Is there switching for replicator dynamics and bimatrix games?

Manuela A. D. Aguiar

Centro de Matemática da Universidade do Porto,*
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
Faculdade de Economia, Universidade do Porto,
Rua Dr Roberto Frias, 4200-464 Porto, Portugal

E-mail: maguiar@fep.up.pt

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Abstract

We consider heteroclinic networks for replicator dynamics and bimatrix games; that is, in a simplex or product of simplices; with equilibria at the vertices and connections at the edges – edge networks. Switching dynamics near a heteroclinic network occurs whenever every (infinite) sequence of connections in the network is shadowed by at least one trajectory in its neighbourhood. Aguiar and Castro (Chaotic switching in a two-person game, *Physica D* **239** (16), 1598–1609) prove switching near an edge network for the dynamics of the Rock-Scissors-Paper game. Here we give conditions for switching dynamics in general bimatrix games and show that switching near an edge network can never occur for replicator dynamics.

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1 Introduction

Let ξ_i and ξ_j be saddle equilibria for the flow of a smooth vector field. By *saddle equilibria* we mean that the equilibria ξ_k, $k = i, j$, have non-trivial stable and unstable manifolds, $W^s(\xi_k) \neq \{\xi_k\}$ and $W^u(\xi_k) \neq \{\xi_k\}$. A *heteroclinic connection* from ξ_i to ξ_j, denoted by $[\xi_i \to \xi_j]$, is a trajectory in $W^u(\xi_i) \cap W^s(\xi_j)$. A *heteroclinic network* corresponds to an

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assembly of saddle equilibria and heteroclinic connections between them that is strongly connected. Thus, every equilibria in a heteroclinic network has at least one incoming and one outgoing connection; and given any two equilibria in the network, there is a sequence of connections taking one to the other.

A particular example of a heteroclinic network occurs when the equilibria are cyclically connected. That is, up to a reordering of the equilibria, there are heteroclinic connections $[\xi_i \to \xi_{i+1}]$, for $i = 1, \ldots, r-1$, and $[\xi_r \to \xi_1]$. This is called a *heteroclinic cycle*. Note that a heteroclinic network can be defined as a connected union of heteroclinic cycles. A subset of a heteroclinic network Σ that is a network is called a *heteroclinic subnetwork* of Σ.

The existence of heteroclinic networks is a common phenomenon in the presence of invariant spaces. This can be a consequence of symmetry, see Krupa [25] and Field [11], of the coupling structure in coupled cell systems, see Aguiar *et al.* [1], or of the formulation of the problem itself, as is the case of games or population dynamics, see Hofbauer [15] and Hofbauer and Sigmund [17].

Here we are interested in heteroclinic networks in games and replicator dynamics and thus contained in a simplex or product of simplices.

In particular, we consider the replicator equation

$$\dot{x}_j = x_j \left[(A\mathbf{x})_j - \mathbf{x}^T A \mathbf{x} \right] \tag{1}$$

with $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in the n-dimensional simplex

$$S_n = \left\{ (x_1, ..., x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1,\ x_i \geq 0, i = 1, \ldots, n+1 \right\},$$

and $A = [a_{ij}]$ an $(n+1) \times (n+1)$ real matrix, and we consider the equations for the dynamics of bimatrix games

$$\begin{aligned} \dot{x}_i &= x_i \left[(A\mathbf{y})_i - \mathbf{x}^T A \mathbf{y} \right] \\ \dot{y}_j &= y_j \left[(B\mathbf{x})_j - \mathbf{y}^T B \mathbf{x} \right] \end{aligned} \tag{2}$$

with $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m+1})$ in the product of simplices $S_n \times S_m$, and $A = [a_{ij}]$ an $(n+1) \times (m+1)$ real matrix and $B = [b_{ji}]$ an $(m+1) \times (n+1)$ real matrix.

Replicator equation and bimatrix games play a fundamental role in describing evolutionary dynamics in a broad variety of modelling situations in different and seemingly unrelated fields ranging from biology to economics, see, for example, Weibull [35], Fudenberg and Levine [12], Hofbauer and Sigmund [17] and Nowak [27]. For more details on replicator and evolutionary game dynamics see also Hofbauer and Sigmund [18] and Szabó and Fáth [32].

In both settings all the vertices of the simplex or product of simplices are equilibria with eigendirections at the edges. Here, we assume that there are no other equilibria at the edges, besides the ones at the vertices, and we assume that they are all hyperbolic saddles. Thus, the edges of the simplex or product of simplices correspond to heteroclinic connections and form a heteroclinic network between the equilibria at the vertices. This type of network, that we will call *edge network*, is common in replicator dynamics and bimatrix games.

As we will see in Proposition 9, there can be continua of heteroclinic connections between the equilibria at the vertices. We extend the definition of edge heteroclinic network to those cases. However, we are excluding the situations where there are heteroclinic connections involving the equilibria at the vertices and other equilibria at the boundary of the simplex or product of simplices.

In [7], Brannath observes that for the case of an edge network in the simplex S_3 there is no orbit whose ω-limit contains more than one heteroclinic cycle in the network. He claims that 'there should be a quite general (but non-trivial) argument which excludes such a 'switching' for networks on S_3 consisting of corners and edges.' Moreover, he argues that 'it would be interesting to find (more complicated or higher dimensional) networks with orbits 'jumping' between the cycles because these orbits could form an invariant set with quite unusual dynamical behaviour.'

At the same time, Kirk and Silber [24] study the competition between two heteroclinic cycles that have a heteroclinic connection in common. More specifically, let ξ_i and ξ_j be two equilibria such that there is a connection $[\xi_i \to \xi_j]$. Let ξ_k and ξ_l be other two equilibria such that there are the connections $[\xi_k \to \xi_i]$ and $[\xi_l \to \xi_i]$ and the connections $[\xi_j \to \xi_k]$ and $[\xi_j \to \xi_l]$. The four equilibria and the heteroclinic connections between them form a heteroclinic network with two heteroclinic cycles with one heteroclinic connection in common (see Figure 1). We shall call such a network a *Kirk&Silber network*. The conclusion in [24] is that if one of the cycles is unstable and the other attracts nearly all trajectories lying in an open neighbourhood of the heteroclinic network, then switching between the cycles can occur in only one direction; a trajectory passing near the common heteroclinic connection may make excursions around the unstable cycle and then switch to the attracting one but thereafter cannot switch back to the unstable one. The dynamics does not allow for more complicated sequences of visits to the equilibria on the network with a *random switching* between the cycles comprising the network.

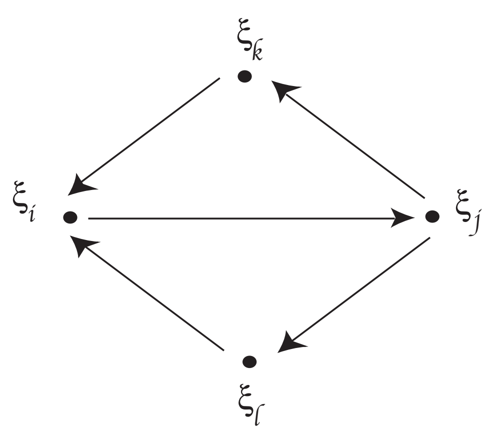

Figure 1: A Kirk&Silber network.

We now make the definiton of switching more precise.

A (finite) *path* on a network Σ is a sequence of connections (c_i), $i = 1, \ldots, s$ in Σ such that $c_i = [\xi_i \rightarrow \xi_j]$ and $c_{i+1} = [\xi_j \rightarrow \xi_k]$, with ξ_i, ξ_j and ξ_k equilibria in Σ. An *infinite path* corresponds to an infinite sequence of connections (c_i), $i \in \mathbb{Z}$.

Given a path on a network Σ, we say that there is a trajectory for the flow of a vector field f that *follows* or *shadows* that path, if for every neighbourhood V of the sequence of connections in Σ defining that path, there is a trajectory for the flow of f contained in V. That is to say, there is a trajectory for the flow of f as close as required to the sequence of connections in Σ defining the path.

We say there is finite (infinite) *switching* near a network if *for every* finite (infinite) path on the network there is a trajectory, near the network, for the flow of f that follows that path. If most, but not all, of the paths on the network can be shadowed by a trajectory for the flow of f, we can say there is *partial switching*. Here we are interested in finite (infinite) switching and not in partial switching, so when we say that *there isn't switching* we may still have partial switching. We will prove switching only for $t \geq 0$. This is sometimes called as *forward switching*, as in Homburg and Knobloch [20]. Here we will simplify and drop the word *forward*.

