $\Lambda^{I_i}_{G}$ and each $\Lambda^{E_{i_j}}_{G_{I_i}}$ is a complete lattice.

Example 4.1 Consider the nonhomogeneous 8-cell network of Figure 2. The network has three $\sim I$-equivalence classes of cells, $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$ and $I_3 = \{6, 7, 8\}$, and the networks $G^{I_i}$ are represented in Figure 3.

\diamond
Figure 2: One example of a nonhomogeneous network of 8 cells with three $\sim_I$-equivalence classes of cells: $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$ and $I_3 = \{6, 7, 8\}$.

Figure 3: The three networks $G^{I_i}$ for each $\sim_I$-equivalence class $I_i$ of cells of the 8-cell nonhomogeneous network of Figure 2.

**Theorem 4.2** Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}})$ be a coupled cell network and consider the lattices $\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{G}}^{I_i}, \Lambda_{\mathcal{G}}^{E_{ij}}$ and the notation defined above. Then the following holds:

(i) The set inclusions:
$$\Lambda_{\mathcal{G}} \subseteq \Lambda_{\mathcal{G}}^{I_i} \subseteq \Lambda_{\mathcal{G}}^{E_{ij}}.$$  

(ii) The lattice $\Lambda_{\mathcal{G}}$ is given by:
$$\Lambda_{\mathcal{G}} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{r_i} \Lambda_{\mathcal{G}}^{E_{ij}}.$$
Proof. The proof of the set inclusions in (i) comes directly from the definition of balanced equivalence relation. We prove now (ii). Let \( \bowtie \) be an equivalence relation on the set of cells \( C \) of the network \( G \) refining \( \sim_I \) and recall Definition 2.4 of balanced equivalence relation. Given two cells \( c, d \) such that \( c \bowtie d \), then as \( c \sim_I d \), there always exists an isomorphism \( \beta : I(c) \to I(d) \) preserving the edge equivalence. To check if it is balanced corresponds to check if we find such an isomorphism \( \beta \) that also preserves the \( \bowtie \)–classes of the tails cells of \( I(c) \) and \( I(d) \). It follows that if we consider the subnetworks obtained from \( G \) by considering only edges of one type, say \( G_i = (C, E_i, \sim C, \sim E_i) \), for \( i = 1, \ldots, l \), where \( \sim E_i \) denotes the relation on \( E_i \) with only the class \( E_i \), then trivially \( \bowtie \) is balanced for \( G \) if and only if \( \bowtie \) is balanced for all \( G_i \), \( i = 1, \ldots, l \). Moreover, given \( i \in \{1, \ldots, l\} \), the relation \( \bowtie \) is balanced for the subnetwork \( G_i \) if and only if it is balanced for all the subnetworks \( G_{E_i}^{I_j} \) where \( I_j \) runs through the \( \sim_I \)-equivalence classes of cells having edges of type \( E_i \) directed to them. Observe that \( \bowtie \) is balanced for \( G_{E_i}^{I_j} \) if \( \bowtie \) is balanced for the cells on \( I_j \). This follows from the fact that if \( c, d \in C \setminus I_j \) and \( c \bowtie d \), then as these cells in the network \( G_{E_i}^{I_j} \) have no edges directed to them, then there is nothing to check. \( \square \)

Corollary 4.3 Let \( G = (C, E, \sim C, \sim E) \) be a homogeneous coupled cell network. Denote by \( E_1, \ldots, E_l \) the \( \sim E \)-equivalence classes. For \( j = 1, \ldots, l \), let \( G_{E_j} \) be the subnetwork of \( G \) given by \( (C, E \cap E_j, \sim C, \sim E) \) and \( \Lambda_{G_j}^{E_j} \) be the set of balanced equivalence relations for \( G_{E_j} \). Then the following holds:

\[
\Lambda_G = \bigcap_{j=1}^l \Lambda_{G_j}^{E_j}.
\]

Example 4.4 Consider the homogeneous 5-cell network of Figure 4 with two types of coupling. The two networks \( G_{E_j} \) are represented in Figure 1. \( \checkmark \)

![Figure 4: A 5-cell homogeneous network with two types of coupling.](image)

From Theorem 4.2, the problem of determining the lattice of synchrony subspaces for a network \( G \) basically reduces to determining the synchrony subspaces of the identical-edge

12
subnetworks $G^l_{C_i}$ of $G$. Note that, for each identical-edge subnetwork associated with an $\sim_I$-equivalence class $I_i$, the cells in $I_i$, receive the same number of inputs (of the same type). Roughly speaking, each such network is regular at the cells in the input class. In Sections 7 and 8 we present some results that show how to optimize the process of obtaining the lattice of synchrony subspaces for these subnetworks using an adaptation of Algorithm 6.3 (for regular networks), and we illustrate that with some examples.

5 Description of the lattice of synchrony subspaces of a regular network

In the next sections we concentrate our attention on how to obtain all the synchrony subspaces for a regular network. We assume the cell phase spaces to be $\mathbb{R}$ and basically describe the polydiagonals that are left invariant by the linear admissible vector fields for the network (Corollary 2.10). In particular, the synchrony subspaces form a subset of the set of all invariant subspaces under the network adjacency matrix.

5.1 Invariant subspaces and eigenvectors of the adjacency matrix

The subspaces invariant by the adjacency matrix $A$ of a regular network $G$ can be described in terms of its eigenvectors and generalized eigenvectors.

In what follows we use the following notation for eigenspaces. If $\lambda \in \mathbb{C}$ is an eigenvalue of a $n \times n$ matrix $A$ with real entries, we define the (real) $\lambda$-eigenspace $E_\lambda$ of $A$ as follows:

$$E_\lambda = \begin{cases} \ker (A - \lambda \text{Id}_n), & (\text{if } \lambda \in \mathbb{R}), \\ \ker [(A - \lambda \text{Id}_n) (A - \bar{\lambda} \text{Id}_n)], & (\text{if } \lambda \notin \mathbb{R}). \end{cases}$$

Trivially, if $\lambda$ is a real eigenvalue of $A$ and $v$ is a nonzero vector of $E_\lambda$, that is, $v$ is an eigenvector of $A$, then the one-dimensional subspace spanned by $v$ is invariant by $A$. If $\lambda$ is a complex eigenvalue of $A$ and $u + iv$ is an eigenvector of the complexification $A_c$ of $A$ (that is, interpreting $A$ as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^n$) then the two-dimensional subspace spanned by $\{v, u\}$ is invariant by $A$. Moreover, as proved by Gohberg et al. [6], every one-dimensional $A$-invariant subspace (one-dimensional $A_c$-invariant subspace) is spanned by some eigenvector $v$ of $A$ (eigenvector $u + iv$ of $A_c$).

Using the eigenvectors of $A$ we can obtain more $A$-invariant subspaces $V$ by considering the subspaces generated by any number of eigenvectors $v_1, \ldots, v_p$ of $A$, and $u_{p+1} + iv_{p+1}, \ldots, u_{p+q} + iv_{p+q}$ of $A_c$:

$$V = \langle v_1 \rangle + \cdots + \langle v_p \rangle + \langle u_{p+1}, u_{p+1} \rangle + \cdots + \langle v_{p+q}, u_{p+q} \rangle.$$

We say that $A$ is semisimple when $A_c$ is diagonalizable and, in that case, these are all the $A$-invariant subspaces.

Otherwise, we have to consider generalized eigenvectors and not all the $A$-invariant subspaces correspond to subspaces spanned by eigenvectors of $A$. We have to consider
Jordan chains. If \( \lambda \) is an eigenvalue of \( A \), the chain of vectors \( v_1, \ldots, v_k \) is a Jordan chain of \( A \) corresponding to \( \lambda \) if
\[
\begin{align*}
v_1 \neq 0, \quad & Av_1 = \lambda v_1 \quad \text{and} \quad (A - \lambda \text{Id}_n) v_{j+1} = v_j, \quad \text{for } j = 1, \ldots, k - 1.
\end{align*}
\]
Thus, \( v_1 \) is an eigenvector of \( A \) associated to \( \lambda \) and \( v_2, \ldots, v_k \) are called generalized eigenvectors of \( A \) corresponding to the eigenvalue \( \lambda \) and the eigenvector \( v_1 \). The subspace generated by a Jordan chain of \( A \) is \( A \)-invariant and the vectors in a Jordan chain are linearly independent [6, Propositions 1.3.1 and 1.3.4].

Notice that, since there is an infinity of eigenvectors of \( A \) associated to \( \lambda \), there is an infinity of Jordan chains of \( A \) corresponding to \( \lambda \). Using Jordan chains of \( A \) we can obtain more \( A \)-invariant subspaces by considering the subspaces generated by any Jordan chain of \( A \). Moreover, the subspaces spanned by any number of eigenvectors and any number of Jordan chains of \( A \) are also \( A \)-invariant.

We denote by \( G_\lambda \) the (real) generalized eigenspace of \( A \) associated to the eigenvalue \( \lambda \), which is defined as follows:
\[
G_\lambda = \begin{cases} 
\ker (A - \lambda \text{Id}_n)^p, & (\text{if } \lambda \in \mathbb{R}), \\
\ker [(A - \lambda \text{Id}_n)^p (A - \lambda \text{Id}_n)]^p, & (\text{if } \lambda \not\in \mathbb{R}),
\end{cases}
\]
with \( p \geq 1 \) the minimal integer such that \( \ker (A - \lambda \text{Id}_n)^i = \ker (A - \lambda \text{Id}_n)^p \), for all \( i > p \). The space \( G_\lambda \) is also called the root subspace of \( A \) corresponding to \( \lambda \) and it contains the vectors from any Jordan chain of \( A \) corresponding to \( \lambda \) [6, Proposition 2.1.1].

We have that if \( \lambda_1, \ldots, \lambda_r \) are all the different eigenvalues of the linear transformation \( A \) then \( \mathbb{R}^n \) is the direct sum of the generalized eigenspaces \( G_{\lambda_1}, \ldots, G_{\lambda_r} \):
\[
\mathbb{R}^n = G_{\lambda_1} \oplus \cdots \oplus G_{\lambda_r}.
\]
Moreover, if \( S \) is an \( A \)-invariant subspace then \( S \) decomposes into a direct sum
\[
S = (G_{\lambda_1} \cap S) \oplus \cdots \oplus (G_{\lambda_r} \cap S).
\]
See for example [6, Theorems 2.1.2 and 2.1.5].

### 5.1.1 Irreducible invariant subspaces

We start by recalling the definition of Jordan subspaces and irreducible invariant subspaces \( V \) for linear maps \( A : \mathbb{R}^n \to \mathbb{R}^n \). If \( A \) has complex eigenvalues, we should read \( A_c : \mathbb{C}^n \to \mathbb{C}^n \) and \( V_c \) in what follows.

**Definition 5.1** (a) An \( A \)-invariant subspace \( V \) is called a Jordan subspace corresponding to the eigenvalue \( \lambda \) of \( A \) if \( V \) has a basis consisting of vectors that form a Jordan chain.
(b) An \( A \)-invariant subspace is called irreducible if it cannot be represented as a direct sum of nonzero \( A \)-invariant subspaces.

By [6, Theorem 2.5.1], an \( A \)-invariant subspace \( V \) is irreducible if and only if there is a unique eigenvector (up to multiplication by a scalar) of \( A \) in \( V \) or, equivalently, if and only if \( V \) is a Jordan subspace.
Remark 5.2 Observe that an invariant subspace $V_c$ of $A_c : \mathbb{C}^n \to \mathbb{C}^n$ is irreducible and contains an eigenvector $u + iv$ (with $v \neq 0$) if and only if the corresponding irreducible over the reals contains $<v, u>$. See [6, Theorem 12.2.4, p. 365] for details on the interpretation of this result in case of complex eigenvalues for real operators.

Since every $A$-invariant subspace is a direct sum of irreducible $A$-invariant subspaces, in order to describe all the $A$-invariant subspaces it is sufficient to describe the irreducible $A$-invariant subspaces, that is, the irreducible $A$-invariant subspaces (Jordan subspaces) contained in the generalized eigenspaces associated to each eigenvalue $\lambda$ of $A$.

5.2 Polydiagonals and eigenvectors of the adjacency matrix

Let $V = \{v_1, \ldots, v_k\}$, with $v_i = (v_{i1}, \ldots, v_{in})$, be a set of (generalized) eigenvectors of the adjacency matrix $A$. Consider the following equivalence relation associated with the set $V$:

$$c \equiv_V d \iff v_{ic} = v_{id} \text{ for all } i \in \{1, \ldots, k\}.$$ 

Denote the corresponding polydiagonal subspace by $\Delta_{\equiv_V}$.

Remark 5.3 Let $V = \{v_1, \ldots, v_k\}$, with $v_i = (v_{i1}, \ldots, v_{in})$, be a set of (generalized) eigenvectors of the adjacency matrix $A$, considering that at least one of the vectors $v_i$ is associated with a complex eigenvalue of $A_c$ (or $A$) and where $\text{Im} v_i \neq 0$. Denote by $\text{Re} V = \{\text{Re} v_1, \ldots, \text{Re} v_k\}$, $\text{Im} V = \{\text{Im} v_1, \ldots, \text{Im} v_k\}$ and $V_R = \text{Re} V \cup \text{Im} V$. Then $\equiv_V = \equiv_{V_R}$ and

$$\Delta_{\equiv_V} \cong \Delta_{\equiv_{V_R}}$$

interpreting $\Delta_{\equiv_V}$ as a complex vector space and $\Delta_{\equiv_{V_R}}$ as real vector space. This follows from the fact that

$$v_{ic} = v_{id} \text{ for all } i \in \{1, \ldots, k\} \iff \begin{cases} \text{Re} v_{ic} = \text{Re} v_{id} \\ \text{Im} v_{ic} = \text{Im} v_{id} \end{cases} \text{ for all } i \in \{1, \ldots, k\}.$$

Remark 5.4 (i) $\Delta_{\equiv_V}$ can be equal to $\mathbb{R}^n$ or $\{v \in \mathbb{R}^n : v_i = v_j, \text{ for all } i, j\}$. These are called the trivial polydiagonal subspaces or trivial synchrony subspaces.

(ii) We can have $\Delta_{\equiv_V} = \Delta_{\equiv_W}$ with $V \neq W$. Moreover, we can have $\Delta_{\equiv_V} = \Delta_{\equiv_W}$ with $<V> \neq <W>$.

(iii) Not all polydiagonal subspaces $\Delta_{\equiv_V}$ are synchrony subspaces.

From the discussion in Sections 5.1 and 5.2, it follows that the synchrony subspaces associated to a regular network can be described in terms of the eigenvectors and Jordan chains of its adjacency matrix with nontrivial polydiagonal subspace.
5.3 Sum-dense set for the lattice of synchrony subspaces

Considering the lattice $V_G$ of the synchrony subspaces of a regular network $G$ with adjacency matrix $A$, we aim to describe a set of synchrony subspaces such that every synchrony subspace in $V_G$ can be obtained from that set, a kind of join-dense set of join-irreducible elements in $V_G$.

Note that in the case of the lattice of the $A$-invariant subspaces, in which the join of two subspaces is given by their sum, the set of irreducible invariant subspaces of $A$ forms the set of join-irreducible elements which is join-dense in that lattice – see Section 5.1.1.

As observed in Remark 3.4, we cannot describe explicitly the join operation for the lattice $V_G$ of synchrony subspaces: the sum of two synchrony subspaces may not be a synchrony subspace. However, every synchrony subspace is given by the sum of (irreducible) $A$-invariant subspaces. In fact, defining the concept of sum-irreducible synchrony subspace, more can be said.

**Definition 5.5** A synchrony subspace of a regular network $G$ with adjacency matrix $A$ is called *sum-irreducible* if it cannot be represented as a sum of proper synchrony subspaces of $G$. A synchrony subspace of $G$ which is not irreducible is called *sum-reducible*.

**Proposition 5.6** Every synchrony subspace associated with a regular network $G$ is a sum of sum-irreducible synchrony subspaces.

**Proof** Let $\Delta_{\infty}$ be a synchrony subspace of a regular network $G$ with adjacency matrix $A$ and recall that the lattice of synchrony subspaces is a subset of the lattice of the $A$-invariant subspaces. If $\Delta_{\infty}$ is a join-irreducible $A$-invariant subspace, then it is a sum-irreducible synchrony subspace. If $\Delta_{\infty}$ is not a join-irreducible $A$-invariant subspace, then it is the join of proper $A$-invariant subspaces. The lattice of the $A$-invariant subspaces gives all the possible decompositions of $\Delta_{\infty}$ as the join of proper $A$-invariant subspaces, that is, as the sum of proper $A$-invariant subspaces. If none of these decompositions is formed by synchrony subspaces ($A$-invariant subspaces that are also polydiagonal) then $\Delta_{\infty}$ is sum-irreducible. Otherwise, $\Delta_{\infty}$ is sum-reducible, that is, a sum of proper synchrony subspaces. Recursively, we obtain the result. \qed

In analogy with the definitions of join-dense set and meet-dense set (recall Section 3.1), we define the concept of *sum-dense set*.

**Definition 5.7** A *sum-dense set* for the lattice $V_G$ of synchrony subspaces for a regular network $G$ is a subset of $V_G$ such that every nontrivial synchrony subspace in $V_G$ can be given by the sum of elements in that subset. By Proposition 5.6, the set of sum-irreducible synchrony subspaces in $V_G$ is a sum-dense set, that we call the *irreducible sum-dense set* of $V_G$ and denote by $I_G$.

Note that, as mention in Remark 5.4 (iii), not every polydiagonal subspace $\Delta_{\infty,v}$ is a synchrony subspace. Nevertheless, that polydiagonal is contained in one or more synchrony subspaces. This motivates the following definition.

**Definition 5.8** Given an eigenvector or Jordan chain $V$, let $S_V$ be the set of synchrony subspaces $\Delta_j$, $j = 1, \ldots, r$ such that $\Delta_{\infty,v} \subseteq \Delta_j$. The *minimal synchrony subspace*,...
\( m_V \), associated to the eigenvector or Jordan chain \( V \) is the intersection of the synchrony subspaces in \( S_V \).

**Remark 5.9** (i) Note that \( m_V \) can be equal to \( \mathbb{R}^n \) (or \( \{ v \in \mathbb{R}^n : v_i = v_j, \text{ for all } i, j \} \)). (ii) A polydiagonal \( \Delta_{\infty} \) associated to an eigenvector or Jordan chain \( V \) is a synchrony subspace if and only if the minimal synchrony subspace \( m_V \) associated to \( V \) is \( \Delta_{\infty} \).

**Definition 5.10** Let \( \lambda \) be an eigenvalue of \( A \) with algebraic and geometric multiplicities \( m^a \) and \( m^g \), respectively.

(i) If \( m^a = m^g \), take \( V^\Delta_{\lambda} \) to be the set of the eigenvectors \( v \) associated with the eigenvalue \( \lambda \) such that \( \Delta \) \( \triangleleft \downarrow \) \( \{ v \} \) is not \( \mathbb{R}^n \). That is, \( V^\Delta_{\lambda} = \{ v \in E_\lambda : v_p = v_l \text{ for some } p \neq l \} \).

(ii) If \( m^a \neq m^g \), take \( V^\Delta_{\lambda} \) to be the set of Jordan chains \( v_1, \ldots, v_k \) corresponding to the eigenvalue \( \lambda \), where \( v_1 \) is an eigenvector associated with \( \lambda \) and \( \Delta \) \( \triangleleft \downarrow \) \( \{ v_1, \ldots, v_k \} \) is not \( \mathbb{R}^n \). That is, \( V^\Delta_{\lambda} = \{ v_1, \ldots, v_k : v_1 \in E_\lambda, (A - \lambda \text{Id})v_{j+1} = v_j \text{ for } j = 1, \ldots, k - 1 \text{ and } v_{jp} = v_{jl} \text{ for all } j = 1, \ldots, k \text{ and some } p \neq l \} \).

If we denote by \( V^\Delta_A \) the set of eigenvectors of \( A \) and corresponding Jordan chains with nontrivial polydiagonal subspace, then

\[
V^\Delta_A = \bigcup_{i=1}^{s} V^\Delta_{\lambda_i}
\]
with \( \lambda_i \), for \( i = 1, \ldots, s \), where \( s \leq n \), the eigenvalues of the matrix \( A \).

**Theorem 5.11** *The minimal synchrony set,*

\[
m_G = \bigcup_{v_i \in V^\Delta_A} m_{V_i},
\]

associated to the eigenvectors and Jordan chains in \( V^\Delta_A \) is a sum-dense set for the lattice \( V_G \) of synchrony subspaces.

**Proof** Let \( \Delta_{\infty} \) be a nontrivial synchrony subspace in \( V_G \). We prove that \( \Delta_{\infty} \) can be given by a sum of synchrony subspaces in \( m_G \).

Since \( \Delta_{\infty} \) is \( A \)-invariant, there is a decomposition of \( \Delta_{\infty} \) into a direct sum of \( A \)-invariant irreducible subspaces, say \( S_i \), for \( i = 1, \ldots, p \),

\[
\Delta_{\infty} = S_1 \oplus \cdots \oplus S_p,
\]
where each \( S_i \) is a Jordan subspace (recall Section 5.1.1). Consider a Jordan chain \( \overline{V}_i \in V^\Delta_A \) which forms a basis of \( S_i \). Thus \( S_i = \langle \overline{V}_i \rangle \subseteq \Delta_{\infty} \subseteq \Delta_{\infty} \). Moreover, by the definition of \( m_{\overline{V}_i} \), we have that \( \Delta_{\infty} \subseteq m_{\overline{V}_i} \subseteq \Delta_{\infty} \). Thus

\[
\Delta_{\infty} = m_{\overline{V}_1} + \cdots + m_{\overline{V}_p}.
\]

\( \square \)
Corollary 5.12 We have the following inclusion of sum-dense sets:

\[ \mathcal{I}_G \subseteq m_G. \]

**Proof** Let \( \Delta_\infty \in \mathcal{I}_G \). By Theorem 5.11, there are minimal synchrony subspaces associated with Jordan chains \( \overline{V_i} \in V^A \) such that

\[ \Delta_\infty = m\overline{V_1} + \cdots + m\overline{V_p}. \]

Since \( \Delta_\infty \) is sum-irreducible, then \( \Delta_\infty = m\overline{V_i} \) for some \( i \) and so \( \Delta_\infty \in m_G \).

Remark 5.13 The algorithm presented in the next section to obtain the lattice \( V_G \) for a given regular network \( G \), is based on the Corollary 5.12. It starts by finding a subset of \( V_G \) containing \( m_G \). It then extracts the subset \( \mathcal{I}_G \) from that subset. Moreover, recall that \( \mathcal{I}_G \) is a sum-dense set of \( V_G \) by Proposition 5.6. Thus, the lattice \( V_G \) is obtained from the set \( \mathcal{I}_G \) using the sum operation of spaces.  

6 Algorithm for regular networks

Based on the results in the previous section, we present an algorithm to find all the nontrivial synchrony subspaces for a regular \( n \)-cell network \( G \), recalling that, by Corollary 2.10, we may assume that the phase space is \( R^n \).

6.1 Fundamental steps of the algorithm

Let \( G \) be a regular \( n \)-cell network with adjacency matrix \( A \). The algorithm that we define contains four fundamental steps:

**Step 1** Find the polydiagonal subspaces associated with the generalized eigenvectors of the adjacency matrix \( A \).

The algorithm starts by finding the polydiagonal subspaces corresponding to equivalence relations associated with the generalized eigenvectors of the matrix \( A \): for each eigenspace \( E_{\lambda_i} \) of \( A \), and for each polydiagonal subspace \( \Delta_\infty \) of \( R^n \) containing generalized eigenvectors associated with eigenvectors in \( E_{\lambda_i} \), step 1 of the algorithm determines the dimension of \( G_{\lambda_i} \cap \Delta_\infty \), that is, the number of linearly independent generalized eigenvectors in \( G_{\lambda_i} \) belonging to \( \Delta_\infty \). Now observe that a vector \( x = (x_1, \ldots, x_n) \in R^n \) belongs to \( \Delta_\infty \) if and only if it satisfies the finite set \( C \) of coordinate equality conditions of the form \( x_i = x_j \) where \( i, j \in \{1, \ldots, n\} \) defining the polydiagonal \( \Delta_\infty \). Thus

\[ G_{\lambda_i} \cap \Delta_\infty = \{x \in G_{\lambda_i} : x \text{ is a solution of } C\}. \]

Equivalently, the dimension of \( G_{\lambda_i} \cap \Delta_\infty \) is the number of linearly independent solutions of the linear homogeneous system \( C \) solved in \( G_{\lambda_i} \). Moreover, we have \( G_{\lambda_i} \cap \Delta_\infty \neq \{0\} \) if and only if that system is undetermined. This step of the algorithm relies on Lemma 6.1 below implying that a system of coordinate equality conditions in \( E_{\lambda_i} \) (or \( G_{\lambda_i} \)) is equivalent to a linear homogeneous system involving the vectors of any basis of \( E_{\lambda_i} \) (or \( G_{\lambda_i} \)).
Note that since the valency $v$ of a regular network is an eigenvalue of the network adjacency matrix and $E_v = \langle (1, \ldots, 1) \rangle$, there is no need for the algorithm to check the equality conditions for the vectors in $E_v$, since they verify all the conditions of equality of coordinates.

From this step, we collect a set of polydiagonal subspaces associated with all the generalized eigenvectors in $V^\Delta_A$.

**Step 2** Find a sum-dense set containing the minimal synchrony set for the lattice $V^\Delta_G$ of synchrony subspaces for $G$.

For each polydiagonal listed in Step 1 of the algorithm, this step checks if it is a synchrony space: a polydiagonal is a synchrony space if and only if the number of generalized eigenvectors found in step 1 associated with that polydiagonal plus one equals the dimension of the polydiagonal. At the end of step 2, we have found a set $S$ of synchrony subspaces containing the minimal synchrony set for the lattice $V^\Delta_G$ of synchrony subspaces. Recall Theorem 5.11 where it is proved that the minimal synchrony set forms a sum-dense set for the lattice $V^\Delta_G$. Thus, the set $S$ is a sum-dense set for $V^\Delta_G$.

**Step 3** Find the irreducible sum-dense set $I^\Delta_G$ of the lattice $V^\Delta_G$.

In this step, we extract the set $I^\Delta_G$ of the irreducible synchrony subspaces from the sum-dense set $S$ of synchrony subspaces obtained in step 2: these are sufficient (and necessary) to generate $V^\Delta_G$. Recall Remark 5.13.

**Step 4** Generate the complete lattice $V^\Delta_G$.

After finding the sum-dense set $I^\Delta_G$ of irreducible synchrony subspaces, the remaining synchrony subspaces in the lattice are given by the possible sums of elements in $I^\Delta_G$ that are polydiagonals.

The next lemma justifies Step 1 of the algorithm.

**Lemma 6.1** Let $E$ be a subspace of $\mathbb{R}^n$ of dimension $m$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and consider a system $C$ with $s$ equations of the form $x_{l_1} = x_{l_2}$, where $l_1, l_2 \in \{1, \ldots, n\}$. Then:

1. Solving the system $C$ in $E$ is equivalent to solve the system $\overline{C}$ with $s$ equations of the form $\sum_{j=1}^m \alpha_j (v_{jl_1} - v_{jl_2}) = 0$, in the unknowns $\alpha_j \in \mathbb{R}$ for $j = 1, \ldots, m$ and $(v_1, v_2, \ldots, v_m)$ a basis of $E$, where $v_j = (v_{j1}, \ldots, v_{jn}) \in \mathbb{R}^n$.

2. If $r$ is the rank of the matrix of the system $\overline{C}$, then there are $m - r$ linearly independent vectors in $E$ satisfying the system $C$.

**Proof** Let $(v_1, v_2, \ldots, v_m)$ be a basis of $E$. Then, for every vector $x = (x_1, \ldots, x_n)$ in $E$ there are unique $\alpha_j \in \mathbb{R}$ with $j = 1, \ldots, m$, such that $x = \sum_{j=1}^m \alpha_j v_j$. Thus, each coordinate equality condition $x_{l_1} = x_{l_2}$ is equivalent to $\sum_{j=1}^m \alpha_j v_{jl_1} = \sum_{j=1}^m \alpha_j v_{jl_2}$ and thus to $\sum_{j=1}^m \alpha_j (v_{jl_1} - v_{jl_2}) = 0$.

Note that the linear systems $C$ and $\overline{C}$ are homogeneous. The degree of indetermination of both systems is $m - r$, with $r$ the rank of the matrix of the system $\overline{C}$, and indicates
the number of independent variables $\alpha_j$ in the solution of system $\mathbf{C}$ and the number of linearly independent vectors in $E$ satisfying the coordinate equality conditions in system $C$.

**Remark 6.2** Let $A$ be a real square matrix of order $n$. For every eigenvalue $\lambda_i$ of $A$, let $m^a_i$ and $m^g_i$ be, respectively, the algebraic and geometric multiplicity of $\lambda_i$. Let $C$ be a set of $s$ conditions of the form $x_{l_1} = x_{l_2}$, where $l_1, l_2 \in \{1, \ldots, n\}$. Applying Lemma 6.1, where $E = E_{\lambda_i}$ and so $m = m^g_i$ indicates the dimension of the eigenspace $E_{\lambda_i}$, we have:

(i) If $m^g_i = 1$ then there is at most one linearly independent eigenvector in $E_{\lambda_i}$ satisfying the set $C$ of $s$ conditions.

(ii) Suppose $m^g_i > 1$. If $s < m^g_i$ then there is at least one linearly independent eigenvector satisfying the conditions. If $s \geq m^g_i$ there can be none.

**6.2 Algorithm**

We present now an algorithm to obtain the lattice of synchrony subspaces of a regular $n$-cell network $\mathcal{G}$. If the adjacency matrix $A$ of $\mathcal{G}$ has complex eigenvalues, the calculations are done on $\mathbb{C}^n$, that is, interpreting $A$ as $A_c : \mathbb{C}^n \rightarrow \mathbb{C}^n$. Recall Remark 5.3.

**Algorithm 6.3** Let $A$ be the adjacency matrix of a regular $n$-cell network $\mathcal{G}$, with valency $v$. Let $\lambda_i$ with $i = 1, \ldots, t$ and $t \leq n$ be the eigenvalues of $A$, with $m^a_i$ and $m^g_i$, respectively, the algebraic and geometric multiplicities, where $\lambda_1 = v$.

1 [Find polydiagonals] For each eigenvalue $\lambda_i$, $i = 2, \ldots, t$ of $A$:

1.1 Let $(v_1, \ldots, v_{m^a_i})$ be a basis of $E_{\lambda_i}$. Consider the matrix $M$ whose columns correspond to the eigenvectors $v_1, \ldots, v_{m^a_i}$.