In Section 2 we prove that the reason for the exclusion of switching near an edge network in S_3, as reported by Brannath [7], is the existence of at least a Kirk&Silber subnetwork. In fact, we show that this is the reason for the absence of switching near an edge network in S_n, for all $n \geq 2$.

In [10], Fatás-Villafranca *et al.*derive a differential equation for the evolution of a dynamical model for discretionary consumption activities with n social groups that reduces to a linear replicator equation. They conclude from the numerical simulations that for $n = 4$ social groups the interior orbits always approach one of the cycles of the heteroclinic network. The 'attracting' cycle depending on the values of the parameters. For $n \geq 5$, the behaviour also looks like the attraction to a heteroclinic cycle. However, it seems that there is irregularity in the order and duration of the visits to the equilibria. As observed, the orbits seem to switch 'randomly' between the cycles of the network in each new return. Although, from our results in Section 2, not all sequences of connections in the network can be realized by an interior trajectory and thus there can only exist partial switching.

In [30, 31], Sato *et al.* present numerical evidence for the existence of complicated behaviour arising from the existence of an edge heteroclinic network for the dynamics of the Rock-Scissors-Paper game with two players. Aguiar and Castro [2] provide an analitycal proof for the numerical results obtained in [30, 31], showing that in the Rock-Scissors-Paper game with two players there is infinite switching, leading to behaviour ranging from almost deterministic actions to chaotic-like dynamics. In Section 3 we consider general bimatrix games finding conditions for the existence of switching near edge networks for the case of two players and more than three actions.

Switching dynamics near homoclinic and heteroclinic networks has been a subject of increasing interest lately, see, for example, [5], [3], [4], [2], [23] and [20]. It is important to identify the different general properties that entail switching; that is, the different mechanisms for switching. According to our knowledge, besides the noise-induced switching in [5], in all the studies so far, the mechanisms for switching are rotating nodes (either equilibria with complex eigenvalues or periodic trajectories) and/or transverse intersections of invariant manifolds.

This paper contributes with a new mechanism for switching: heteroclinic networks in conservative dynamics without a Kirk&Silber subnetwork. We believe that the results in the paper should apply to any such network for a volume preserving flow defined on a compact manifold.

2 There isn't switching in replicator dynamics

We consider the replicator equation

$$\dot{x}_j = x_j \left[\sum_{i=1}^{n+1} a_{ji} x_i - \mathbf{x}^T A \mathbf{x} \right] \tag{3}$$

with $\mathbf{x}^T A \mathbf{x} = \sum_{j=1}^{n+1} x_j \sum_{i=1}^{n+1} a_{ji} x_i$, defined on the n-dimensional simplex S_n.

All coordinate hyperplanes $x_j = 0$, $j = 1, \ldots, n+1$, and all their intersections are invariant by the flow of (3). The intersection of those invariant subspaces with the simplex S_n makes all the boundary n-faces, $n \geq 0$, of the simplex invariant by the flow. Thus, the vertices ξ_i, $i = 1, \ldots, n+1$, are equilibria, which we assume to be saddles. Moreover, if we assume that there are no other equilibria at the edges then they correspond to heteroclinic connections.

We remark that, for each vertex there is at least one incoming and one outgoing connection. Thus, with our assumptions, we conclude that there is an edge heteroclinic network at the boundary of S_n. Moreover, for every pair of equilibria ξ_i and ξ_j at the vertices there is always one of the connections $[\xi_i \to \xi_j]$ or $[\xi_j \to \xi_i]$. The number of connections to and from a vertex add up to n.

The main result in this section is Theorem 6 which states that there can be not switching near an edge network in S_n, for all $n \in \mathbb{N}$. The idea of the proof is to show that every edge network in S_n has a Kirk&Silber network as a subnetwork. As it is shown in Theorem 1, there can be not switching near such a network. The proof of Theorem 1 consists in showing that there isn't switching along the common connection in the Kirk&Silber subnetwork.

We say there is *switching along a connection* $[\xi_k \to \xi_j]$ if, for any neighbourhood of a point in a connection leading to node ξ_k, there are trajectories starting in that neighbourhood that follow along the connection $[\xi_k \to \xi_j]$ and then along all the possible connections forward from ξ_j.

Note that, switching along a connection is a necessary condition for switching near a network.

Theorem 1 *There isn't switching near a heteroclinic network with a Kirk&Silber subnetwork.*

Proof We can assume, without loss of generality, that the Kirk&Silber subnetwork is formed by the equilibria ξ_n, ξ_{n+1}, ξ_1 and ξ_2 and the connections $[\xi_n \to \xi_{n+1}]$, $[\xi_1 \to \xi_n]$, $[\xi_2 \to \xi_n]$, $[\xi_{n+1} \to \xi_1]$ and $[\xi_{n+1} \to \xi_2]$. See Figure 2.

Let $-c_1$ and $-c_2$, with $c_1, c_2 > 0$, be the eigenvalues of ξ_n in the direction of ξ_1 and ξ_2, respectively, and $e > 0$ the eigenvalue in the direction of ξ_{n+1}. Let e_1 and e_2, with $e_1, e_2 > 0$, be the eigenvalues of ξ_{n+1} in the direction of ξ_1 and ξ_2, respectively, and $-c < 0$ the eigenvalue in the direction of ξ_n.

We follow the computations in [2]. By the C^1 extension by Ruelle [29] of Hartman's results [13], the vector field may be linearised around each equilibrium ξ_i, up to a set of measure zero. So, generically, there is a C^1 linearization for all the equilibria in the network.

We note the diagonal structure, in the nonradial directions, of the Jacobian matrix at each equilibrium, see Lemma 21 in Appendix B, which is crucial in what follows. We can choose local coordinates $(w_1, \ldots, w_n) \in S_n$ such that the linearization of the flow near ξ_n is given by

$$\begin{aligned} \dot{w}_1 &= -c_1 w_1 \\ \dot{w}_2 &= -c_2 w_2 \\ \dot{w}_i &= \lambda_i^n w_i \qquad i = 3, \ldots, n-1 \\ \dot{w}_n &= e w_n \end{aligned}$$

with λ_i^n, $i = 3, \ldots, n-1$ the remaining eigenvalues of ξ_n and the linearization of the flow near ξ_{n+1} is given by

$$\begin{aligned} \dot{w}_1 &= e_1 w_1 \\ \dot{w}_2 &= e_2 w_2 \\ \dot{w}_i &= \lambda_i^{n+1} w_i \qquad i = 3, \ldots, n-1 \\ \dot{w}_n &= -c w_n \end{aligned}$$

with λ_i^{n+1}, $i = 3, \ldots, n-1$ the remaining eigenvalues of ξ_{n+1}.

The flow near ξ_n is then given by

$$\mathcal{F}_t^n(w_1, w_2, \ldots, w_{n-1}, w_n) = (w_1 e^{-c_1 t}, w_2 e^{-c_2 t}, \ldots, w_n e^{et}), \tag{4}$$

and the flow near ξ_{n+1} is given by

$$\mathcal{F}_t^{n+1}(w_1, w_2, \ldots, w_{n-1}, w_n) = (w_1 e^{e_1 t}, w_2 e^{e_2 t}, \ldots, w_n e^{-ct}). \tag{5}$$

In a neighbourhood of each node ξ_n and ξ_{n+1} where the flow can be linearized, we define cross-sections for the connection $[\xi_n \to \xi_{n+1}]$ and for each of the connections involving those nodes and the nodes ξ_1 and ξ_2 as follows

$$\begin{aligned} \Sigma_{n,1}^{\text{in}} &= \{(h, w_2, w_3, \ldots, w_n) : \ 0 < w_2, w_3, \ldots, w_n < h\} \\ \Sigma_{n,2}^{\text{in}} &= \{(w_1, h, w_3, \ldots, w_n) : \ 0 < w_1, w_3, \ldots, w_n < h\} \\ \Sigma_{n,n+1}^{\text{out}} &= \{(z_1, z_2, z_3, \ldots, z_{n-1}, h) : \ 0 < z_1, z_2, \ldots, z_{n-1} < h\} \end{aligned}$$

$$\begin{aligned} \Sigma_{n+1,n}^{\text{in}} &= \{(w_1, w_2, w_3, \ldots, w_{n-1}, h) : \ 0 < w_2, w_3, \ldots, w_n < h\} \\ \Sigma_{n+1,1}^{\text{out}} &= \{(h, z_2, z_3, \ldots, z_n) : \ 0 < z_1, z_3, \ldots, z_n < h\} \\ \Sigma_{n+1,2}^{\text{out}} &= \{(z_1, h, z_3, \ldots, z_n) : \ 0 < z_1, z_2, \ldots, z_{n-1} < h\} \end{aligned}$$

where $0 < h < 1$ is a positive number, small enough to guarantee transversality of the flow near each saddle. A two-dimensional representation of these sections is given in Figure 2.