1.2 Let $C = \emptyset$. For every pair of rows $l_j, l_k$ of $M$:

1.2.1 If $l_j = l_k$ then $C = C \cup \{x_j = x_k\}$ and eliminate row $l_k$ of $M$. ¹

1.3 Construct a four-column table for $E_{\lambda_i}$ with a row containing: in the first entry the set of the equality conditions $C$ found in step 1.2; in the second entry the corresponding polydiagonal dimension; in the third entry the basis of eigenspace $E_{\lambda_i}$; in the fourth entry the number of vectors of the basis, $m^g_i$.

1.4 Let $s$ be the number of remaining rows in $M$. Construct a new matrix $\overline{M}$ with rows given by $r_j - r_k$ for $j = 1, \ldots, s$ and $k = j + 1, \ldots, s$ where $r_j, r_k$ are rows in $M$. Thus $\overline{M}$ has $\overline{s} = s(s - 1)/2$ rows, each corresponding to an equality $x_j = x_k$ with $j, k \in \{1, \ldots, n\}$.

1.5 Let $S$ be the set of all the submatrices of $\overline{M}$ with $s - 2$ rows obtained from $\overline{M}$ by elimination of rows. ²

1.6 While $S \neq \emptyset$,

¹If rows $j$ and $k$ of $M$ are equal that means that $x_j = x_k$ for all vectors $x = (x_1, \ldots, x_n) \in E_{\lambda_i}$, and we can eliminate one of those rows. See Remark 6.6 (i).

²See Remark 6.6 (ii).
1.6.1 Let $N$ be a submatrix in $S$ and $S = S \setminus \{N\}$.

1.6.2 Let $r$ be the rank of $N$;

1.6.3 If $r < m_i^g$ then:

1.6.3.1 Let $C_N$ be the set of equalities given by the rows of $N$ and $C$ be the set of equalities obtained in step 1.2. If there is no row in the table of $E_{\lambda_i}$ corresponding to set of equalities $C \cup C_N$ then add a new row to the table containing: in the first entry $C \cup C_N$; in the second entry the corresponding polydiagonal dimension; in the third entry a basis of the subspace of the eigenvectors in $E_{\lambda_i}$ that satisfy the set of equality conditions (obtained from the solution set of the homogeneous system with the coefficient matrix $N$), and in the fourth entry the number of vectors of the basis, $m_i^g - r$.

1.6.4 Otherwise, $r = m_i^g$:

1.6.4.1 Consider the set $S_N$ of all the submatrices of $N$ obtained by eliminating one row of $N$.

1.6.4.2 Let $S = S \cup S_N$.

1.7 If $m_i^g < m_i^g$ then:

1.7.1 Compute a basis of $\text{Im} (A - \lambda_i \text{Id}_n)$.

1.7.2 For each row in the table for $E_{\lambda_i}$,

1.7.2.1 If the intersection of the subspace corresponding to the basis in that row with $\text{Im} (A - \lambda_i \text{Id}_n)$ is a nonzero subspace then:

Let $B_1$ be a basis of that intersection;

Let $C$ be the first entry of the row (the set of equality conditions);

JordanChain($B_1, C, 2$).

2 [Find sum-dense set] Consider the empty set $S$. For each table, for each row of the table:

2.1 Let $C$ be the set of equality conditions in that row and $d$ the dimension of the polydiagonal subspace $\Delta_{\infty}$ given by those conditions.

2.2 If the number of vectors in that row of the table equals $d - 1$ then there is an eigenvector basis of $\Delta_{\infty}$ (considering also the eigenvector $(1, ...1)$) and thus $\Delta_{\infty}$ is a synchrony subspace. Let $S = S \cup \{\Delta_{\infty}\}$.

2.3 If the number of vectors in that row of the table is less than $d - 1$ and there are more tables, then look at the other tables to find all the rows whose equality conditions include the set $C$ of equality conditions.

2.3.1 If the total sum of the number of vectors equals $d - 1$ then there is an eigenvector basis of $\Delta_{\infty}$ (considering also the eigenvector $(1, ...1)$) and thus $\Delta_{\infty}$ is a synchrony subspace. Let $S = S \cup \{\Delta_{\infty}\}$.

\[3\text{See Remark 6.2.}\]

\[4\text{This step finds a sum-dense set containing the minimal synchrony set for the lattice of synchrony subspaces. Recall Theorem 5.11 where it is proved that the minimal synchrony set forms a sum-dense set for the lattice of synchrony subspaces.}\]
2.3.2 If the total sum of the number of vectors is still less than \( d - 1 \) then:

2.3.2.1 Eliminate that row of the table.

2.3.2.2 Let \( c = \#C \). For each subset of \( c - 1 \) conditions of the initial set \( C \) of \( c \) conditions:

If there is no row at the table with that set of \( c - 1 \) conditions then add a new row to the end of the table differing from the deleted row only at the first and second entries: the first entry contains the set of the \( c - 1 \) conditions and the second entry is \( n - c + 1 \), the dimension of the corresponding polydiagonal.

Otherwise, change the corresponding row: replacing the third entry by a basis of the subspace generated by the union of the bases in this row and the one in the deleted row; changing the fourth entry by the number of vectors of that basis. Move that row to the end of the table.

3 [Find the irreducible sum-dense set] Decompose \( S \) into the disjoint union \( \bigcup_{i=1}^{r} S_{j_i} \), where each set \( S_{j_i} \) contains the synchrony subspaces in \( S \) of dimension \( j_i \), with \( j_{i-1} < j_i \), for \( i = 2, \ldots, r \). Let \( \mathcal{I}_G = S_{j_1} \).

3.1 For \( i = 2 \) to \( r \):

3.1.1 For each subspace \( E \) in \( S_{j_i} \), if it is not a sum of subspaces in \( \mathcal{I}_G \), then let \( \mathcal{I}_G = \mathcal{I}_G \cup E \).

4 [Find the lattice] Let \( V_G = \text{Sum}(\mathcal{I}_G) \). Return(\( \mathcal{I}_G, V_G \))

Algorithm 6.4 [JordanChain(\( B_{k-1}, C, k \))]

1 Let \( V_{k-1} \) be the subspace generated by the basis \( B_{k-1} \).

2 Let \( V_k \) be the subspace of vectors \( v_k \) that satisfy \((A - \lambda_i \text{Id}_n) v_k = v_{k-1} \) for some \( v_{k-1} \in V_{k-1} \).\(^5\)

3 Let \( B_C \) be the basis at the third entry in the table for \( E_{\lambda_i} \) corresponding to the set of equality conditions \( C \).

4 If \( B_C \) is a basis of \( V_k \), then exit the JordanChain routine.

5 Complete the basis \( B_C \) with a set \( \overline{B}_k \) of vectors forming a basis of \( V_k \).\(^6\)

6 Consider the matrix \( M \) whose columns are the vectors of the basis \( \overline{B}_k \).

7 Construct a new matrix \( \overline{M} \) with rows given by \( r_j - r_k \), with \( r_j \) and \( r_k \) rows in \( M \), whenever \( x_j = x_k \) is in \( C \).\(^7\)

\(^5\) \( V_k \) is a subspace of \( \ker (A - \lambda_i \text{Id}_n)^k \).

\(^6\) \( < B_C > \subseteq \ker (A - \lambda_i \text{Id}_n)^{k-1} \subseteq V_k \).

\(^7\) If row \( r_j - r_k \) of \( \overline{M} \) is zero, that means that \( x_j = x_k \) for all vectors in \( V_k \).
Let $S$ be the set of all submatrices of $\overline{M}$ of rank less than $\#B_k$ obtained from $\overline{M}$ by elimination of rows.

Decompose $S$ into disjoint union $\bigcup_{i=0}^{\#B_k-1} S_i$, where each $S_i$ is the set of all matrices in $S$ with rank $i$. For each $S_i$ remove any matrix $N$ that is a submatrix of a matrix in $S_i$ different from $N$.

If $S \neq \emptyset$ then, for $i = 0$ to $\#B_k - 1$:

10.1 While $S_i \neq \emptyset$ do:

10.1.1 Let $N \in S_i$ and $S_i = S_i \setminus \{N\}$.

10.1.2 Let $\overline{B}_k$ be a basis of the subspace of $<\overline{B}_k>$ obtained from the solution set of the homogeneous system with the coefficient matrix $N$.

10.1.3 Let $C_N$ be the set of equality conditions corresponding to the rows of $N$.

10.1.4 If $C_N = C$, then change the row corresponding to the set $C$: replacing the third entry by the basis $B = B_C \cup \overline{B}_k$ and the fourth entry by $\#B$.

Otherwise, if there is no row at the table with the set of conditions $C_N$, then add a new row at the top of the table containing: in the first entry $C_N$; in the second entry the corresponding polydiagonal dimension; in the third entry the basis $B = B_C \cup \overline{B}_k$; in the fourth entry $\#B$.

Else, go to step 10.1.

10.1.5 If the intersection of the subspace corresponding to the basis $\overline{B}_k$ with $\text{Im}(A - \lambda_i \text{Id}_n)$ is a nonzero subspace then:

10.1.5.1 Let $B_k$ be a basis of the intersection $<B> \cap \text{Im}(A - \lambda_i \text{Id}_n)$.

10.1.5.2 JordanChain($B_k, C_N, k + 1$).

\[\diamond\]

**Algorithm 6.5 [Sum($\mathcal{IG}$)]**

The set $\mathcal{IG}$ contains the irreducible sum-dense set of the lattice $V_G$.

1 Let $V_G = \mathcal{IG}$.

2 Let $s = \#\mathcal{IG}$.

3 For $i = 2$ to $s$,

3.1 For every (possible) subset $\{\Delta_{\infty j_1}, \ldots, \Delta_{\infty j_l}\}$, with $j_k \neq j_l$, of $i$ synchrony subspaces in $\mathcal{IG}$,

3.1.1 Let $\Delta_{\infty} = \Delta_{\infty j_1} + \cdots + \Delta_{\infty j_l}$.

\[\text{Equivalently, } C_N \text{ is the set of equality conditions satisfied by the vectors in } <\overline{B}_k>\text{.}\]
3.1.2 If $\Delta_\infty$ is a polydiagonal subspace then let $V_G = V_G \cup \{\Delta_\infty\}$.

4 Return $V_G \cup \{\Delta_0\} \cup \{P\}$ where $\Delta_0$ is the full synchronous polydiagonal space.

\[ \Diamond \]

**Remark 6.6** (i) The procedure in step 1.2 of Algorithm 6.3 optimizes the execution of the algorithm in the case where there is a set $C$ of coordinate equalities satisfied by all the vectors in $E_{\lambda_i}$. Thus, given an eigenvector $v \in E_{\lambda_i}$, the polydiagonal $\Delta_{\infty v}$ is a subspace of the polydiagonal corresponding to the set $C$. Now recall that the minimal synchrony set $m_G$ is a sum-dense set for the lattice $V_G$ (Theorem 5.11). This guarantees that at step 2 of the algorithm we obtain a set of synchrony subspaces containing the minimal synchrony set for the lattice of synchrony subspaces and so forming a sum-dense set for that lattice.

(ii) Each row $r_j - r_k$ of the matrix $\overline{M}$ in step 1.4 of the Algorithm 6.3 corresponds to a set of coordinate equalities of the form $x_j = x_k$ for $j \neq k$. (Recall that if $x_{l_1} = x_{l_2} \in C$ then all vectors $x = (x_1, \ldots, x_n) \in E_{\lambda_i}$ satisfy $x_{l_1} = x_{l_2}$.) So, each submatrix of $\overline{M}$ in step 1.5 of the algorithm corresponds to a system of coordinate equalities. Moreover, the reason to consider only submatrices of $\overline{M}$ with rows up to $s - 2$ is that the minimum dimension for any nontrivial polysynchronous subspace is 2 and there are $s$ independent coordinate variables. The condition that the rank of the submatrices of $\overline{M}$ is less than $m_i^2$ guarantees that there is at least one nonzero vector in $E_{\lambda_i}$ satisfying the equality conditions corresponding to the submatrix, see 2. of Lemma 6.1.

\[ \Diamond \]

### 6.3 Examples

In this section we illustrate the implementation of the Algorithm 6.3 with five regular network examples: three where the adjacency matrix is semi-simple and two where it is not the case.

#### 6.3.1 Semi-simple adjacency matrix

**Example 6.7** Consider the 5-cell regular network $G$ on the left of Figure 1 and recall the associated coupled cell systems at Example 2.3. Using Algorithm 6.3, we obtain all the nontrivial synchrony spaces associated to the network $G$, see Table 1.

| $\Delta_1 = \{x : x_2 = x_3\}$ | $\Delta_4 = \{x : x_2 = x_3 = x_5\}$ | $\Delta_5 = \{x : x_3 = x_4 = x_5\}$ |
| $\Delta_2 = \{x : x_3 = x_5\}$ | $\Delta_6 = \{x : x_4 = x_5\}$ | $\Delta_{10} = \{x : x_1 = x_2, x_3 = x_4 = x_5\}$ |
| $\Delta_3 = \{x : x_4 = x_5\}$ | $\Delta_7 = \{x : x_1 = x_4, x_2 = x_3\}$ | $\Delta_{11} = \{x : x_1 = x_4, x_2 = x_3 = x_5\}$ |
| $\Delta_8 = \{x : x_2 = x_3, x_4 = x_5\}$ | $\Delta_{12} = \{x : x_1 = x_2 = x_3, x_4 = x_5\}$ |
| $\Delta_9 = \{x : x_2 = x_4, x_3 = x_5\}$ | $\Delta_{13} = \{x : x_1 = x_4 = x_5, x_2 = x_3\}$ |
| $\Delta_{14} = \{x : x_2 = x_3 = x_4 = x_5\}$ |

Table 1: Nontrivial synchrony subspaces for the network on the left of Figure 1.
We illustrate now the implementation of Algorithm 6.3.

**Step 1.** The adjacency matrix of $G$ is

$$A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}$$

and it has eigenvalues $2, -1, 0$, with algebraic (and geometric) multiplicities $1, 2, 2$, respectively. The associated eigenspaces (in $\mathbb{R}^5$) are $E_2 = \langle (1, 1, 1, 1, 1) \rangle$, $E_{-1} = \langle (1, -1, -1, 0, 0), (1, 0, 0, -1, -1) \rangle$ and $E_0 = \langle (1, 0, -1, 0, -1), (0, 1, 0, -1, 0) \rangle$.

**Steps 1.1-1.6 for $E_{-1}$:** let

$$M = \begin{bmatrix}
1 & 1 \\
-1 & 0 \\
-1 & 0 \\
0 & -1 \\
0 & -1
\end{bmatrix}.$$

As rows 2, 3 are equal and rows 4, 5 are equal we have that vectors $x = (x_1, \ldots, x_5) \in E_{-1}$ satisfy $x_2 = x_3$ and $x_4 = x_5$. We eliminate rows 3 and 5 of $M$ and create Table 2. Thus,

$$M = \begin{bmatrix}
1 & 1 \\
-1 & 0 \\
0 & -1
\end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix}
2 & 1 \\
1 & 2 \\
-1 & 1
\end{bmatrix}.$$

As $M$ has three rows, we have $s = 3$, and so, it is sufficient to consider the submatrices of $\overline{M}$ with $3 - 2 = 1$ row. The first row of $\overline{M}$ corresponds to the equality $x_1 = x_2$, the second row to $x_1 = x_4$ and the third to $x_2 = x_4$. Considering all the submatrices formed by one row of $\overline{M}$, that is,

$$S = \{ [21], [12], [-11] \},$$

we have that Table 2 transforms to Table 3.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = x_3, x_4 = x_5$</td>
<td>3</td>
<td>$((1, -1, -1, 0, 0), (1, 0, 0, -1, -1))$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2: Incomplete table for $E_{-1}$.