We can do a linear rescaling of the local coordinates near each equilibrium such that $h = 1$. Thus, for ease of computations and without loss of generality, we will consider $h = 1$.

The points in $\Sigma_{k,j}^{\text{out}}$ follow the connection from saddle ξ_k to saddle ξ_j. Analogously, the points in $\Sigma_{k,j}^{\text{in}}$ come from a neighbourhood of saddle ξ_j and are taken close to saddle ξ_k.

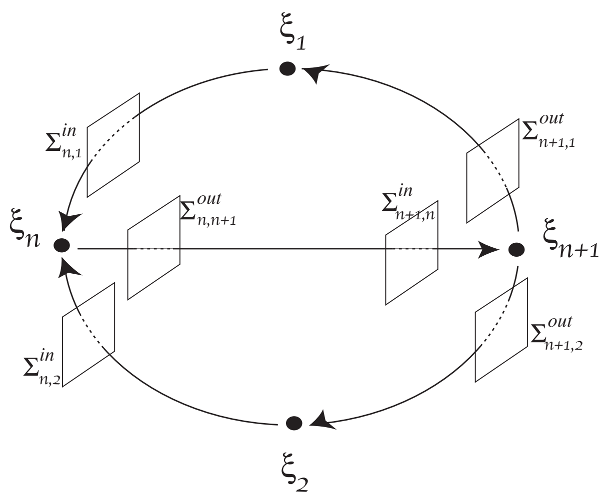

Figure 2: Two-dimensional representation of the cross-sections.

Using the linear flow near ξ_n we define the maps $\Psi_{i,n,n+1} : \Sigma_{n,i}^{\text{in}} \to \Sigma_{n,n+1}^{\text{out}}$, $i = 1, 2$, by the following rules,

$$\Psi_{1,n,n+1}(w_2, w_3, \ldots, w_{n-1}, w_n) = \left(w_n^{\frac{c_1}{e}}, w_2 w_n^{\frac{c_2}{e}}, w_3 w_n^{-\frac{\lambda_3^n}{e}}, \ldots, w_{n-1} w_n^{-\frac{\lambda_{n-1}^n}{e}} \right) = (z_1, \ldots, z_n),$$

$$\Psi_{2,n,n+1}(w_1, w_3, \ldots, w_{n-1}, w_n) = \left(w_1 w_n^{\frac{c_1}{e}}, w_n^{\frac{c_2}{e}}, w_3 w_n^{-\frac{\lambda_3^n}{e}}, \ldots, w_{n-1} w_n^{-\frac{\lambda_{n-1}^n}{e}} \right) = (z_1, \ldots, z_n),$$

Thus, the points in $\Sigma_{n,n+1}^{\text{out}}$ with $w_2 < w_1^{\frac{c_2}{c_1}}$ come from $\Sigma_{n,1}^{\text{in}}$, and the ones satisfying $w_2 > w_1^{\frac{c_2}{c_1}}$ come from $\Sigma_{n,2}^{\text{in}}$.

In the neighbourhood of ξ_{n+1}, using the linearized flow, we define the maps $\Psi_{n,n+1,i} : \Sigma_{n+1,n}^{\text{in}} \to \Sigma_{n+1,i}^{\text{out}}$, $i = 1, 2$ by

$$\Psi_{n,n+1,1}(w_1, w_2, \ldots, w_{n-1}) = \left(w_2 w_1^{-\frac{e_2}{e_1}}, w_3 w_1^{-\frac{\lambda_3^{n+1}}{e_1}}, \ldots, w_{n-1} w_1^{-\frac{\lambda_{n-1}^{n+1}}{e_1}}, w_1^{\frac{c}{e_1}} \right) = (z_1, \ldots, z_n),$$

$$\Psi_{n,n+1,2}(w_1, w_2, \ldots, w_{n-1}) = \left(w_1 w_2^{-\frac{e_1}{e_2}}, w_3 w_2^{-\frac{\lambda_3^{n+1}}{e_2}}, \ldots, w_{n-1} w_2^{-\frac{\lambda_{n-1}^{n+1}}{e_2}}, w_2^{\frac{c}{e_2}} \right) = (z_1, \ldots, z_n),$$

Thus, the points in $\Sigma^{\text{in}}_{n+1,n}$ that go to $\Sigma^{\text{out}}_{n+1,1}$ satisfy $w_2 < w_1^{\frac{e_2}{e_1}}$ and the points that go to $\Sigma^{\text{out}}_{n+1,2}$ satisfy $w_2 > w_1^{\frac{e_2}{e_1}}$.

We can consider the transition map $\Phi_{n,n+1}$ from $\Sigma^{\text{out}}_{n,n+1}$ to $\Sigma^{\text{in}}_{n+1,n}$ as

$$\Phi_{n,n+1}(z_1, \ldots, z_{n-1}) = (a_1 z_1, \ldots, a_{n-1} z_{n-1})$$

with a_i, $i = 1, \ldots, n-1$ constants.

As a simplification, and without loss of generality, we can approximate the transition map $\Phi_{n,n+1}$ by the identity map.

So, in $\Sigma^{\text{in}}_{n+1,n}$, the surface $w_2 = w_1^{\frac{c_2}{c_1}}$ divides the region of points that come from ξ_1 from the region of points that come from ξ_2, and the surface $w_2 = w_1^{\frac{e_2}{e_1}}$ divides the region of points that go to ξ_1 from the region of points that go to ξ_2.

If $\frac{e_2}{e_1} \neq \frac{c_2}{c_1}$, the two curves divide $\Sigma^{\text{in}}_{n+1,n}$ in three regions of points and if $\frac{e_2}{e_1} = \frac{c_2}{c_1}$, in two regions.

Thus, independently of the magnitude of the eigenvalues $e_i, -c_i$, $i = 1, 2$, there is always one transition $\xi_p \to \xi_n \to \xi_{n+1} \to \xi_q$, $p, q = 1, 2$ that is not followed by points near the heteroclinic connection $[\xi_n \to \xi_{n+1}]$.

Then, there isn't switching along the connection $[\xi_i \to \xi_j]$ and thus, there isn't switching near the network. □

The following three lemmas allow to prove the result in Theorem 5 that every edge network in S_n has a Kirk&Silber subnetwork.

The *type of a saddle equilibrium* ξ_i is the tuple (s, u), where $s = \dim W^s(\xi_i)$ and $u = \dim W^u(\xi_i)$.

Lemma 2 *Consider an edge network Σ in S_n with two equilibria ξ_i and ξ_j of type (s^i, u^i) and (s^j, u^j), respectively, such that there is a connection $[\xi_i \to \xi_j]$. If $s^i + u^j - (n-1) \geq 2$ then Σ has a Kirk&Silber subnetwork.*

Proof Let $E = \{\xi_1, \ldots, \xi_{n+1}\} \setminus \{\xi_i, \xi_j\}$, then $\#E = n-1$. We have that ξ_i receives a connection from s^i equilibria in E and ξ_j sends a connection to u^j equilibria in E.

If $s^i + u^j - (n-1) \geq 2$ then there are at least two equilibria, say ξ_k and ξ_l, such that there is a connection from ξ_k and from ξ_l to ξ_i and there is a connection from ξ_j to ξ_k and to ξ_l.

The equilibria ξ_i, ξ_j ,ξ_k and ξ_l with the heteroclinic connections between them form a Kirk&Silber subnetwork of Σ. □

Lemma 3 *Let Σ be an edge network in S_n. For each equilibrium ξ_k in Σ, let (s^k, u^k) be the type of ξ_k. We have that, $s^i + u^j - (n-1) < 2$ for every connection $[\xi_i \to \xi_j]$ if and only if n is even and all saddles are of type $(\frac{n}{2}, \frac{n}{2})$.*

Proof If all saddles are of type $(\frac{n}{2}, \frac{n}{2})$, then $s^i + u^j - (n-1) = 1 < 2$, for all connections $[\xi_i \to \xi_j]$.

Now, suppose that for every connection $[\xi_i \to \xi_j]$ in Σ we have $s^i + u^j - (n-1) < 2$. We start by observing that

(i) the network Σ cannot have connections $[\xi_i \to \xi_j]$ such that ξ_i is of type $(s^i, n-s^i)$ and ξ_j is of type $(s^j, n-s^j) = (s^i - m, n - s^i + m)$, with $m \geq 1$. Otherwise, $s^i + u^j - (n-1) = m + 1 \geq 2$.

Thus, saddles of type $(1, n-1)$, if any, can only receive a connection from a saddle of type $(1, n-1)$. We show that there can be no saddle of type $(1, n-1)$, otherwise there would be a subnetwork of Σ with only saddles of type $(1, n-1)$, which is impossible. To see this, suppose there was a saddle of type $(1, n-1)$ in Σ. Without loss of generality, we can assume that it is ξ_1 and that there is the heteroclinic connection $[\xi_{n+1} \to \xi_1]$, and so, the heteroclinic connections $[\xi_1 \to \xi_2], \ldots, [\xi_1 \to \xi_n]$. Consequently, ξ_{n+1} must also be of type $(1, n-1)$. Without loss of generality, we can assume that there is the heteroclinic connection $[\xi_n \to \xi_{n+1}]$, and thus, the heteroclinic connections $[\xi_{n+1} \to \xi_2], \ldots, [\xi_{n+1} \to \xi_{n-1}]$. But then, ξ_n must also be of type $(1, n-1)$. We have already the connections $[\xi_1 \to \xi_n]$ and $[\xi_n \to \xi_{n+1}]$, thus we must have the heteroclinic connections $[\xi_n \to \xi_2], \ldots, [\xi_n \to \xi_{n-1}]$. We have then the cycle of heteroclinic connections $[\xi_{n+1} \to \xi_1]$, $[\xi_1 \to \xi_n]$ and $[\xi_n \to \xi_{n+1}]$ with the saddles ξ_1, ξ_n and ξ_{n+1} of type $(1, n-1)$. In consequence Σ cannot be a heteroclinic network since there is no sequence of connections joining any of the equilibria ξ_i, $i \neq 1, n, n+1$ to any of the equilibria ξ_1, ξ_n, ξ_{n+1}. Summarizing, if there was one saddle of type $(1, n-1)$ then there would be a group of three saddles of type $(1, n-1)$ forming a heteroclinic subnetwork such that there is no heteroclinic connection from the remaining saddles to any of the saddles of that group. An analogous argument applies to saddles of type $(n-1, 1)$, since saddles of this type can only send a connection to a saddle of type $(n-1, 1)$.

Since there can be no saddle of type $(1, n-1)$, by (i) the saddles of type $(2, n-2)$ can only receive connections from saddles of type $(2, n-2)$. Using a reasoning analogous to the one for the saddles of type $(1, n-1)$, we conclude that there can be no saddle of type $(2, n-2)$. If there was one saddle of type $(2, n-2)$ then there would be a group of at least five saddles of type $(2, n-2)$ forming a heteroclinic subnetwork such that there is no heteroclinic connection from the remaining saddles to any of the saddles of that group. Note that not all the $n+1$ saddles in Σ can be of type $(1, n-1)$ since we must have $\sum_{k=1}^{n+1} s^k = \sum_{k=1}^{n+1} u^k$. An analogous argument applies to saddles of type $(n-2, 2)$, since saddles of this type can only send a connection to a saddle of type $(n-1, 1)$ or $(n-2, 2)$ and we have already concluded that there can be no saddle of type $(n-1, 1)$.

The above reasoning applies recursively until we endup with no saddles if n is odd and only with saddles of type $(\frac{n}{2}, \frac{n}{2})$ in Σ if n is even. □

Lemma 4 *Every edge network Σ in S_n, with $n > 2$ even, with all the saddles of the type $(\frac{n}{2}, \frac{n}{2})$ has a Kirk&Silber subnetwork.*

Proof Let ξ_i and ξ_j be any two equilibria in Σ. Then, there exists one of the connections $[\xi_i \to \xi_j]$ or $[\xi_j \to \xi_i]$ in Σ. Without loss of generality, assume that there exists the connection $[\xi_i \to \xi_j]$. There are $n-1$ more equilibria in Σ. We have that ξ_i receives a connection from $\frac{n}{2}$ equilibria $\eta_1, \ldots, \eta_{\frac{n}{2}}$ and ξ_j sends a connection to $\frac{n}{2}$ equilibria $\zeta_1, \ldots, \zeta_{\frac{n}{2}}$. See Figure 3 a).

Thus, there is at least one equality $\eta_r = \zeta_s$, $r, s \in \{1, \ldots, \frac{n}{2}\}$, since there are only $n-1$ equilibria in Σ, besides ξ_i and ξ_j. If there is more than one such equality then we get a Kirk&Silber subnetwork. Let's assume then that there is only one such equality, which we can assume, without loss of generality, to be $\eta_1 = \zeta_1 = \xi_k$. And so, $\eta_p \neq \zeta_q$, for all $p, q \in \{2, \ldots, \frac{n}{2}\}$. Moreover, ξ_j receives a connection from $\xi_i, \eta_2, \ldots, \eta_{\frac{n}{2}}$ and ξ_i sends a connection to $\xi_j, \zeta_2, \ldots, \zeta_{\frac{n}{2}}$.

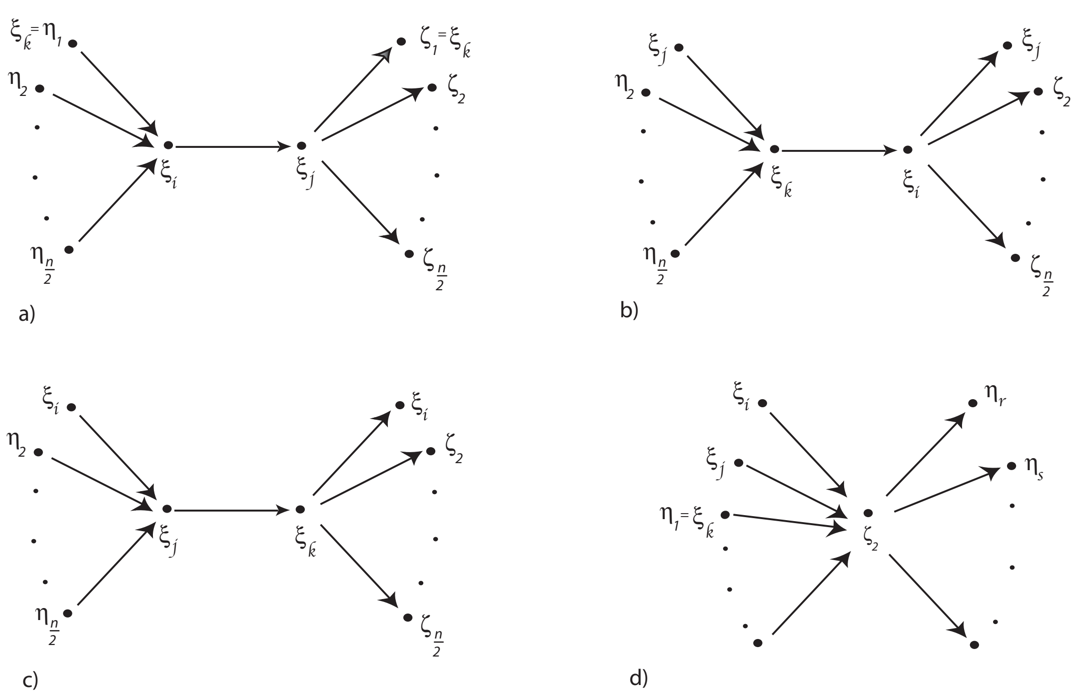

Figure 3: Illustration of the heteroclinic connections mentioned in the proof of Lemma 4.

In order for a Kirk&Silber subnetwork not to exist we must have that ξ_k receives a connection from $\xi_j, \eta_2, \ldots, \eta_{\frac{n}{2}}$ and sends a connection to $\xi_i, \zeta_2, \ldots, \zeta_{\frac{n}{2}}$. See Figure 3 b) and c).

We have then that, for example, the equilibria ζ_2 receives a connection from ξ_i, ξ_j and ξ_k and thus must send a connection to two of the η's, lets say, η_r and η_s. See Figure 3 d).

And so, the equilibria ξ_i, ζ_2, η_r and η_s and the connections between them form a Kirk&Silber subnetwork. See Figure 4.