**Steps 1.1-1.6 for $E_0$:** let

$$M = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 \\
0 & -1 \\
-1 & 0
\end{bmatrix}.$$
<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = x_3, x_4 = x_5$</td>
<td>3</td>
<td>$((1, -1, -1, 0, 0), (1, 0, 0, -1, -1))$</td>
<td>2</td>
</tr>
<tr>
<td>$x_1 = x_2 = x_3, x_4 = x_5$</td>
<td>2</td>
<td>$((1, 1, 1, -2, -2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5, x_2 = x_3$</td>
<td>2</td>
<td>$((1, -2, -2, 1, 1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_3 = x_4 = x_5$</td>
<td>2</td>
<td>$((2, -1, -1, -1, -1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3: Complete table for $E_{-1}$.

and observe that row 3 is equal to row 5. Therefore $x_3 = x_5$ for the eigenvectors $x$ in $E_0$.

We then eliminate row 5 of $M$ creating Table 4 and obtaining

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3 = x_5$</td>
<td>4</td>
<td>$((1, 0, -1, 0, -1), (0, 1, 0, -1, 0))$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4: Incomplete table for $E_0$.

We have $s = 4$, and so, it is sufficient to consider the submatrices of $\overline{M}$ with $4 - 2 = 2$ rows. The rows of $\overline{M}$ correspond, respectively, to the equalities $x_1 = x_2, x_1 = x_3, x_1 = x_4, x_2 = x_3, x_2 = x_4$ and $x_3 = x_4$. The submatrices of $\overline{M}$ with 2 rows and rank less than 2 are

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

These submatrices correspond, respectively, to the conditions $x_1 = x_2, x_3 = x_4 = x_5$ and $x_1 = x_4, x_2 = x_3 = x_5$. We have then that the set $S$ in step 1.6 is

$$S = \left\{ [1 \ 1], \ [2 \ 0], \ [1 \ 1], \ [1 \ 1], \ [0 \ 2], \ [-1 \ 1], \ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$ 

Considering this information, Table 4 transforms to Table 5.

Steps 2.1-2.3 Consider the first row in Table 3 of $E_{-1}$. The polydiagonal subspace of $\mathbb{R}^5$ of the vectors $(x_1, x_2, x_3, x_4, x_5)$ satisfying the conditions $x_2 = x_3, x_4 = x_5$ is 3-dimensional. The two linearly independent eigenvectors of $E_{-1}$ that verify those conditions and the eigenvector $(1, 1, \ldots, 1)$ form a basis of that polydiagonal subspace. Thus, it is a synchrony subspace. The same holds for the other three rows of Table 3 obtaining so the following synchrony subspaces:

$$\Delta_8 = \{ x : x_2 = x_3, x_4 = x_5 \}, \quad \Delta_{12} = \{ x : x_1 = x_2 = x_3, x_4 = x_5 \},$$

$$\Delta_{13} = \{ x : x_1 = x_4 = x_5, x_2 = x_3 \}, \quad \Delta_{14} = \{ x : x_2 = x_3 = x_4 = x_5 \}.$$
<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_3 = x_5)</td>
<td>4</td>
<td>(((1, 0, -1, 0, -1), (0, 1, 0, -1, 0)))</td>
<td>2</td>
</tr>
<tr>
<td>(x_1 = x_2, x_3 = x_5)</td>
<td>3</td>
<td>(((1, 1, -1, -1, -1)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_1 = x_3 = x_5)</td>
<td>3</td>
<td>(((0, 1, 0, -1, 0)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_1 = x_4, x_3 = x_5)</td>
<td>3</td>
<td>(((1, -1, -1, 1, -1)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_2 = x_3 = x_5)</td>
<td>3</td>
<td>(((1, -1, -1, 1, -1)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_2 = x_4, x_3 = x_5)</td>
<td>3</td>
<td>(((1, 0, -1, 0, -1)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_3 = x_4 = x_5)</td>
<td>3</td>
<td>(((1, 1, -1, -1, -1)))</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5: Complete table for \(E_0\).

The polydiagonal subspace \(F\) of the vectors \((x_1, x_2, x_3, x_4, x_5)\) (in \(\mathbb{R}^5\)) satisfying the condition \(x_3 = x_5\) which appears in the first row of Table 5 for \(E_0\) is 4-dimensional. Besides the eigenvector \((1, 1, \ldots, 1)\) there are two linearly independent eigenvectors of \(E_0\) in \(F\) and one linearly independent eigenvector of \(E_{-1}\) (satisfying \(x_2 = x_3 = x_4 = x_5\)) in \(F\) (fourth row of Table 3). Thus, there is an eigenvector basis for the subspace and so, it is a synchrony subspace, \(\Delta_2\).

The subspace \(F\) of the vectors \((x_1, x_2, x_3, x_4, x_5)\) in \(\mathbb{R}^5\) satisfying the conditions \(x_1 = x_2, x_3 = x_5\) in the second row of Table 5 for \(E_0\) is 3-dimensional. Besides the eigenvector \((1, 1, \ldots, 1)\), there is only one linearly independent eigenvector of \(E_0\) in \(F\) and no eigenvector of \(E_{-1}\) in \(F\). Thus, there is no eigenvector basis for the subspace and so, it is not a synchrony subspace. The same happens for the next two rows of Table 5 for \(E_0\). Applying the step 2.3.2 to these three rows, we have to add the following five rows to the Table 5:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1 = x_2)</td>
<td>4</td>
<td>(((1, 1, -1, -1, -1)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_1 = x_3)</td>
<td>4</td>
<td>(((0, 0, -1, 0)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_1 = x_5)</td>
<td>4</td>
<td>(((0, 1, 0, -1, 0)))</td>
<td>1</td>
</tr>
<tr>
<td>(x_1 = x_4)</td>
<td>4</td>
<td>(((1, -1, -1, 1, -1)))</td>
<td>1</td>
</tr>
</tbody>
</table>

However, applying the steps 2.1-2.3 to these rows we obtain no synchrony subspaces.

The subspace \(F\) of the vectors \((x_1, x_2, x_3, x_4, x_5)\) in \(\mathbb{R}^5\) satisfying the conditions \(x_2 = x_3 = x_5\) in row five of Table 5 for \(E_0\) is 3-dimensional. Besides the eigenvector \((1, 1, \ldots, 1)\) there is one linearly independent eigenvector of \(E_0\) in \(F\) and one linearly independent eigenvector of \(E_{-1}\) (satisfying \(x_2 = x_3 = x_4 = x_5\)) in \(F\). Thus, there is an eigenvector basis for the subspace and so, it is a synchrony subspace, \(\Delta_4\). The same happens for the subspaces of \(\mathbb{R}^5\) defined by the conditions in the next two rows of the table for \(E_0\): we obtain \(\Delta_9\) and \(\Delta_5\).

For the last two rows of Table 5 for \(E_0\), the subspace of \(\mathbb{R}^5\) of the vectors \((x_1, x_2, x_3, x_4, x_5)\) satisfying the conditions in each row is 2-dimensional. Besides the eigenvector \((1, 1, \ldots, 1)\) there is one more linearly independent eigenvector in \(E_0\) that verifies those conditions.
Thus, it is a synchrony subspace. We have then more six synchrony subspaces:

\[ \Delta_2 = \{ x \ : x_3 = x_5 \}, \quad \Delta_4 = \{ x \ : x_2 = x_3 = x_5 \}, \]
\[ \Delta_9 = \{ x \ : x_2 = x_4, x_3 = x_5 \}, \quad \Delta_5 = \{ x \ : x_3 = x_4 = x_5 \}, \]
\[ \Delta_{10} = \{ x \ : x_1 = x_2, x_3 = x_4 = x_5 \}, \quad \Delta_{11} = \{ x \ : x_1 = x_4, x_2 = x_3 = x_5 \}. \]

At the end of the step 2.3, the sum-dense set \( S \) of synchrony subspaces (which contains the minimal synchrony set \( m_G \) as defined in Theorem 5.11), obtained directly from Tables 3 and 5, is

\[ S = \{ \Delta_2, \Delta_4, \Delta_5, \Delta_8, \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14} \}. \]

**Step 3** Observe that each of the synchrony subspaces \( \Delta_{10}, \ldots, \Delta_{14} \) is two-dimensional and so it is sum-irreducible. Moreover,

\[ \Delta_2 = \Delta_{10} + \Delta_{11} + \Delta_{14}, \quad \Delta_4 = \Delta_{11} + \Delta_{14}, \quad \Delta_5 = \Delta_{10} + \Delta_{14} \quad \text{and} \quad \Delta_8 = \Delta_{12} + \Delta_{14}. \]

It follows then that at the end of step 3, we get the irreducible sum-dense set

\[ I_G = \{ \Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14} \}. \]

![Figure 5: The lattice of synchrony subspaces for the 5-cell regular network \( G \) on the left of Figure 1: the nontrivial synchrony subspaces \( \Delta_i \), for \( i = 1, \ldots, 14 \), are listed in Table 1. The top element is the total phase space \( P \) (the total asynchronous polydiagonal space) and the bottom element \( \Delta_0 \) is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set \( I_G \) are in green.](image)

**Step 4** Applying \( \text{Sum}(I_G) \), we get the lattice of synchrony subspaces listed in Table 1. See the lattice in Figure 5. \( \Diamond \)
Example 6.8 Consider the 5-cell regular network $G$ on the right of Figure 1. The coupled cell systems associated to the network $G$ satisfy
\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_1) \\
\dot{x}_2 &= f(x_2, x_1, x_4) \\
\dot{x}_3 &= f(x_3, x_1, x_5) \\
\dot{x}_4 &= f(x_4, x_1, x_2) \\
\dot{x}_5 &= f(x_5, x_1, x_3)
\end{align*}
\]
where $f(u, v, w)$ is a smooth function invariant under the permutation of the variables $v$ and $w$. Using Algorithm 6.3, we obtain all the nontrivial synchrony spaces associated to the network $G$, see Table 6.

| $\Delta_1 = \{ \mathbf{x} : x_1 = x_2 \}$ | $\Delta_5 = \{ \mathbf{x} : x_1 = x_2 = x_4 \}$ | $\Delta_{11} = \{ \mathbf{x} : x_1 = x_2 = x_3, x_4 = x_5 \}$ |
| $\Delta_2 = \{ \mathbf{x} : x_1 = x_4 \}$ | $\Delta_6 = \{ \mathbf{x} : x_2 = x_3, x_4 = x_5 \}$ | $\Delta_{12} = \{ \mathbf{x} : x_1 = x_2 = x_4, x_3 = x_5 \}$ |
| $\Delta_3 = \{ \mathbf{x} : x_2 = x_4 \}$ | $\Delta_7 = \{ \mathbf{x} : x_1 = x_2, x_3 = x_5 \}$ | $\Delta_{13} = \{ \mathbf{x} : x_1 = x_2 = x_5, x_3 = x_4 \}$ |
| $\Delta_4 = \{ \mathbf{x} : x_3 = x_5 \}$ | $\Delta_8 = \{ \mathbf{x} : x_2 = x_4, x_3 = x_5 \}$ | $\Delta_{14} = \{ \mathbf{x} : x_1 = x_3 = x_4, x_2 = x_5 \}$ |
| $\Delta_9 = \{ \mathbf{x} : x_2 = x_5, x_3 = x_4 \}$ | $\Delta_9 = \{ \mathbf{x} : x_2 = x_5, x_3 = x_4 \}$ | $\Delta_{15} = \{ \mathbf{x} : x_1 = x_4 = x_5, x_2 = x_3 \}$ |
| $\Delta_{10} = \{ \mathbf{x} : x_1 = x_4, x_3 = x_5 \}$ | $\Delta_{10} = \{ \mathbf{x} : x_1 = x_4, x_3 = x_5 \}$ | $\Delta_{16} = \{ \mathbf{x} : x_2 = x_3 = x_4 = x_5 \}$ |

Table 6: Nontrivial synchrony subspaces for the network on the right of Figure 1.

We illustrate now the implementation of Algorithm 6.3.

**Step 1.** The adjacency matrix of $G$,
\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix},
\]
has eigenvalues $2, -1, 1$, with multiplicities $1, 3, 1$, respectively. The associated eigenspaces (in $\mathbb{R}^5$) are $E_2 = \langle (1, 1, 1, 1, 1) \rangle$,
\[
E_{-1} = \langle (1, -1, -1, 0, 0), (1, 0, 0, -1, -1), (1, 0, -1, -1, 0) \rangle \quad \text{and} \quad E_1 = \langle (0, 0, 1, 0, 1) \rangle.
\]

**Steps 1.1-1.6** for $E_{-1}$: let
\[
M = \begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 0 \\
-1 & 0 & -1 \\
0 & -1 & -1 \\
0 & -1 & 0
\end{bmatrix}
\quad \text{and} \quad
\overline{M} = \begin{bmatrix}
2 & 1 & 1 \\
2 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 1 \\
0 & 0 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & 0 \\
-1 & 1 & 0 \\
-1 & 1 & -1 \\
0 & 0 & -1
\end{bmatrix}.
\]

29
Note that the rows in $M$ are all different, so there is no row we can eliminate from $M$. The rows of $\overline{M}$ correspond, respectively, to the equalities $x_1 = x_2$, $x_1 = x_3$, $x_1 = x_4$, $x_1 = x_5$, $x_2 = x_3$, $x_2 = x_4$, $x_2 = x_5$, $x_3 = x_4$, $x_3 = x_5$ and $x_4 = x_5$. Since $s = n = 5$, it is sufficient to consider the submatrices of $\overline{M}$ formed by 2 or 3 rows and rank 2. We obtain Table 7.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l. i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2 = x_3$</td>
<td>3</td>
<td>$((1,1,1,-2,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2 = x_4$</td>
<td>3</td>
<td>$((0,0,1,0,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2 = x_5$</td>
<td>3</td>
<td>$((1,1,-2,-2,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3 = x_4$</td>
<td>3</td>
<td>$((1,-2,1,1,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3 = x_5$</td>
<td>3</td>
<td>$((0,1,0,-1,0))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5$</td>
<td>3</td>
<td>$((1,-2,-2,1,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_3 = x_4$</td>
<td>3</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_3 = x_5$</td>
<td>3</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_4 = x_5$</td>
<td>3</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_3 = x_4 = x_5$</td>
<td>3</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2, x_3 = x_4$</td>
<td>3</td>
<td>$((1,1,-2,-2,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2, x_3 = x_5$</td>
<td>3</td>
<td>$((2,2,-1,-4,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2, x_4 = x_5$</td>
<td>3</td>
<td>$((1,1,1,-2,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3, x_2 = x_4$</td>
<td>3</td>
<td>$((2,-1,2,-1,-4))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3, x_2 = x_5$</td>
<td>3</td>
<td>$((1,-2,1,1,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3, x_4 = x_5$</td>
<td>3</td>
<td>$((1,1,1,-2,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4, x_2 = x_3$</td>
<td>3</td>
<td>$((1,-2,-2,1,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4, x_2 = x_5$</td>
<td>3</td>
<td>$((1,-2,1,1,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4, x_3 = x_5$</td>
<td>3</td>
<td>$((2,-4,-1,2,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_5, x_2 = x_3$</td>
<td>3</td>
<td>$((1,-2,-2,1,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_5, x_2 = x_4$</td>
<td>3</td>
<td>$((2,-1,-4,-1,2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_5, x_3 = x_4$</td>
<td>3</td>
<td>$((1,1,-2,-2,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_4, x_3 = x_5$</td>
<td>3</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5, x_2 = x_3$</td>
<td>2</td>
<td>$((1,-2,-2,1,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_3 = x_4 = x_5$</td>
<td>2</td>
<td>$((2,-1,-1,-1,-1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2 = x_3, x_4 = x_5$</td>
<td>2</td>
<td>$((1,1,1,-2,-2))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2 = x_5, x_3 = x_4$</td>
<td>2</td>
<td>$((1,1,-2,-2,1))$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3 = x_4, x_2 = x_5$</td>
<td>2</td>
<td>$((1,-2,1,1,-2))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7: Complete table for $E_{-1}$. 
Steps 1.1-1.6 for $E_1$: let

$$M = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$  

Rows 1, 2, 4 are equal and rows 3, 5 are equal. That means that $x_1 = x_2 = x_4$ and $x_3 = x_5$ for the eigenvectors in $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in E_1$. Eliminate rows 2, 4, 5 from $M$, obtaining

$$M = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} -1 \end{bmatrix}$$

and create Table 8. The row of $\overline{M}$ corresponds to the equality $x_1 = x_3$. We have $s = 2$, and since $2 - 2 = 0$, there is no need to consider any submatrix of $\overline{M}$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2 = x_4$, $x_3 = x_5$</td>
<td>2</td>
<td>$((0,0,1,0,1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8: Complete table for $E_1$.