$\square$

Theorem 5 *Every edge network in S_n, for all $n > 2$ has a Kirk&Silber subnetwork.*

Proof Let Σ be an edge network in S_n, with $n > 2$. Each saddle equilibrium ξ_i of Σ is of the type $(s^i, n - s^i)$, with $s^i = 1, \ldots, n-1$.

According to Lemma 2, Σ has a Kirk&Silber subnetwork if there are connections $[\xi_i \rightarrow \xi_j]$ such that $s^i + (n - s^j) - (n-1) \geq 2$.

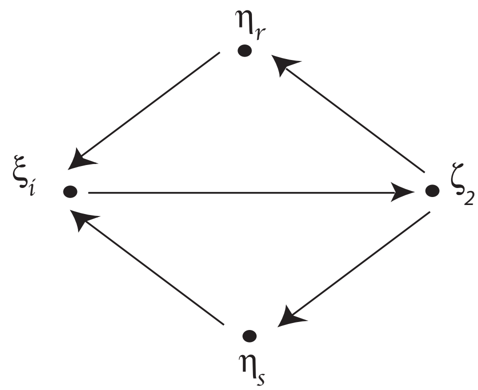

Figure 4: Kirk&Silber subnetwork obtained in the proof of Lemma 4.

By Lemma 3, for such subnetwork not to exist, n must be even and the network Σ can only have saddles of type $(\frac{n}{2}, \frac{n}{2})$.

By Lemma 4, an edge network with all saddles of the type $(\frac{n}{2}, \frac{n}{2})$ contains a Kirk&Silber subnetwork. □

Theorem 6 *There isn't switching near an edge network in S_n, for all n.*

Proof If $n = 2$ then the edge heteroclinic network is a heteroclinic cycle and so switching makes no sense. For $n > 2$ the result follows from Theorem 1 and Theorem 5. □

3 Switching in bimatrix games

We consider the analog of the replicator equation for bimatrix games,

$$\begin{aligned} \dot{x}_i &= x_i \left[\sum_{j=1}^{m+1} a_{ij} y_j - \mathbf{x}^T A \mathbf{y} \right] \\ \dot{y}_j &= y_j \left[\sum_{i=1}^{n+1} b_{ji} x_i - \mathbf{y}^T B \mathbf{x} \right] \end{aligned} \tag{6}$$

with $\mathbf{x}^T A \mathbf{y} = \sum_{j=1}^{m+1} y_j \sum_{i=1}^{n+1} a_{ij} x_i$ and $\mathbf{y}^T B \mathbf{x} = \sum_{i=1}^{n+1} x_i \sum_{j=1}^{m+1} b_{ji} y_j$, for $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$.

For more on this evolutionary dynamics for bimatrix games, see Hofbauer [16] and Hofbauer and Sigmund [17] and [18]. In particular, this bimatrix dynamics preserves volume inside $S_n \times S_m$. Conservative systems tend to be complicated because they do not contract. In fact, there can be no compact set contained in the interior of the product of simplices $S_n \times S_m$ which attracts all nearby trajectories, but they can occur in the boundary, Eshel and Akin [9].

We start by finding, in Theorem 10, conditions for the existence of edge heteroclinic networks in products of simplices and then show in Proposition 12 that this networks can never have a Kirk&Silber subnetwork. On account of the absence of a Kirk&Silber subnetwork, by the results in Section 2, there is the possibility of switching in the neighbourhood of these networks. First, we find conditions for the existence of switching along the connections of an edge network, as this is a necessary condition for switching near the network, Theorem 13 and Corollary 14. Since the flow of (6) in int$S_n \times S_m$ preserves volume (see Hofbauer and Sigmund [17]), we argue in Conjecture 16 the existence in int$S_n \times S_m$ of a flow invariant neighbourhood for edge heteroclinic networks in $S_n \times S_m$. Under the assumption of this conjecture and that the flow is topologically mixing, we show switching dynamics near the networks with the condition that $n = m$.

The boundary of the product of simplices $S_n \times S_m$ is invariant by the flow of (6) as it corresponds to the intersection of $S_n \times S_m$ with all the coordinate hyperplanes $x_i = 0 \wedge y_j = 0$, which are all flow-invariant. In particular, the vertices and the edges of $S_n \times S_m$ are flow-invariant.

The equilibria ξ_{ij} at the vertices are given by $\xi_{ij} = (\xi_i^n, \xi_j^m)$ with ξ_i^n a vertex in S_n and ξ_j^m a vertex in S_m, for $1 \leq i \leq n+1$ and $1 \leq j \leq m+1$.

Proposition 7 *If the matrices A and B in (6) satisfy for each pair (i, j), with $1 \leq i \leq n+1$, $1 \leq j \leq m+1$, that $a_{kj} \neq a_{ij}$ and $b_{li} \neq b_{ji}$, for $1 \leq k \leq n+1$, $k \neq i$, $1 \leq l \leq m+1$ and $l \neq j$, then all the edges of $S_n \times S_m$ correspond to heteroclinic connections connecting the equilibria ξ_{ij} at the vertices. All the edge heteroclinic connections involving the equilibrium ξ_{ij} are to or from the equilibria ξ_{kj} and ξ_{il}, and are as follows:*

- *If $a_{kj} - a_{ij} < 0$ then there is the edge heteroclinic connection $[\xi_{kj} \to \xi_{ij}]$. Otherwise there is the connection $[\xi_{ij} \to \xi_{kj}]$.*
- *If $b_{li} - b_{ji} < 0$ then there is the edge heteroclinic connection $[\xi_{il} \to \xi_{ij}]$. Otherwise there is the connection $[\xi_{ij} \to \xi_{il}]$.*

.

Proof The partial derivatives of (6) are given by

$$\frac{\partial \dot{x}_i}{\partial x_i} = (1 - x_i) \sum_{j=1}^{m+1} a_{ij} y_j - \mathbf{x}^T A \mathbf{y} \qquad \frac{\partial \dot{y}_j}{\partial y_j} = (1 - y_j) \sum_{i=1}^{n+1} b_{ji} x_i - \mathbf{y}^T B \mathbf{x}$$

$$\frac{\partial \dot{x}_i}{\partial x_k} = -x_i \sum_{j=1}^{m+1} a_{kj} y_j, \text{ for } k \neq i \qquad \frac{\partial \dot{y}_j}{\partial y_k} = -y_j \sum_{i=1}^{n+1} b_{ki} x_i, \text{ for } k \neq j$$

$$\frac{\partial \dot{x}_i}{\partial y_j} = x_i \left[a_{ij} - \sum_{l=1}^{n+1} a_{lj} x_l \right] \qquad \frac{\partial \dot{y}_j}{\partial x_i} = y_j \left[b_{ji} - \sum_{l=1}^{m+1} b_{li} y_l \right]$$

The Jacobian matrix at a vertex equilibria ξ_{ij} is a block matrix of the form

$$\begin{bmatrix} J_A & 0_{(n+1)\times(m+1)} \\ 0_{(m+1)\times(n+1)} & J_B \end{bmatrix}$$

with

$$J_A = \begin{bmatrix} a_{1j}-a_{ij} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{2j}-a_{ij} & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & .. \\ 0 & 0 & \dots & a_{i-1j}-a_{ij} & 0 & 0 & \dots & 0 \\ -a_{1j} & -a_{2j} & \dots & -a_{i-1j} & -a_{ij} & -a_{i+1j} & \dots & -a_{(n+1)j} \\ 0 & 0 & \dots & 0 & 0 & a_{i+1j}-a_{ij} & .. & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & .. \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{(n+1)j}-a_{ij} \end{bmatrix}$$

and

$$J_B = \begin{bmatrix} b_{1i}-b_{ji} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & b_{2i}-b_{ji} & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & .. \\ 0 & 0 & \dots & b_{j-1i}-b_{ji} & 0 & 0 & \dots & 0 \\ -b_{1i} & -b_{2i} & \dots & -b_{j-1i} & -b_{ji} & -b_{j+1i} & \dots & -b_{(m+1)i} \\ 0 & 0 & \dots & 0 & 0 & b_{j+1i}-b_{ji} & .. & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & .. \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{(m+1)i}-b_{ji} \end{bmatrix}$$

Simple computations show that there are no other equilibria at the edge joining the equilibrium ξ_{ij} to the equilibrium ξ_{kj} if and only if $a_{ij} \neq a_{kj}$. Analogously, the edge joining the equilibrium ξ_{ij} to the equilibrium ξ_{il} has no other equilibria if and only if $b_{ji} \neq b_{li}$.