Steps 2.1-2.3 Consider the sets of equality conditions corresponding to the two-dimensional polydiagonals in Table 7 for $E_{-1}$ (last five rows) and Table 8 for $E_1$. Besides the eigenvector $(1,1,\ldots,1)$ there is one more linearly independent eigenvector in those subspaces. Thus, they are synchrony subspaces and correspond to the synchrony subspaces $\Delta_{15}, \Delta_{16}, \Delta_{11}, \Delta_{13}, \Delta_{14}$ and $\Delta_{12}$.

Each subspace of the vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ satisfying the conditions in rows 1 - 23 of the Table 7 for $E_{-1}$ is 3-dimensional. Besides the eigenvector $(1,1,\ldots,1)$ there is only one linearly independent eigenvector in $E_{-1}$ contained in those subspaces. Since the conditions in rows 2, 12, 19 and 23 of the table of $E_{-1}$ are a subset of the conditions in the row of the Table 8 for $E_1$, there is also one linearly independent eigenvector in $E_1$ contained in those subspaces and so they are synchrony subspaces. They correspond to the synchrony subspaces $\Delta_5, \Delta_7, \Delta_{10}$ and $\Delta_8$. The remaining 19 rows do not correspond to synchrony subspaces.

Observe that, the implementation of the step 2.3.2 to the rows of the Table 7 for $E_{-1}$ that are not corresponding to synchrony subspaces will imply the addition of several rows to the Table 7 for $E_{-1}$ but not deriving more synchrony subspaces. At the end of the step 2.3, we have obtained the set of synchrony subspaces

$$S = \{ \Delta_5, \Delta_7, \Delta_8, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16} \}$$

which contains the minimal synchrony set $m_G$ as defined in Theorem 5.11.

Step 3 Each of the synchrony subspaces $\Delta_{11}, \ldots, \Delta_{16}$ is two-dimensional and so it is sum-irreducible. Moreover, the synchrony subspaces $\Delta_5, \Delta_7, \Delta_{10}$ are also sum-irreducible. As
\( \Delta_8 = \Delta_{12} + \Delta_{16} \), it follows then that at the end of step 3, we get the sum-dense set of irreducible subspaces (in the minimal synchrony set),

\[
\mathcal{I}_G = \{ \Delta_5, \Delta_7, \Delta_{10}, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16} \}.
\]

Figure 6: The lattice of synchrony subspaces for the 5-cell regular network \( \mathcal{G} \) on the right of Figure 1: the nontrivial synchrony subspaces \( \Delta_i \), for \( i = 1, \ldots, 16 \), are listed in Table 6. The top element is the total phase space \( P \) (the total asynchronous polydiagonal space) and the bottom element \( \Delta_0 \) is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set \( \mathcal{I}_G \) are in green.

**Step 4** Applying \( \text{Sum}(\mathcal{I}_G) \), we get the lattice of synchrony subspaces listed in Table 6. See the lattice in Figure 6.

\[ \Diamond \]

**Example 6.9** Consider the 5-cell regular network \( \mathcal{G} \) in Figure 7. The coupled cell systems associated to the network \( \mathcal{G} \) satisfy

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_1, x_3) \\
\dot{x}_2 &= f(x_2, x_1, x_2) \\
\dot{x}_3 &= f(x_3, x_2, x_2) \\
\dot{x}_4 &= f(x_4, x_3, x_5) \\
\dot{x}_5 &= f(x_5, x_1, x_5)
\end{align*}
\]

where \( f(u, v, w) \) is a smooth function invariant under permutation of the last two variables. Let \( \mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \). Using Algorithm 6.3, we obtain all the nontrivial synchrony spaces associated to the network \( \mathcal{G} \), see Table 9.
Figure 7: A 5-cell regular network.

\[ \Delta_1 = \{ x : x_2 = x_5 \} \]
\[ \Delta_2 = \{ x : x_1 = x_2 = x_3 \} \]
\[ \Delta_3 = \{ x : x_1 = x_2 = x_3 = x_5 \} \]
\[ \Delta_4 = \{ x : x_1 = x_2 = x_3, x_4 = x_5 \} \]

Table 9: Nontrivial synchrony subspaces for the network in Figure 7.

We illustrate very briefly the implementation of Algorithm 6.3.

Step 1. The adjacency matrix of $G$ is
\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
and it has eigenvalues $2, 0, 1$ and $\pm i$. We consider thus the complexification $A_c$ of $A$ (see Remark 5.3). The associated eigenspaces (in $\mathbb{C}^5$) are
\[
E_2 = \langle (1, 1, 1, 1, 1) \rangle, \quad E_0 = \langle (0, 0, 0, 1, 0) \rangle, \quad E_1 = \langle (0, 0, 1, 1) \rangle, \quad E_i = \langle (0, 0, 0, 1) \rangle,
\]
\[
E_{-i} = \langle (0, 0, 1, 1) \rangle, \quad E_{-i} = \langle (0, 0, 0, 1) \rangle.
\]

Steps 1.1-1.6: at the end of Step 1 we get Tables 10-13.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2 = x_3 = x_5$</td>
<td>2</td>
<td>$((0, 0, 0, 1, 0))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 10: Complete table for $E_0$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2 = x_3, x_4 = x_5$</td>
<td>2</td>
<td>$((0, 0, 0, 1, 1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11: Complete table for $E_1$.

Steps 2.1-2.3 Since all the polydiagonals in Tables 10-13 have an eigenvector basis, at the end of step 2 we get the synchrony subspaces $\Delta_1, \Delta_3$ and $\Delta_4$. 

33
<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = x_5$</td>
<td>4</td>
<td>$((-1 + i, 1, -2i, -2 - i, 1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 12: Complete table for $E_i$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = x_5$</td>
<td>4</td>
<td>$((-1 - i, 1, 2i, -2 + i, 1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 13: Complete table for $E_{-i}$.

Figure 8: The lattice of synchrony subspaces for the 5-cell regular network $G$ of Figure 7: the nontrivial synchrony subspaces $\Delta_i$, for $i = 1, \ldots, 4$, are listed in Table 9. The top element is the total phase space $P$ (the total asynchronous polydiagonal space) and the bottom element $\Delta_0$ is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set $\mathcal{I}_G$ are in green.

**Steps 3 and 4** We get the irreducible sum-dense set

$$\mathcal{I}_G = \{\Delta_1, \Delta_3, \Delta_4\}.$$ 

Applying $\text{Sum}(\mathcal{I}_G)$, we get the lattice of synchrony subspaces listed in Table 9. See the lattice in Figure 8.

6.3.2 Non semi-simple adjacency matrix

**Example 6.10** Consider the 6-cell regular network $G$ of Figure 9. The coupled cell
systems associated to the network \( G \) satisfy
\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_6) \\
\dot{x}_2 &= f(x_2, x_1, x_4) \\
\dot{x}_3 &= f(x_3, x_2, x_5) \\
\dot{x}_4 &= f(x_4, x_2, x_3) \\
\dot{x}_5 &= f(x_5, x_2, x_3) \\
\dot{x}_6 &= f(x_6, x_2, x_4)
\end{align*}
\]

where \( f(u, v, w) \) is a smooth function invariant under permutation of the last two variables. Let \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \). Using Algorithm 6.3, we obtain all the nontrivial synchrony spaces associated to the network \( G \), see Table 14.

<table>
<thead>
<tr>
<th>( \Delta_1 )</th>
<th>( {x : x_3 = x_5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_2 )</td>
<td>( {x : x_4 = x_5} )</td>
</tr>
<tr>
<td>( \Delta_3 )</td>
<td>( {x : x_3 = x_6, x_4 = x_5} )</td>
</tr>
<tr>
<td>( \Delta_4 )</td>
<td>( {x : x_3 = x_4 = x_5} )</td>
</tr>
<tr>
<td>( \Delta_5 )</td>
<td>( {x : x_1 = x_4 = x_5, x_3 = x_6} )</td>
</tr>
<tr>
<td>( \Delta_6 )</td>
<td>( {x : x_3 = x_4 = x_5 = x_6} )</td>
</tr>
<tr>
<td>( \Delta_7 )</td>
<td>( {x : x_1 = x_2, x_3 = x_4 = x_5 = x_6} )</td>
</tr>
<tr>
<td>( \Delta_8 )</td>
<td>( {x : x_1 = x_3 = x_4 = x_5 = x_6} )</td>
</tr>
</tbody>
</table>

Table 14: Nontrivial synchrony subspaces for the network of Figure 9.

We illustrate briefly the implementation of Algorithm 6.3.

**Step 1.** The adjacency matrix of \( G \) is
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 
\end{bmatrix}.
\]

The eigenvalues of \( A \) are 2, -1 and 0, with algebraic multiplicities 1, 2 and 3, respectively. The associated eigenspaces (in \( \mathbb{R}^6 \)) are 
\( E_2 = \langle (1, 1, 1, 1, 1, 1) \rangle \),
\( E_{-1} = \langle (1, -2, 1, 1, 1, 1) \rangle \) and \( E_0 = \ker A = \langle (1, 1, -1, -1, -1, -1) \rangle \).
Steps 1.1-1.6 for $E_{-1}$: we get Table 15 and we identify the synchrony subspace $\Delta_8$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_3 = x_4 = x_5 = x_6$</td>
<td>2</td>
<td>$((1, -2, 1, 1, 1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 15: Complete table for $E_{-1}$.

Step 1.7

Step 1.7.1 The vectors in $\text{Im}(A + I_6)$ satisfy $x_3 = x_5$ and $\text{Im}(A + I_6)$ is 5-dimensional. It follows then that $\text{Im}(A + I_6)$ is the polydiagonal $\{x : x_3 = x_5\}$.

Step 1.7.2

Step 1.7.2.1 for the row of Table 15

As $<(1, -2, 1, 1, 1, 1) > \cap \text{Im}(A + I_6) \neq \{0\}$, take the following basis:

$$B_1 = ((1, -2, 1, 1, 1, 1)),$$

and call:

**JordanChain**$(((1, -2, 1, 1, 1, 1)), \{x_1 = x_3 = x_4 = x_5 = x_6\}, 2)$

We obtain the following data:

$$V_1 =<(1, -2, 1, 1, 1, 1)>, \quad B_C = ((1, -2, 1, 1, 1, 1)) ;$$

The subspace $V_2$ of the vectors $v_2 \in \ker(A + I_6)^2$ such that $(A + I_6)v_2 \in V_1$ is:

$$V_2 = \{(3\alpha, -2\beta - 4\alpha, 3\beta, 3\alpha, 3\alpha, 3\beta) : \alpha, \beta \in \mathbb{R}\}.$$

Choosing

$$\overline{B}_2 = ((3, -4, 0, 3, 3, 0)),$$

we have that

$$B_C \cup \overline{B}_2 = ((1, -2, 1, 1, 1, 1), (3, -4, 0, 3, 3, 0))$$

is a basis of $V_2$,

$$M = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ -3 \\ -3 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}.$$
In step 9 of the JordanChain routine, $S = S_0$, and executing step 10, we add a new row to the Table 15, obtaining Table 16 and identifying the synchrony subspace $\Delta_5$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5, x_3 = x_6$</td>
<td>3</td>
<td>$((1, -2, 1, 1, 1, 1), (3, -4, 0, 3, 3, 0))$</td>
<td>2</td>
</tr>
<tr>
<td>Old</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1 = x_3 = x_4 = x_5 = x_6$</td>
<td>2</td>
<td>$((1, -2, 1, 1, 1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 16: After performing the JordanChain routine, we add one row to the Table 15 for $E_{-1}$.

**Steps 1.1-1.6** for $E_0$: we get Table 17 and we identify the synchrony subspace $\Delta_7$.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2, x_3 = x_4 = x_5 = x_6$</td>
<td>2</td>
<td>$((1, 1, -1, -1, -1, -1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 17: Complete table for $E_0$.

**Step 1.7**

**Step 1.7.1** The space $\text{Im}A$ is the polydiagonal $\{x : x_4 = x_5\}$.

**Step 1.7.2**

**Step 1.7.2.1 for the row of Table 17**
As $< (1, 1, -1, -1, -1, -1) > \cap \text{Im}A \neq \{0\}$, take the following basis: $B_1 = (1, 1, -1, -1, -1, -1)$, and call:

**JordanChain**($((1, 1, -1, -1, -1, -1)), \{x_1 = x_2, x_3 = x_4 = x_5 = x_6\}, 2$)

We obtain the following data:

$V_1 = < (1, 1, -1, -1, -1, -1) >$, $B_C = (1, 1, -1, -1, -1, -1)$ ;

The subspace $V_2$ of the vectors $v_2 \in \ker A^2$ such that $Av_2 \in V_1$ is $\ker A^2$:

$V_2 = \{-3\alpha + \beta, -\alpha - \beta, 2\alpha, 2\alpha, 2\alpha, 2\beta : \alpha, \beta \in \mathbb{R}\} .

Choosing $\overline{B}_2 = (1, -1, 0, 0, 0, 2)$,
we have that
\[ BC \cup \overline{B}_2 = ((1, 1, -1, -1, -1, 1, -1, 0, 0, 0, 2)) \]
is a basis of \( V_2 \),
\[
M = \begin{bmatrix}
1 \\
-1 \\
0 \\
0 \\
0 \\
2
\end{bmatrix}
\quad \text{and} \quad
\overline{M} = \begin{bmatrix}
2 \\
0 \\
0 \\
-2 \\
-2 \\
-2
\end{bmatrix}.
\]

In step 9 of the JordanChain routine, \( S = S_0 \), and executing step 10, we add a new row to the Table 17, obtaining Table 18.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_3 = x_4 = x_5 )</td>
<td>4</td>
<td>((1, 1, -1, -1, -1, 1, -1, 0, 0, 0, 2))</td>
<td>2</td>
</tr>
<tr>
<td>Old</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 = x_2, x_3 = x_4 = x_5 = x_6 )</td>
<td>2</td>
<td>((1, 1, -1, -1, -1, -1))</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 18: After performing the JordanChain routine, we add one row to the Table 17 for \( E_0 \).