Consequently, if $a_{ij} \neq a_{kj}$ and $b_{ji} \neq b_{li}$, the equilibrium ξ_{ij} has an edge heteroclinic connection to or from the equilibria ξ_{kj} and ξ_{il} with $1 \leq k \leq n+1$, $k \neq i$ and $1 \leq l \leq m+1$, $l \neq j$.

From the Jacobian matrix at each vertex equilibria ξ_{ij} it follows immediatly that, if $a_{kj} - a_{ij} < 0$ then there is the edge heteroclinic connection $[\xi_{kj} \to \xi_{ij}]$. Otherwise there is the connection $[\xi_{ij} \to \xi_{kj}]$. If $b_{li} - b_{ji} < 0$ then there is the edge heteroclinic connection $[\xi_{il} \to \xi_{ij}]$. Otherwise there is the connection $[\xi_{ij} \to \xi_{il}]$. □

Remark 1 *Note that:*

1. *All the eigenvalues of the equilibria at the vertices are real.*

2. *For each $1 \leq j \leq m+1$ there is an n-dimensional simplex S_{nj} contained in $S_n \times S_m$ with vertices $\xi_{1j}, \dots, \xi_{(n+1)j}$. For each $1 \leq i \leq n+1$ there is an m-dimensional simplex S_{im} contained in $S_n \times S_m$ with vertices $\xi_{i1}, \dots, \xi_{i(m+1)}$.*

Proposition 8 *Under the conditions of Theorem 7, that is, if $a_{kj} \neq a_{ij}$ and $b_{li} \neq b_{ji}$, for $1 \leq j, l \leq m+1$ and $1 \leq i, k \leq n+1$ with $j \neq l$ and $i \neq k$, there are no equilibria contained in the simplices S_{nj} and S_{im} and the dynamics inside each simplex and inside every face of each simplex corresponds to a continuum of connections from one vertex to another.*

Proof Tedious computations show that there are no equilibria contained in the simplices S_{nj} and S_{im}, if $a_{kj} \neq a_{ij}$ and $b_{li} \neq b_{ji}$, for $1 \leq j, l \leq m+1$ and $1 \leq i, k \leq n+1$ with $j \neq l$ and $i \neq k$.

Due to the condition $a_{kj} \neq a_{ij}$, for $1 \leq j \leq m+1$, it follows that for each column j of matrix A there is a strict ordering of its elements , that is, we have

$$a_{i_1 j} < a_{i_2 j} < \ldots < a_{i_{n+1} j}, \quad \text{with } \{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n+1\}.$$

In consequence, the equilibrium $\xi_{i_1 j}$ is a repeller and the equilibrium $\xi_{i_{n+1} j}$ is an attractor in the restriction to the simplex S_{nj} and thus the dynamics inside the simplex S_{nj} corresponds to a continuum of connections from $\xi_{i_1 j}$ to $\xi_{i_{n+1} j}$. Analogously for the faces of S_{nj}.

Due to the condition $b_{li} \neq b_{ji}$, for $1 \leq i \leq n+1$, we have that for each column i of the matrix B there is a strict ordering of its elements, that is, we have

$$b_{j_1 i} < b_{j_2 i} < \ldots < b_{j_{m+1} i}, \quad \text{with } \{j_1, \ldots, j_{m+1}\} = \{1, \ldots, m+1\}.$$

So, the equilibrium ξ_{ij_1} is a repeller and the equilibrium $\xi_{ij_{m+1}}$ is an attractor in the restriction to the simplex S_{im} and thus the dynamics inside the simplex S_{im} corresponds to a continuum of connections from ξ_{ij_1} to $\xi_{ij_{m+1}}$. Analogously for the faces of S_{im}. □

Proposition 9 *Under the conditions of Theorem 7, that is, if $a_{kj} \neq a_{ij}$ and $b_{li} \neq b_{ji}$, for $1 \leq j, l \leq m+1$ and $1 \leq i, k \leq n+1$ with $j \neq l$ and $i \neq k$, the dynamics at any square 2-face of $S_n \times S_m$ is conjugated to one of the dynamics pictured in Figure 5.*

Proof Consider a square 2-face of $S_n \times S_m$ with vertices ξ_{ij}, ξ_{il}, ξ_{kj} and ξ_{kl}, with $i, k \in \{1, \ldots, n+1\}$ and $j, l \in \{1, \ldots, m+1\}$. This corresponds to have $x_r = 0$ for all $r \neq i, k$ and $y_s = 0$ for all $s \neq j, l$ and, moreover, $x_k = 1 - x_i$ and $y_l = 1 - y_j$.

We find conditions for the existence of equilibria at the interior of the square 2-face, taking into account that $a_{ij} \neq a_{kj}$, $a_{il} \neq a_{kl}$, $b_{ji} \neq b_{li}$ and $b_{jk} \neq b_{lk}$.

Tedious computations show that, if $(a_{kj} - a_{ij})(a_{il} - a_{kl}) > 0$ and $(b_{li} - b_{ji})(b_{jk} - b_{lk}) > 0$, then there is only one equilibrium in the interior of the square 2-face whose coordinates satisfy

$$x_i = \frac{b_{jk} - b_{lk}}{b_{li} + b_{jk} - b_{ji} - b_{lk}}, \quad x_k = \frac{b_{li} - b_{ji}}{b_{li} + b_{jk} - b_{ji} - b_{lk}}, \quad x_r = 0 \text{ for all } r \neq i, k,$$

and

$$y_j = \frac{a_{il} - a_{kl}}{a_{il} + a_{kj} - a_{ij} - a_{kl}}, \quad y_l = \frac{a_{kj} - a_{ij}}{a_{il} + a_{kj} - a_{ij} - a_{kl}}, \quad y_s = 0 \text{ for all } s \neq j, l.$$

The Jacobian matrix at the equilibrium in the restriction to the square 2-face has eigenvalues given by

$$\pm\sqrt{\frac{(a_{kj} - a_{ij})(a_{il} - a_{kl})(b_{li} - b_{ji})(b_{jk} - b_{lk})}{a_{kj} + a_{il} - a_{ij} - a_{kl})(b_{li} + b_{jk} - b_{ji} - b_{lk})}}$$

which are symmetric real eigenvalues if $(a_{kj} - a_{ij})$, $(a_{il} - a_{kl})$, $(b_{li} - b_{ji})$ and $(b_{jk} - b_{lk})$ have all the same sign, in which case the dynamics at the square 2-face is conjugated to the

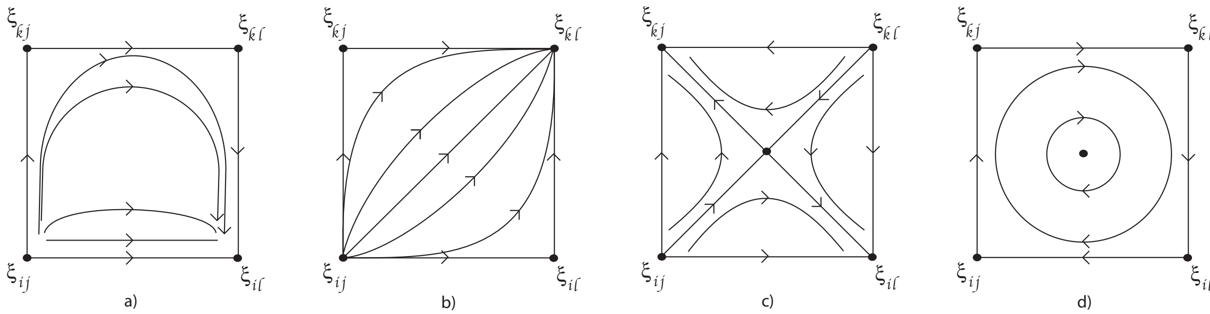

Figure 5: Possible dynamics, up to conjugacy, at the square 2-faces of $S_n \times S_m$, when there is an edge heteroclinic network at the boundary of $S_n \times S_m$.

dynamics in Figure 5 c). Otherwise, the eigenvalues are a pair of pure imaginary complex numbers and the dynamics at the square 2-face is conjugated to the dynamics in Figure 5 d).

If $(a_{kj} - a_{ij})(a_{il} - a_{kl}) < 0$ or $(b_{li} - b_{ji})(b_{jk} - b_{lk}) < 0$, then there are no equilibria in the interior of the square 2-face. If only one of the products is negative, the dynamics at the square 2-face is conjugated to the dynamics in Figure 5 a). Otherwise, if both are negative the dynamics is conjugated to the one in Figure 5 b).