All the vectors \( v_2 \in V_2 \) satisfy the condition \( x_4 = x_5 \) defining \( \text{Im}A \). Thus \( V_2 \cap \text{Im}A = V_2 \). Calling

**JordanChain**\(((1, 1, -1, -1, -1, 1, -1, 0, 0, 0, 2)), \{x_3 = x_4 = x_5\}, 3)\),

we obtain the following data:

\( V_2 = \langle (1, 1, -1, -1, -1, -1, 1, -1, 0, 0, 0, 2) \rangle, \quad BC = ((1, 1, -1, -1, -1, 1, -1, 0, 0, 0, 2)) \); 

The subspace \( V_3 \) of the vectors \( v_3 \in \ker A^3 \) such that \( Av_3 \in V_2 \) is \( \ker A^3 \):

\[
V_3 = \{ (-7\alpha + \beta + \gamma, \alpha - 3\beta - \gamma, 4\beta, 4\alpha, 4\beta, 2\gamma) : \alpha, \beta, \gamma \in \mathbb{R} \}.
\]

Choosing

\[ \overline{B}_3 = ((-7, 1, 0, 4, 0, 0)) \],

we have that

\[ BC \cup \overline{B}_3 \]

38
We add one row to the Table 18, obtaining Table 19.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_3 = x_5$</td>
<td>5</td>
<td>$((1, 1, -1, -1, -1, -1), (1, -1, 0, 0, 0, 2)$</td>
<td>3</td>
</tr>
<tr>
<td>$x_3 = x_4 = x_5$</td>
<td>4</td>
<td>$((1, 1, -1, -1, -1, -1), (1, -1, 0, 0, 0, 2))$</td>
<td>2</td>
</tr>
<tr>
<td>Old</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1 = x_2, x_3 = x_4 = x_5 = x_6$</td>
<td>2</td>
<td>$((1, 1, -1, -1, -1, -1))$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 19: After performing the JordanChain routine, we add one row to the Table 18 for $E_0$.

**Step 2.** Using Table 16 and Table 19, we identify the sum-dense set of synchrony subspaces

$$S = \{ \Delta_1, \Delta_4, \Delta_5, \Delta_7, \Delta_8 \}.$$  

**Step 3.** As all the synchrony subspaces in the set $S$ are sum-irreducible, we have that

$$\mathcal{I}_G = \{ \Delta_1, \Delta_4, \Delta_5, \Delta_7, \Delta_8 \}.$$  

**Step 4.** We obtain the lattice $V_G$ formed by the synchrony subspaces in Figure 10 and listed in Table 14.

**Example 6.11** Consider the 6-cell regular network $G$ of Figure 11.

The coupled cell systems associated to the network $G$ satisfy

$$
\begin{align*}
\dot{x}_1 &= f(x_1, x_1, x_6) \\
\dot{x}_2 &= f(x_2, x_2, x_5) \\
\dot{x}_3 &= f(x_3, x_2, x_5) \\
\dot{x}_4 &= f(x_4, x_1, x_3) \\
\dot{x}_5 &= f(x_5, x_1, x_3) \\
\dot{x}_6 &= f(x_6, x_2, x_4)
\end{align*}
$$

where $f(u, v, w)$ is a smooth function invariant under permutation of the last two variables.

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6)$. Using Algorithm 6.3, we obtain all the nontrivial synchrony spaces associated to the network $G$, see Table 20.
Figure 10: The lattice of synchrony subspaces for the 6-cell regular network $\mathcal{G}$ of Figure 9: the nontrivial synchrony subspaces $\Delta_i$, for $i = 1, \ldots, 8$, are listed in Table 14. The top element is the total phase space $P$ (the total asynchronous polydiagonal space) and the bottom element $\Delta_0$ is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set $\mathcal{I}_\mathcal{G}$ are in green.

Figure 11: A 6-cell regular network.

We illustrate briefly the implementation of Algorithm 6.3.
Step 1. The adjacency matrix of $G$ is

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
$$

The eigenvalues of $A$ are 2 and 0, with algebraic multiplicities 1 and 5, respectively. The associated eigenspaces (in $\mathbb{R}^6$) are $E_2 = \langle (1, 1, 1, 1, 1, 1) \rangle$ and

$$E_0 = \ker A = \langle (1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0) \rangle.$$

Steps 1.1-1.6 for $E_0$: we get Table 21 and we identify the synchrony subspaces $\Delta_8$ and $\Delta_9$ (first two lines of the table).

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = x_2, x_3 = x_4 = x_5 = x_6$</td>
<td>2</td>
<td>$\langle (1, 1, -1, -1, -1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5, x_2 = x_3 = x_6$</td>
<td>2</td>
<td>$\langle (1, -1, -1, 1, 1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_2, x_3 = x_6, x_4 = x_5$</td>
<td>3</td>
<td>$\langle (1, -1, -1, -1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_3 = x_6, x_4 = x_5$</td>
<td>3</td>
<td>$\langle (0, 1, 0, -1, -1, 0) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_1 = x_4 = x_5, x_3 = x_6$</td>
<td>3</td>
<td>$\langle (1, -1, -1, 1, 1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_3 = x_6, x_4 = x_5$</td>
<td>3</td>
<td>$\langle (1, -1, -1, 1, 1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_2 = x_4 = x_5, x_3 = x_6$</td>
<td>3</td>
<td>$\langle (1, 0, -1, 0, 0, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_3 = x_4 = x_5 = x_6$</td>
<td>3</td>
<td>$\langle (1, 1, -1, -1, -1, -1) \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$x_3 = x_6, x_4 = x_5$</td>
<td>4</td>
<td>$\langle (1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0) \rangle$</td>
<td>2</td>
</tr>
</tbody>
</table>

Step 1.7

Step 1.7.1 Since $\text{Im}A = \{x : x_2 = x_3, x_4 = x_5\}$, we can take the following basis for $\text{Im}A$:

$$\langle (1, 0, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 0), (0, 0, 0, 0, 0, 1) \rangle.$$
Step 1.7.2 Observe that only the rows 2, 5, 6, 9 of Table 21 for $E_0$ associated, respectively, with the following sets of equality conditions,
\[
\begin{align*}
  x_1 &= x_4 = x_5, x_2 = x_3 = x_6; \\
  x_1 &= x_4 = x_5, x_3 = x_6; \\
  x_2 &= x_3 = x_6, x_4 = x_5; \\
  x_3 &= x_6, x_4 = x_5;
\end{align*}
\]
are such that the intersection of the subspace corresponding to the basis in the row with $\text{Im}A$ is a nonzero subspace.

Step 1.7.2.1 for row 2 of Table 21
Take the following basis for the intersection of the subspace corresponding to the basis in the row with $\text{Im}A$,
\[
B_1 = ((1, -1, -1, 1, 1, -1)),
\]
and call:
\[
\text{JordanChain}(((1, -1, -1, 1, 1, -1)), \{x_1 = x_4 = x_5, x_2 = x_3 = x_6\}, 2)
\]
We obtain the following data:
\[
V_1 = \langle (1, -1, -1, 1, 1, -1) \rangle, \quad B_C = ((1, -1, -1, 1, 1, -1)) ;
\]
The subspace $V_2$ of the vectors $v_2 \in \ker A^2$ satisfying $Av_2 \in V_1$ is $\ker A^2$:
\[
V_2 = \{ (\beta, \gamma, \alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha - \beta) : \alpha, \beta, \gamma \in \mathbb{R} \}.
\]
Choosing
\[
\overline{B}_2 = ((1, 0, -1, 0, 0, -1), (0, 0, 1, -1, 1, 0)) ,
\]
we have that
\[
B_C \cup \overline{B}_2 = ((1, -1, -1, 1, 1, -1), (1, 0, -1, 0, 0, -1), (0, 0, 1, -1, -1, 1) )
\]
is a basis of $V_2$ and
\[
M = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
-1 & 1 \\
0 & -1 \\
0 & -1 \\
-1 & 1
\end{bmatrix}.
\]
The matrix $\overline{M}$ for the set of equality conditions $C = \{x_1 = x_4 = x_5, x_2 = x_3 = x_6\}$ is
\[
\overline{M} = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & -1 \\
1 & -1 \\
0 & 0 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad \text{rank}\overline{M} = 2;
\]
42
Taking the set \( S = S_1 \cup S_2 \) of the submatrices of \( \overline{M} \) with rank 1 or 0, as described in step 9 of the JordanChain routine, we have

\[
S_1 = \left\{ N_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}
\]

and

\[
S_2 = \left\{ N_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.
\]

\( N_1 \) **Step 10.1** The rows of

\[
N_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

correspond to the set of equality conditions \( C_{N_1} = \{ x_1 = x_4 = x_5, x_3 = x_6 \} \) of row 5 of Table 21 for \( E_0 \).

\( N_2 \) **Step 10.1** The rows of

\[
N_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

correspond to the set of the equality conditions \( C_{N_2} = \{ x_2 = x_3 = x_6, x_4 = x_5 \} \) of row 6 of Table 21 for \( E_0 \).

\( N_3 \) **Step 10.1** The rows of

\[
N_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

correspond to the set of equality conditions \( C_{N_3} = \{ x_3 = x_6, x_4 = x_5 \} \) of row 9 of Table 21 for \( E_0 \).

**Step 1.7.2.1 for row 5 of Table 21**

Take the following basis for the intersection of the subspace corresponding to the basis in the row with \( \text{Im}A \),

\[
B_1 = ((1, -1, -1, 1, 1, -1)),
\]

and call:

\[
\text{JordanChain}(((1, -1, -1, 1, 1, -1)), \{ x_1 = x_4 = x_5, x_3 = x_6 \}, 2)
\]
As before, we obtain the following data:

\[ V_1 = \langle (1, -1, -1, 1, 1, -1) \rangle, \quad B_C = ((1, -1, 1, 1, 1, -1)) ; \]

\[ V_2 = \ker A^2 = \{ (\beta, \gamma, \alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha - \beta) \colon \alpha, \beta, \gamma \in \mathbb{R} \}; \]

\[ \overline{B}_2 = ((1, 0, -1, 0, 0, -1), (0, 0, 1, -1, 1, 1)) ; \]

\[ B_C \cup \overline{B}_2 = ((1, -1, 1, 1, 1, -1), (1, 0, -1, 0, 0, -1), (0, 0, 1, -1, 1, 1)) \) is a basis of \( V_2 \);

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} . \]

The matrix \( \overline{M} \) for the set of equality conditions \( C = \{ x_1 = x_4 = x_5, x_3 = x_6 \} \) is

\[ \overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \text{rank} \overline{M} = 1 < 2 . \]

Taking the set \( S = S_1 \cup S_0 \) of the relevant submatrices of \( \overline{M} \) with rank 1 or 0,

\[ S_1 = \left\{ N_1 = \overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \]

and

\[ S_0 = \left\{ N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} . \]

\( N_1 \) **Step 10.1** The rows of

\[ N_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]

correspond to set of equality conditions \( C_{N_1} = C = \{ x_1 = x_4 = x_5, x_3 = x_6 \} \) of row 5 of Table 21 for \( E_0 \). We take

\[ \overline{B}_2 = ((1, 0, -2, 1, 1, -2)) \]

and we change that row: taking the basis

\[ B = ((1, -1, -1, 1, 1, -1), (1, 0, -2, 1, 1, -2)) \]
in the third entry and \#B = 2 in the fourth entry - see row 5 of the Old group of rows of the Table 22. We also identify the synchrony subspace \( \Delta_6 \) and we don’t call the JordanChain routine at step 10.1.5 since

\[
< \overline{B}_2 > \cap \text{Im}A = \{0\}.
\]

\( N_2 \) Step 10.1 The rows of

\[
N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

correspond to the set of equality conditions \( C_{N_2} = \{x_3 = x_6, x_4 = x_5\} \) of row 9 of Table 21 for \( E_0 \).

Step 1.7.2.1 for row 6 of Table 21

Take the following basis for the intersection of the subspace corresponding to the basis in the row with \( \text{Im}A \),

\[
B_1 = ((1, -1, -1, 1, 1, -1)),
\]

and call:

\[
\text{JordanChain}(((1, -1, -1, 1, 1, -1)), \{x_2 = x_3 = x_6, x_4 = x_5\}, 2)
\]

As before, we obtain the following data:

\[
V_1 = < (1, -1, -1, 1, 1, -1) > , \quad B_C = ((1, -1, -1, 1, 1, -1));
\]

\[
V_2 = \ker A^2 = \{ (\beta, \gamma, \alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha - \beta) : \alpha, \beta, \gamma \in \mathbb{R} \};
\]

\[
\overline{B}_2 = ((1, 0, -1, 0, 0, -1), (0, 0, 1, -1, -1, 1));
\]

\[
B_C \cup \overline{B}_2 = ((1, -1, -1, 1, 1, -1), (1, 0, -1, 0, 0, -1), (0, 0, 1, -1, -1, 1)) \) is a basis of \( V_2 \);

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}
\]

The matrix \( \overline{M} \) for the set of equality conditions \( \{x_2 = x_3 = x_6, x_4 = x_5\} \) is

\[
\overline{M} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

and \( \text{rank} \overline{M} = 1 < 2 \).
Taking the set $S = S_1 \cup S_0$ of the relevant submatrices of $\overline{M}$ with rank 1 or 0, we have

$$S_1 = \left\{ N_1 = \overline{M} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$S_0 = \left\{ N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

**N1 Step 10.1** The rows of

$$N_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

correspond to the set of equality conditions $C_{N_1} = C = \{ x_2 = x_3 = x_6, x_4 = x_5 \}$ of row 6 of Table 21 for $E_0$. We take

$$\overline{B}_2 = ((1, 0, 0, -1, -1, 0))$$

and we change that row: taking the basis

$$B = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0))$$

in the third entry and $\#B = 2$ in the fourth entry - see row 6 of the Old group of rows of the Table 22. We identify the synchrony subspace $\Delta_7$. As

$$< \overline{B}_2 > \cap \text{Im} A = < \overline{B}_2 >,$$

and $< B > \subseteq \text{Im} A$, we have that $< B > \cap \text{Im} A = < B >$. Take

$$B_2 = B = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0)),$$

and execute:

**JordanChain**$(B_2, C, 3)$
We obtain the following data:

\( B_2 = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0)) \);

\( V_2 = \langle B_2 \rangle \);

\[ V_3 = \{ v_3 \in \ker A^3 : Av_3 \in V_2 \} = \ker A^3 \]

\[ = \{ (\alpha + \beta + \gamma + \tau, -\alpha, -2\beta, -\gamma, -\gamma, -2\tau) : \alpha, \beta, \gamma, \tau \in \mathbb{R} \} \);

\( B_C = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0)) \);

\( \overline{B}_3 = ((1, 0, -1, 0, 0, -1), (2, -1, -2, 0, 0, 0)) \);

\( B_C \cup \overline{B}_3 \) is a basis of \( V_3 \);

\[
M = \begin{bmatrix}
1 & 2 \\
0 & -1 \\
-1 & -2 \\
0 & 0 \\
0 & 0 \\
-1 & 0
\end{bmatrix}.
\]