□

Remark 2 *In our study we are excluding the existence of heteroclinic connections between the equilibria at the vertices and other equilibria at the boundary of $S_n \times S_m$, and thus we are not considering the case c) in Figure 5.*

Theorem 10 *If the matrices A and B in (6) satisfy for each pair (i, j), with $1 \leq i \leq n+1$, $1 \leq j \leq m+1$, that*

- *$a_{kj} - a_{ij}$ and $b_{li} - b_{ji}$, are all nonzero and are not all of the same sign,*
- *$(a_{kj} - a_{ij}), (a_{il} - a_{kl})$, $(b_{li} - b_{ji})$ and $(b_{jk} - b_{lk})$ are not all of the same sign,*

for $1 \leq k \leq n+1$, $k \neq i$, $1 \leq l \leq m+1$ and $l \neq j$, then there is an edge heteroclinic network at the boundary of $S_n \times S_m$ connecting the equilibria ξ_{ij} at the vertices.

Proof From the proof of Propositions 7, 8 and 9, the stable and unstable manifolds of all the vertex equilibria are contained in the union of all the simplices S_{nj}, S_{im}, for $1 \leq j \leq m+1$ and $1 \leq i \leq n+1$, and all the square 2-faces.

If for all $1 \leq i \leq n+1$, $1 \leq j \leq m+1$, we have that $a_{kj} - a_{ij}$ and $b_{li} - b_{ji}$ are all nonzero then, by Proposition 7, there are edge heteroclinic connections between the equilibria ξ_{ij} at the vertices

Moreover, if $a_{kj} - a_{ij}$ and $b_{li} - b_{ji}$ are not all of the same sign then it follows that all the equilibria ξ_{ij} at the vertices are hyperbolic saddles and thus those equilibria and the heteroclinic connections between them form an heteroclinic network at the boundary of $S_n \times S_m$.

The condition that $(a_{kj} - a_{ij}), (a_{il} - a_{kl})$, $(b_{li} - b_{ji})$ and $(b_{jk} - b_{lk})$ are not all of the same sign ensures that there is no square 2-face with dynamics conjugated to the dynamics in Figure 5 c) and so the heteroclinic network at the boundary of $S_n \times S_m$ is an edge heteroclinic network. □

Example 11 Consider equations (6) defined in $S_2 \times S_2$ with the entries of the matrices A and B satisfying, respectively,

$$\begin{array}{ccc} a_{21} < a_{11} < a_{31} & & b_{21} < b_{11} < b_{31} \\ a_{32} < a_{22} < a_{12} & \text{and} & b_{32} < b_{22} < b_{12} \\ a_{13} < a_{33} < a_{23} & & b_{13} < b_{33} < b_{23}. \end{array}$$

Then, there is and edge heteroclinic network at the boundary of $S_2 \times S_2$ and the dynamics at the square 2-faces is conjugated to the dynamics shown in Figure 6. ◊

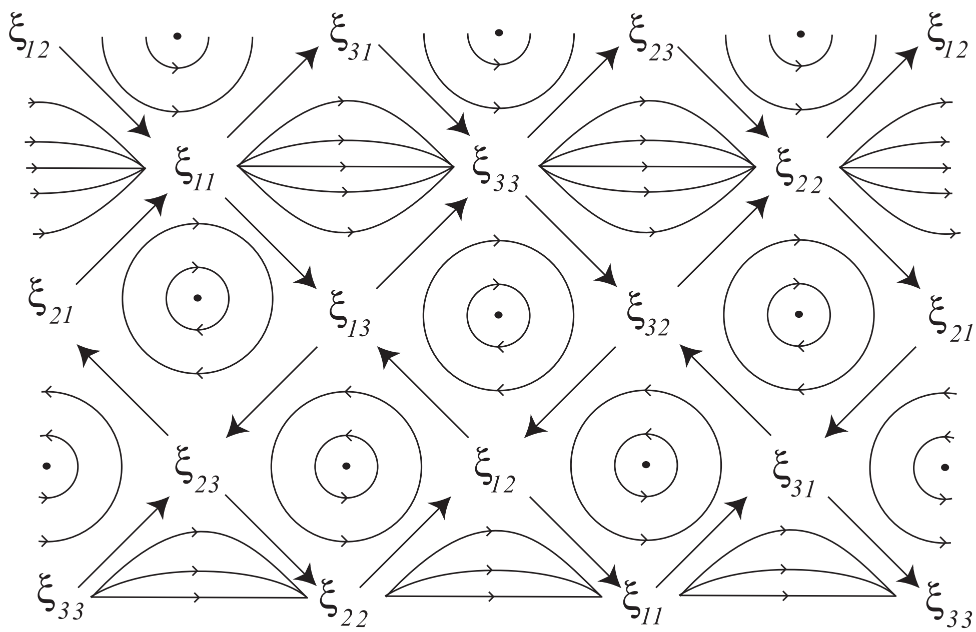

Figure 6: Dynamics, up to conjugacy, at the square 2-faces of $S_2 \times S_2$ for Example 11 and at square 2-faces of $S_3 \times S_3$ for Example 15.

Next we prove that in a product of simplices no edge heteroclinic network has a Kirk&Silber subnetwork. This leaves to believe that there can be switching along the connections of an edge network. We identify those connections in Theorem 13.

Proposition 12 *An edge heteroclinic network Σ in $S_n \times S_m$ never has a Kirk&Silber subnetwork.*

Proof We can ignore the continua of connections in the simplices S_{nj} and S_{im}, for $1 \leq j \leq m+1$ and $1 \leq i \leq n+1$, since there is a continuum of connections between two vertex equilibria in S_{nj} (or S_{im}) if and only if there is an edge heteroclinic connection between them. Thus, we will consider only edge heteroclinic connections and 2-dimensional continua of connections contained in square 2-faces of $S_n \times S_m$.

Suppose that an edge heteroclinic network Σ has a Kirk&Silber subnetwork involving only edge connections. Then, there are four equilibria $e_{i_k j_k}$, $k = 1, \ldots, 4$, in Σ such that there are the following edge heteroclinic connections in Σ

$$[e_{i_1 j_1} \to e_{i_2 j_2}] \quad [e_{i_2 j_2} \to e_{i_3 j_3}] \quad [e_{i_2 j_2} \to e_{i_4 j_4}] \quad [e_{i_3 j_3} \to e_{i_1 j_1}] \quad [e_{i_4 j_4} \to e_{i_1 j_1}]$$

In the edge connection $[e_{i_1 j_1} \to e_{i_2 j_2}]$, by Theorem 7, we must have $i_1 = i_2$ or $j_1 = j_2$. Let us assume, without loss of generality, that $j_1 = j_2$ and thus $i_1 \neq i_2$. In that case, we must have $j_3 = j_1 = j_2$ and $i_3 \neq i_1$ and $i_3 \neq i_2$. Otherwise, if $j_3 \neq j_1 = j_2$ then, by Theorem 7, we should have $i_1 = i_2 = i_3$, which is impossible. By the same argument, we must have $j_4 = j_1 = j_2$ and $i_4 \neq i_1$.

We would then have the following edge heteroclinic connections in Σ

$$[e_{i_1 j_1} \to e_{i_2 j_1}], \quad [e_{i_2 j_1} \to e_{i_3 j_1}], \quad [e_{i_2 j_1} \to e_{i_4 j_1}], \quad [e_{i_3 j_1} \to e_{i_1 j_1}] \text{ and } [e_{i_4 j_1} \to e_{i_1 j_1}],$$

and thus we would have

$$a_{i_2 j_1} - a_{i_1 j_1} > 0, \quad a_{i_3 j_1} - a_{i_2 j_1} > 0, \quad a_{i_4 j_1} - a_{i_2 j_1} > 0, \quad a_{i_1 j_1} - a_{i_3 j_1} > 0 \text{ and } a_{i_1 j_1} - a_{i_4 j_1} > 0. \quad (7)$$

But,

$$(a_{i_2 j_1} - a_{i_1 j_1}) + (a_{i_3 j_1} - a_{i_2 j_1}) + (a_{i_1 j_1} - a_{i_3 j_1}) = 0$$

and

$$(a_{i_2 j_1} - a_{i_1 j_1}) + (a_{i_4 j_1} - a_{i_2 j_1}) + (a_{i_1 j_1} - a_{i_4 j_1}) = 0.$$

So, (7) is impossible. Thus, there cannot be any Kirk&Silber subnetwork with only edge connections for the edge network Σ.