The matrix \( \overline{M} \) for the set of equality conditions \( C_{N_1} = C = \{ x_2 = x_3 = x_6, x_4 = x_5 \} \) is

\[
\overline{M} = \begin{bmatrix}
1 & 1 & -1 \\
1 & 0 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

and \( \text{rank} \overline{M} = 2 \);

Taking the set \( S = S_1 \cup S_0 \) of the relevant submatrices of \( \overline{M} \) with rank 1 or 0,

\[
S_1 = \left\{ N_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, N_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, N_{13} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \right\}
\]

and

\[
S_0 = \left\{ N_{14} = \begin{bmatrix} 0 & 0 \end{bmatrix} \right\}.
\]

**Step 10.1** The rows of

\[
N_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

correspond to the set of equality conditions \( \{ x_2 = x_3, x_4 = x_5 \} \). We add one row to the Table 21, obtaining the fourth row of the New group of rows of the Table 22, identifying the synchrony subspace \( \Delta_3 \)
and we have to execute

\textbf{JordanChain}(((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1)), \{x_2 = x_3, x_4 = x_5\}, 4)

We have

\[ B_3 = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1)) ; \]

\[ V_3 = < B_3 > ; \]

\[ B_4 = ((0, 0, 0, -2, 0, 1), (0, 0, 1, -7, 1, 1)) ; \]

\[ B_C = B_3 = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1)) ; \]

\[ \overline{B}_4 = ((0, 0, 0, -2, 0, 1), (0, 0, 1, -7, 1, 1)) ; \]

\[ B_C \cup \overline{B}_4 \text{ is a basis of } V_4 ; \]

\[ M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -2 & -7 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} . \]

The matrix \( M \) for the set of equality conditions \( \{x_2 = x_3, x_4 = x_5\} \) is

\[ M = \begin{bmatrix} 0 & -1 \\ -2 & -8 \end{bmatrix} \text{ and } \text{rank} M = 2. \]

We need to consider the submatrices \([0, -1]\) and \([-2, -8]\) of \( M \), of rank 1. From the first one, we add one row to the Table 21, obtaining the third row of the New group of rows of the Table 22, and identifying the synchrony subspace \( \Delta_1 \). From the second one, we add one row to the Table 21, obtaining the second row of the New group of rows of the Table 22, identifying the synchrony subspace \( \Delta_2 \) and we don’t call the JordanChain routine in step 10.1.5.

\( N_{12} \) \textbf{Step 10.1} The rows of

\[ N_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \]

correspond to the set of equality conditions \( x_2 = x_6, x_4 = x_5 \). We add one row to the Table 21, obtaining the first row of the New group of rows of the Table 22, identifying the synchrony subspace \( \Delta_4 \) and we don’t call the JordanChain routine in step 10.1.5.
Step 10.1 The rows of
\[ N_{13} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \]
correspond to the set of equality conditions \( x_3 = x_6, x_4 = x_5 \) of row 9 of Table 21.

Step 10.1 The row of
\[ N_{14} = \begin{bmatrix} 0 & 0 \end{bmatrix} \]
corresponds to the equality condition \( x_4 = x_5 \), which is the condition in the second row of the New group of rows of the Table 22.

Step 10.1 The rows of
\[ N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]
correspond to the set of equality conditions \( C_{N_2} = \{ x_3 = x_6, x_4 = x_5 \} \) of row 9 of Table 21 for \( E_0 \).

Step 1.7.2.1 for row 9 of Table 21
Take the following basis for the intersection of the subspace corresponding to the basis in the row with \( \text{Im}A \),
\[ B_1 = ((1, -1, -1, 1, 1, -1)) , \]
and call:
\[ \text{JordanChain}(((1, -1, -1, 1, 1, -1)), \{ x_3 = x_6, x_4 = x_5 \}, 2) \]
We obtain the following data:
\[ V_1 = \langle (1, -1, -1, 1, 1, -1) \rangle ; \]
\[ B_C = ((1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0)) ; \]
\[ V_2 = \{ v_2 \in \ker A^2 : Av_2 \in V_1 \} = \ker A^2 \]
\[ = \{ (\beta, \gamma, -\alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha + \beta) : \alpha, \beta, \gamma \in \mathbb{R} \} ; \]
\[ \overline{B}_2 = ((0, 0, 1, -1, -1, 1)) ; \]
\[ B_C \cup \overline{B}_2 \text{ is a basis of } V_2 ; \]
\[ M = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \text{ and } \overline{M} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \]
As \( \text{rank} \bar{M} = 0 \), we have \( S = S_0 \) and in \( S_0 \) we just have to take into consideration the matrix \( N = \bar{M} \) that corresponds to the set of equality conditions \( C_N = C = \{ x_3 = x_6, x_4 = x_5 \} \) of row 9 of Table 21 for \( E_0 \). We change that row, taking the basis

\[
B = B_C \cup \bar{B}_2
\]

in the third entry and \( \#B = 3 \) in the fourth entry - see row 9 of the \( O \) group of rows of the Table 22. We identify the synchrony subspace \( \Delta_5 \) and we don’t call the JordanChain routine since

\[
< \bar{B}_2 > \cap \text{Im} A = \{ 0 \}.
\]

There are no more rows in Table 21 such that the corresponding basis intersects nontrivially \( \text{Im} A \). Thus step 1.7.2 is concluded and we proceed to step 2 of the Algorithm 6.3 considering now Table 22 that was obtained from Table 21, with changes at rows 5,6 and 9, and four new rows.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Polydiag. dim.</th>
<th>Basis</th>
<th>Nr. l.i. vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_2 = x_6, x_4 = x_5 )</td>
<td>4</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (3, -1, -3, 0, 0, -1))</td>
<td>3</td>
</tr>
<tr>
<td>( x_4 = x_5 )</td>
<td>5</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1), (0, 0, 1, 1, 1, -3))</td>
<td>4</td>
</tr>
<tr>
<td>( x_2 = x_3 )</td>
<td>5</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1), (0, 0, 0, -2, 0, 1))</td>
<td>4</td>
</tr>
<tr>
<td>( x_2 = x_3, x_4 = x_5 )</td>
<td>4</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0), (-1, 1, 1, 0, 0, -1))</td>
<td>3</td>
</tr>
<tr>
<td>Old</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 = x_2, x_3 = x_4 = x_5 = x_6 )</td>
<td>2</td>
<td>((1, -1, -1, -1, -1, -1))</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 = x_4, x_2 = x_3 = x_6 )</td>
<td>2</td>
<td>((1, -1, -1, 1, 1, -1))</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 = x_2, x_3 = x_6, x_4 = x_5 )</td>
<td>3</td>
<td>((1, -1, -1, -1, -1, -1))</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 = x_3 = x_6, x_4 = x_5 )</td>
<td>3</td>
<td>((0, 1, 0, -1, -1, 0))</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 = x_4 = x_5, x_3 = x_6 )</td>
<td>3</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, -2, 1, 1, -2))</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 = x_3 = x_6, x_4 = x_5 )</td>
<td>3</td>
<td>((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0))</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 = x_4 = x_5, x_3 = x_6 )</td>
<td>3</td>
<td>((1, 0, -1, 0, 0, -1))</td>
<td>1</td>
</tr>
<tr>
<td>( x_3 = x_4 = x_5 = x_6 )</td>
<td>3</td>
<td>((1, -1, -1, -1, -1))</td>
<td>1</td>
</tr>
<tr>
<td>( x_3 = x_6, x_4 = x_5 )</td>
<td>4</td>
<td>((1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0), (0, 0, 1, -1, -1, 1))</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 22: At the end of the execution of the step 1 of the Algorithm 6.3, rows 5,6,9 of the Table 21 for \( E_0 \) were changed and four rows were added (the first four rows of this table).

**Step 2.** We identify the sum-dense set of synchrony subspaces

\[
S = \{ \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9 \}.
\]
Step 3. From the set $S$, we have that $\Delta_5 = \Delta_6 + \Delta_8$ and $\Delta_2 = \Delta_3 + \Delta_4$. Moreover, all the others synchrony subspaces are sum-irreducible. It follows then that

$$\mathcal{I}_G = \{\Delta_1, \Delta_3, \Delta_4, \Delta_6, \Delta_7, \Delta_8, \Delta_9\}.$$ 

Figure 12: The lattice of synchrony subspaces for the 6-cell regular network $G$ of Figure 11. The synchrony subspaces $\Delta_i$, for $i = 1, \ldots, 9$, are listed in Table 20. The top element is the total phase space $P$ (the total asynchronous polydiagonal space) and the bottom element $\Delta_0$ is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense $\mathcal{I}_G$ are in green.

Step 4. We obtain the lattice $V_G$ formed by the synchrony subspaces in Figure 12 and listed in Table 20.

7 More on the lattice of synchrony subspaces for non-regular homogeneous networks

Recall Corollary 4.3 in Section 4 which shows that, a polydiagonal subspace for a homogeneous network $G$ with edge-types $E_1, \ldots, E_l$, is of synchrony, if and only if it is a synchrony subspace for all its regular subnetworks $G_{E_j}$. Thus we can use Algorithm 6.3 to obtain the lattice $V_{G_{E_j}}$ for the subnetworks $G_{E_j}$ and then the lattice $V_G$ is given by the intersection of those lattices.
Example 7.1 Consider the 5-cell homogeneous network $\mathcal{G}$ in Figure 4. The coupled cell systems associated to the network $\mathcal{G}$ satisfy

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_4, x_2, x_4) \\
\dot{x}_2 &= f(x_2, x_1, x_5, x_1, x_4) \\
\dot{x}_3 &= f(x_3, x_1, x_5, x_1, x_5) \\
\dot{x}_4 &= f(x_4, x_1, x_3, x_1, x_2) \\
\dot{x}_5 &= f(x_5, x_1, x_3, x_1, x_3)
\end{align*}
\]

where $f(u, v, w, z, t)$ is a smooth function invariant under permutation of the variables $v$ and $w$ and under the permutation of the variables $z$ and $t$. There are two edge types, $\mathcal{E}_1$ and $\mathcal{E}_2$ with adjacency matrices, respectively,

\[
A_1 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
A_2 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

Note that, the subnetworks $\mathcal{G}_{\mathcal{E}_1}$ and $\mathcal{G}_{\mathcal{E}_2}$, with adjacency matrices $A_1$ and $A_2$, respectively, are the ones in the Examples 6.7 and 6.8. The set of the nontrivial synchrony subspaces for the subnetwork $\mathcal{G}_{\mathcal{E}_1}$ is given in Table 1, and the ones for the subnetwork $\mathcal{G}_{\mathcal{E}_2}$ appear in Table 6. By Corollary 4.3, the nontrivial synchrony subspaces for $\mathcal{G}$ are the ones listed in Table 23.

<table>
<thead>
<tr>
<th>${x : x_3 = x_5}$</th>
<th>${x : x_2 = x_3, \ x_4 = x_5}$</th>
<th>${x : x_1 = x_2 = x_3, \ x_4 = x_5}$</th>
<th>${x : x_1 = x_4 = x_5, \ x_2 = x_3}$</th>
<th>${x : x_2 = x_3 = x_4 = x_5}$</th>
</tr>
</thead>
</table>

Table 23: Nontrivial synchrony subspaces for the network of Figure 4.

One way to implement an efficient algorithm to obtain the lattice $V_{\mathcal{G}}$ is to find the lattice $V_{\mathcal{G}_{\mathcal{E}_j}}$ for some subnetwork $\mathcal{G}_{\mathcal{E}_j}$ executing Algorithm 6.3, and then finding the subset of subspaces in $V_{\mathcal{G}_{\mathcal{E}_j}}$ that are left invariant by the adjacency matrices of the other subnetworks $\mathcal{G}_{\mathcal{E}_k}$.

Another way to implement an optimized algorithm to obtain the lattice $V_{\mathcal{G}}$ is not to make distinction of the edge-types and consider a ‘global adjacency matrix’ (that includes the arrows of all types). Obviously, the synchrony subspaces in $V_{\mathcal{G}}$ are synchrony subspaces for the regular network $\overline{\mathcal{G}}$ with adjacency matrix given by that global matrix. Since the reverse is not true, after executing Algorithm 6.3 for $\overline{\mathcal{G}}$, it is then necessary to find the subset of subspaces in $V_{\overline{\mathcal{G}}}$ that are left invariant by all the adjacency matrices of $\mathcal{G}$. 

52
8 More on the lattice of synchrony subspaces for non-homogeneous networks

Theorem 4.2 in Section 4 relates the lattice of balanced equivalence relations of a non-homogeneous network \( \mathcal{G} \) with the lattices of balanced equivalence relations of its subnetworks by input class and edge-type class \( \mathcal{G}^{\mathcal{I}_j} \) and \( \mathcal{G}^{\mathcal{I}_j}_{\mathcal{E}_{ij}} \) (recall the definition in Section 4). Here we go further and show how to optimize the process to obtain the nontrivial balanced equivalence relations in \( \Lambda_\mathcal{G} \) given the balanced equivalence relations in \( \mathcal{G}^{\mathcal{I}_j}_{\mathcal{E}_{ij}} \). For that, we need to introduce the following definition.

Definition 8.1 Let \( H \) be a subnetwork of a network \( \mathcal{G} \) and \( \bowtie \) be an equivalence relation on the set \( \mathcal{C}_H \) of the cells of \( H \). Using \( \bowtie \) we define an equivalence relation \( \bowtie \bowtie \) on the set \( \mathcal{C} \supseteq \mathcal{C}_H \) of the cells of \( \mathcal{G} \) by \( [c]_{\bowtie\bowtie} = [c]_\bowtie \), for \( c \in \mathcal{C}_H \) and \( [c]_{\bowtie\bowtie} = \{ c \} \) for \( c \in \mathcal{C} \setminus \mathcal{C}_H \). We say that \( \bowtie \bowtie \) is the unfolding of \( \bowtie \bowtie \) on \( \mathcal{C} \). Analogously, if \( \bowtie \bowtie \) is an equivalence relation on the set \( \mathcal{C} \), then the equivalence relation \( \bowtie \bowtie \) on the set of cells \( \mathcal{C}_H \subset \mathcal{C} \) of the cells in \( H \) defined by \( [c]_{\bowtie\bowtie} = [c]_\bowtie \cap \mathcal{C}_H \) is called the restriction of \( \bowtie \bowtie \) to \( \mathcal{C}_H \).

Let \( \tilde{G}^I_i \) and \( \tilde{G}^{E}_{E_{ij}} \) be the subnetworks of \( G^I_i \) and \( G^{E}_{E_{ij}} \), respectively.

Theorem 8.2 Let \( \mathcal{G} \) be a non-homogeneous network. For an input equivalence class \( I_i \) of \( \mathcal{G} \), consider the corresponding subnetwork \( \tilde{G}^I_i \) and the associated identical-edge subnetworks \( \tilde{G}^{E}_{E_{ij}} \), for \( j = 1, \ldots, r_i \). Let \( \mathcal{B}_i \) be the set of balanced equivalence relations \( \bowtie_i \) on the cells of \( \tilde{G}^I_i \) such that there is at least one cell \( c \in I_i \) with \( \# [c]_{\bowtie_i} > 1 \) and not admitting a balanced refinement \( \bowtie_{i}^i \) such that for \( c \in I_i \) we have \( [c]_{\bowtie_{i}^i} = [c]_{\bowtie_i} \). Let \( \mathcal{B}_{E_{ij}} \) be the set of balanced equivalence relations \( \bowtie_{E_{ij}} \) on the cells of \( \tilde{G}^{E}_{E_{ij}} \) such that there is at least one cell \( c \in I_i \) with \( \# [c]_{\bowtie_{E_{ij}}} > 1 \) and not admitting a balanced refinement \( \bowtie_{E_{ij}}^i \) such that for \( c \in I_i \) we have \( [c]_{\bowtie_{E_{ij}}^i} = [c]_{\bowtie_{E_{ij}}} \). For each \( \bowtie_{E_{ij}} \in \mathcal{B}_{E_{ij}} \), consider its unfolding \( \bowtie_{E_{ij}} \) to the set of cells of \( \tilde{G}^{E}_{E_{ij}} \).

An equivalence relation \( \bowtie_i \) is in \( \mathcal{B}_i \) if and only if

\[
\bowtie_i = \bowtie_{i_1} \lor \cdots \lor \bowtie_{i_{r_i}}
\]

for \( \bowtie_{i_1} \in \mathcal{B}_{i_1}, \ldots, \bowtie_{i_{r_i}} \in \mathcal{B}_{i_{r_i}} \) satisfying \( [c]_{\bowtie_{i_1}} = [c]_{\bowtie_{i_{r_i}}} \), for all \( j, k \in \{ 1, \ldots, r_i \} \) and \( c \in I_i \).

Proof Let \( \bowtie_i \in \mathcal{B}_i \) and \( \bowtie_{E_{ij}} \), for \( j = 1, \ldots, r_i \), be the restriction of \( \bowtie_i \) to the set of cells of \( \tilde{G}^{E}_{E_{ij}} \). Trivially each \( \bowtie_{E_{ij}} \) is a balanced equivalence relation in \( \mathcal{B}_i \) and \( \bowtie_i = \bowtie_{i_1} \lor \cdots \lor \bowtie_{i_{r_i}} \), with \( \bowtie_{E_{ij}} \) the unfolding of \( \bowtie_{E_{ij}} \) to the cells of \( \tilde{G}^{I}_i \). Moreover, since for all \( j = 1, \ldots, r_i \), the input equivalence class \( I_i \) is a subset of the set of cells of \( \tilde{G}^{E}_{E_{ij}} \), we have \( [c]_{\bowtie_{i_1}} = [c]_{\bowtie_{i_{r_i}}} \), for all \( j, k \in \{ 1, \ldots, r_i \} \) and \( c \in I_i \).
Now, let \( \triangleright \alpha_{i_j} \in \tilde{B}_i, \ldots, \triangleright \alpha_{i_j} \in \tilde{B}_{i_{r_i}} \) satisfying \( [c]_{\triangleright \alpha_{i_j}} = [c]_{\triangleright \alpha_{i_k}} \), for all \( j, k \in \{1, \ldots, r_i\} \) and \( c \in I_i \). First, we prove that \( \triangleright \alpha = \triangleright \alpha_1 \lor \cdots \lor \triangleright \alpha_{r_i} \) is a balanced equivalence relation on the cells of \( \tilde{G}_I \). For that we have to show that given any two cells \( c, d \) of \( \tilde{G}_I \) such that \( c \triangleright \alpha, d \), there is an edge-type preserving isomorphism \( \beta_i(c, d) : I(c) \rightarrow I(d) \), between the input sets of \( c \) and \( d \) in \( \tilde{G}_I \), respectively, such that for all \( \alpha \in I(c) \), the tail cells of \( \alpha \) and \( \beta_i(c, d)(\alpha) \) are in the same \( \triangleright \alpha \) class. Note that \( c \triangleright \alpha, d \) only if \( c \) and \( d \) belong to the same input equivalence class. Consider first the case \( c, d \in I_i \). If \( c \triangleright \alpha, d \), then \( c \triangleright \alpha_j, d \) (and thus \( c \triangleright \alpha_{i_j}, d \)), for all \( j = 1, \ldots, r_i \). Moreover, since \( \triangleright \alpha_{i_j} \) is balanced, as \( [c]_{\triangleright \alpha_{i_j}} = [c]_{\triangleright \alpha_{i_k}} \), we define the isomorphism \( \beta_i(c, d) \) as \( \beta_i(c, d)(e) = \beta_i(c, d)(e) \), with \( i_j \) the edge type of \( e \). Since for every \( \beta_i(c, d) \), for \( j = 1, \ldots, r_i \), the tail cells of \( \alpha \) and \( \beta_i(c, d)(\alpha) \) are in the same \( \triangleright \alpha \) class, the same follows for \( \beta_i(c, d) \) and \( \triangleright \alpha_i \). In the case \( c \triangleright \alpha, d \) with \( c, d \in I_i \), for \( l \neq i \), we have \( I(c) = I(d) = \emptyset \) and thus there is nothing to prove. That \( \triangleright \alpha \) belongs to \( \tilde{B}_i \) follows trivially from the fact that \( \triangleright \alpha_j, i_j \in \tilde{B}_i \), for \( j = 1, \ldots, r_i \). □

**Theorem 8.3** Let \( G \) be a nonhomogeneous network and for \( i = 1, \ldots, k \), consider \( \tilde{G}_I^i \) the subnetwork of \( G \) corresponding to the input equivalence class \( I_i \). Let \( B_i \) be the set of the unfoldings \( \triangleright \alpha \) to \( C \) of all the balanced equivalence relations \( \triangleright \alpha_i \in \tilde{B}_i \) defined in Theorem 8.2, together with the trivial relation (where each class is formed by a unique cell), on the set of cells of \( \tilde{G}_I^i \).

An equivalence relation \( \triangleright \alpha \) on the set of cells of \( G \) is balanced if and only if

\[
\triangleright \alpha = \triangleright \alpha_1 \lor \cdots \lor \triangleright \alpha_k,
\]

with \( \triangleright \alpha_i \in B_i \) such that \( [c]_{\triangleright \alpha_j} \subseteq [c]_{\triangleright \alpha_l} \), for \( c \in I_i \), for all \( 1 \leq j, l \leq k \).

**Proof** Let \( \triangleright \alpha \) be a balanced equivalence relation on the set of cells of \( G \). Let \( \triangleright \alpha_i \), for \( i = 1, \ldots, k \), be the restriction of \( \triangleright \alpha \) to the set of cells of \( \tilde{G}_I^i \). Trivially each \( \triangleright \alpha_i \) is a balanced equivalence relation on the cells of \( \tilde{G}_I^i \), for \( i = 1, \ldots, k \). If for all \( c \in I_i \) we have \( \#[c]_{\triangleright \alpha_i} = 1 \) then we can take \( \triangleright \alpha_i \) to be the trivial relation on \( C \). Otherwise, if \( \triangleright \alpha_i \notin \tilde{B}_i \) then let \( \triangleright \alpha_i^* \in \tilde{B}_i \) such that \( \triangleright \alpha_i^* \sqsubset \triangleright \alpha_i \) and take \( \triangleright \alpha = \triangleright \alpha_i \cup \triangleright \alpha_i^* \). It follows trivially that \( \triangleright \alpha = \triangleright \alpha_1 \lor \cdots \lor \triangleright \alpha_k \). Moreover, observe that \( \tilde{G}_I^i \) contains the cells in \( I_i \) and that \( \triangleright \alpha \) refines \( \sim_I \) since it is balanced. Thus, for \( c \in I_i \), we have that \( [c]_{\triangleright \alpha} \subseteq [c]_{\sim_I} \) and so \( [c]_{\triangleright \alpha} = [c]_{\sim_I} = [c]_{\triangleright \alpha_i} \). If \( j \neq l \) then \( [c]_{\triangleright \alpha_j} \subseteq [c]_{\triangleright \alpha_l} \).

Let \( \triangleright \alpha_i \in B_i \) such that \( [c]_{\triangleright \alpha_j} \subseteq [c]_{\triangleright \alpha_l} \), for \( c \in I_i \), for all \( 1 \leq j, l \leq k \). To prove that \( \triangleright \alpha = \triangleright \alpha_1 \lor \cdots \lor \triangleright \alpha_k \) is a balanced equivalence relation on the cells of \( G \) we have to show that given any two cells \( c, d \) of \( G \) such that \( c \triangleright \alpha, d \), there is an edge-type preserving isomorphism \( \beta(c, d) : I(c) \rightarrow I(d) \), between the input sets of \( c \) and \( d \), respectively, such that for all \( \alpha \in I(c) \), the tail cells of \( \alpha \) and \( \beta(c, d)(\alpha) \) are in the same \( \triangleright \alpha \) class.

Note that \( c \triangleright \alpha \) only if \( c \) and \( d \) belong to the same input equivalence class since each \( \triangleright \alpha_i \in B_i \) refines \( \sim_I \). If \( c, d \in I_i \), for \( l \in \{1, \ldots, k\} \), and \( c \triangleright \alpha, d \), then \( c \triangleright \alpha d \) and thus \( c \triangleright \alpha d \), since, for all \( j \neq l \), \( [c]_{\triangleright \alpha_j} \subseteq [c]_{\triangleright \alpha_l} \). As \( \triangleright \alpha_i \) is balanced, there is an edge-type preserving isomorphism \( \beta_i(c, d) : I(c) \rightarrow I(d) \) such that for all \( \alpha \in I(c) \), the tail cells of \( \alpha \) and \( \beta_i(c, d)(\alpha) \) are in the same \( \triangleright \alpha \) class. We define \( \beta(c, d)(e) = \beta_i(c, d)(e) \). Since the input set of the cells in \( I_I \) is empty in the subnetworks \( \tilde{G}_I^I \), for \( j \neq l \), we conclude that the
isomorphism $\beta(c, d)$ satisfies for all $\alpha \in I(c)$, that the tail cells of $\alpha$ and $\beta(c, d)(\alpha)$ are in the same $\propto$ class.

Using the one-to-one correspondence between balanced equivalence relations and synchrony subspaces, we have that we can use the above results and an adaptation of Algorithm 6.3 to obtain the lattice of synchrony subspaces of a nonhomogeneous network. We illustrate that with a network example.

**Example 8.4** Consider the nonhomogeneous network $G$ in Figure 2. The coupled cell systems associated to the network $G$ satisfy

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_3, x_8) \\
\dot{x}_2 &= f(x_2, x_1, x_4, x_7) \\
\dot{x}_3 &= g(x_3, x_4) \\
\dot{x}_4 &= g(x_4, x_5) \\
\dot{x}_5 &= g(x_5, x_3) \\
\dot{x}_6 &= h(x_6, x_1, x_5) \\
\dot{x}_7 &= h(x_7, x_2, x_5) \\
\dot{x}_8 &= h(x_8, x_2, x_5)
\end{align*}
\]

where $f(u, v, w, z, t)$ is a smooth function.

The nontrivial synchrony subspaces for the network $G$ are given in Table 24 and we show now how to use the above results in order to obtain this table.

<table>
<thead>
<tr>
<th>$\Delta_1$</th>
<th>${x : x_7 = x_8}$</th>
<th>$\Delta_4$</th>
<th>${x : x_1 = x_2, x_3 = x_4 = x_5, x_7 = x_8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_2$</td>
<td>${x : x_3 = x_4 = x_5}$</td>
<td>$\Delta_5$</td>
<td>${x : x_1 = x_2, x_3 = x_4 = x_5, x_6 = x_7 = x_8}$</td>
</tr>
<tr>
<td>$\Delta_3$</td>
<td>${x : x_3 = x_4 = x_5, x_7 = x_8}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 24:** Nontrivial synchrony subspaces for the network of Figure 2.

The $\sim_\tau$-equivalence classes of $G$ are

$\tau = \{1, 2\}, \quad \tau = \{3, 4, 5\}$ and $\tau = \{6, 7, 8\}$.

The subnetworks $G_1^{\tau}$, for $i = 1, 2, 3$, of $G$ are shown in Figure 3. Recall that each subnetwork $G_1^{\tau}$ is obtained from $G$ considering only the edges that are directed to cells in $I_i$. Looking at $G_1^{\tau}$ we see that: there are three edge-types with head cells in $I_1$ - let $E_1$, $E_2$ and $E_3$ be the classes of the edges with head cells in $I_1$ and tail cells in $I_1$, $I_2$ and $I_3$, respectively; there is only one type of edge with head cells in $I_2$ - let $E_4$ be the class of the edges with head and tail cells in $I_2$; there are two edge-types with head cells in $I_3$ - let $E_5$ and $E_6$ be the classes of the edges with head cells in $I_3$ and tail cells in $I_1$ and $I_2$, respectively.

Let’s consider now the subnetworks $G_1^{\tau}$ of $G_1^{\tau}$, for $i = 1, 2, 3$. Thus each $G_1^{\tau}$ is obtained from $G_1^{\tau}$ considering only the cells in $I_i$ and in their input sets. It follows then that the subnetwork $G_1^{\tau}$ contains the cells $1, 2, 3, 4, 7$ and $8$; the subnetwork $G_1^{\tau}$ contains only the cells in $I_2$; and $G_1^{\tau}$ contains the cells $6, 7, 8, 1, 2$ and $5$. 

55
We get
\[ V_{\mathcal{G}_{\mathcal{E}_1}} = \{ x : x_1 = x_2 \} , \]
\[ V_{\mathcal{G}_{\mathcal{E}_2}} = \{ x : x_1 = x_2, x_3 = x_4 \} , \]
\[ V_{\mathcal{G}_{\mathcal{E}_3}} = \{ x : x_1 = x_2, x_7 = x_8 \} , \]
using an adaptation of Algorithm 6.3. By Theorem 8.2, taking
\[ \Delta_1 = \{ x : x_1 = x_2 \} \cap \{ x : x_1 = x_2, x_3 = x_4 \} \cap \{ x : x_1 = x_2, x_7 = x_8 \} \]
we have that
\[ V_{\mathcal{G}_{\mathcal{I}_1}} = \{ \Delta_1 \} . \]
Analogously, we obtain
\[ V_{\mathcal{G}_{\mathcal{I}_2}} = \{ \Delta_2 \} \]
with
\[ \Delta_2 = \{ x : x_3 = x_4 = x_5 \} . \]
Moreover, we have that
\[ V_{\mathcal{G}_{\mathcal{I}_3}} = \{ \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8 \} , \]
 taking
\[ \Delta_3 = \{ x : x_1 = x_2, x_6 = x_7 \} , \]
\[ \Delta_4 = \{ x : x_1 = x_2, x_6 = x_8 \} , \]
\[ \Delta_5 = \{ x : x_1 = x_2, x_7 = x_8 \} , \]
\[ \Delta_6 = \{ x : x_1 = x_2, x_6 = x_7 = x_8 \} , \]
\[ \Delta_7 = \{ x : x_7 = x_8 \} . \]
By Theorem 8.3, we get the nontrivial synchrony subspaces in \( V_{\mathcal{G}} \):
\[ \Delta_4 = \Delta_1 \cap \Delta_2 \cap \Delta_3 ; \]
\[ \Delta_5 = \Delta_1 \cap \Delta_4 ; \]
\[ \Delta_3 = \Delta_0 \cap \Delta_2 \cap \Delta_5 ; \]
\[ \Delta_2 = \Delta_0 \cap \Delta_1 \cap \Delta_3 ; \]
\[ \Delta_1 = \Delta_0 \cap \Delta_5 . \]
Here, each \( \Delta_i \) corresponds to the total asynchronous polydiagonal subspace (corresponding to the trivial relation where each class is formed by a unique cell).

References