Therefore, there cannot be a Kirk&Silber subnetwork involving edge connections and 2-dimensional continua of connections. This comes from the fact that all the connections must be in the same square 2-face and it is impossible to have a Kirk&Silber subnetwork in that case (see Figure 5 b)).

We can see geometrically that there cannot be a Kirk&Silber subnetwork with only 2-dimensional continua of connections using the fact that two square 2-faces either do not intersect at all or they have only one edge in common. The closest we could have to a Kirk&Silber subnetwork with only 2-dimensional continua of connections is pictured in Figure 7, but even this situation is impossible since the dynamics in the edges connecting the equilibrium $\xi_{i_3 j_3}$ to the equilibria $\xi_{i_1 j_1}$ and $\xi_{i_8 j_8}$ cannot be defined.

□

Theorem 13 *Let Σ be an edge network in $S_n \times S_m$. There is switching along an edge heteroclinic connection $[\xi_i \to \xi_j]$ if and only if $s^i \leq s^j$, with (s^k, u^k) the type of ξ_k, for $k = i, j$.*

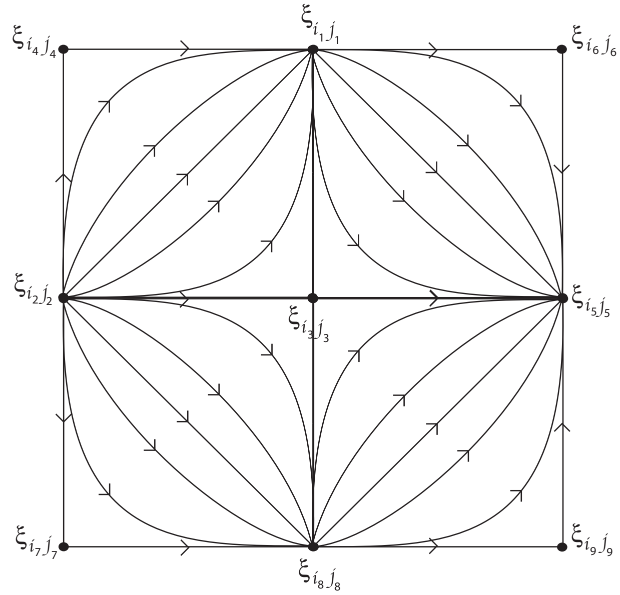

Figure 7: Geometric justification for the absence of a Kirk&Silber subnetwork with only 2-dimensional continua of connections. The dynamics pictured is impossible as the dynamics in the edges connecting the equilibrium $\xi_{i_3 j_3}$ to the equilibria $\xi_{i_1 j_1}$ and $\xi_{i_8 j_8}$ cannot be defined.

Proof The network Σ contains $p = (n+1) \times (m+1)$ equilibria and each equilibrium ξ_k has $r = m + n = s^k + u^k$ eigendirections at the product of simplices $S_n \times S_m$.

In the first part of the proof we ignore the existence of 2-dimensional continua of face heteroclinic connections in Σ, if any, and consider only the edge heteroclinic connections.

We can consider, without loss of generality, the heteroclinic connection $[\xi_{p-1} \to \xi_p]$. Let $s = s^{p-1}$, $u = u^{p-1}$, $\overline{s} = s^p$ and $\overline{u} = u^p$.

Let $-c_1, \ldots, -c_s < 0$ be the eigenvalues of ξ_{p-1} in the direction of $\xi_{i_1}, \ldots, \xi_{i_s}$, respectively, with $i_k \in \{1, \ldots, p-2\}$, for $k = 1, \ldots, s$, and $e > 0$ the eigenvalue in the direction of ξ_p. Let $\overline{e}_1, \ldots, \overline{e}_{\overline{u}} > 0$ be the eigenvalues of ξ_p in the direction of $\xi_{j_1}, \ldots, \xi_{j_{\overline{u}}}$, respectively, with $j_l \in \{1, \ldots, p-2\}$, for $l = 1, \ldots, \overline{u}$, and $-\overline{c} < 0$ the eigenvalue in the direction of ξ_{p-1}.

We follow the computations in [2] and in the proof of Theorem 1. By the C^1 extension by Ruelle [29] of Hartman's results [13], the vector field may be linearised around each equilibrium ξ_i, up to a set of measure zero.

We can choose local coordinates $(w_1, \ldots, w_n) \in S_n$ such that the linearization of the flow near ξ_{p-1} is given by

$$\begin{aligned} \dot{w}_k &= -c_k w_k && k = 1, \ldots, s \\ \dot{w}_{s+l} &= e_l w_{s+l} && l = 1, \ldots, u-1 \\ \dot{w}_r &= e w_r && \end{aligned}$$

with $e_l > 0$, $l = 1, \ldots, u-1$ the remaining eigenvalues of ξ_{p-1} and the linearization of the flow near ξ_p is given by

$$\begin{aligned}
&\dot{w}_k = -\overline{c}_k w_k \qquad k = 1, \dots, \overline{s} - 1\\
&\dot{w}_{\overline{s}-1+l} = \overline{e}_l w_{\overline{s}-1+l} \qquad l = 1, \dots, \overline{u}\\
&\dot{w}_r = -c w_r
\end{aligned}$$

with $-\overline{c}_k < 0$, $k = 1, \dots, \overline{s} - 1$ the remaining eigenvalues of ξ_p.

The flow near ξ_{p-1} is then given by

$$\mathcal{F}_t^{p-1}(w_1, w_2, \dots, w_{r-1}, w_r) = (w_1 e^{-c_1 t}, \dots, w_s e^{-c_s t}, w_{s+1} e^{e_1 t}, \dots, w_{r-1} e^{e_{u-1} t}, w_r e^{et}), \quad (8)$$

and the flow near ξ_p is then given by

$$\mathcal{F}_t^{p}(w_1, w_2, \dots, w_{r-1}, w_r) = (w_1 e^{-\overline{c}_1 t}, \dots, w_{\overline{s}-1} e^{-\overline{c}_{\overline{s}-1} t}, w_{\overline{s}} e^{\overline{e}_1 t}, \dots, w_{r-1} e^{\overline{e}_{\overline{u}} t}, w_r e^{-ct}). \quad (9)$$

In a neighbourhood of the nodes ξ_{p-1} and ξ_p where the flow can be linearized, we define cross-sections for the connection $[\xi_{p-1} \to \xi_p]$ and for each of the connections involving those nodes and the nodes $\xi_{i_1}, \dots, \xi_{i_s}$ and the nodes $\xi_{j_1}, \dots, \xi_{j_{\overline{u}}}$ as follows

$$\begin{aligned}
\Sigma^{\text{in}}_{p-1,i_k} &= \{(w_1, \dots, w_{k-1}, h, w_{k+1}, \dots, w_r) : \ 0 < w_1, w_2, \dots, w_r < h\} \qquad k = 1, \dots, s\\
\Sigma^{\text{out}}_{p-1,p} &= \{(z_1, z_2, z_3, \dots, z_{r-1}, h) : \ 0 < z_1, z_2, \dots, z_{r-1} < h\}
\end{aligned}$$

$$\begin{aligned}
\Sigma^{\text{in}}_{p,p-1} &= \{(w_1, w_2, w_3, \dots, w_{r-1}, h) : \ 0 < w_2, w_3, \dots, w_{r-1} < h\}\\
\Sigma^{\text{out}}_{p,j_1} &= \{(z_1, z_2, \dots, z_{\overline{s}-1}, h, z_{\overline{s}+1}, \dots, z_n) : \ 0 < z_1, z_3, \dots, z_r < h\}\\
&\dots\\
\Sigma^{\text{out}}_{p,j_{\overline{u}}} &= \{(z_1, z_2, z_3, \dots, h, z_r) : \ 0 < z_1, z_2, \dots, z_r < h\}
\end{aligned}$$

where $0 < h < 1$ is a positive number, small enough to guarantee transversality of the flow near each saddle. We can do a linear rescaling of the local coordinates near each each equilibrium such that $h = 1$. For ease of computations, and without loss of generality, we will consider $h = 1$.

Note that, for each equilibrium ξ_{p-1} and ξ_p, the union of the cross sections, as defined above, is the boundary of a neighbourhood of the equilibrium and thus it intersects all the trajectories approaching or leaving that neighbourhood of the equilibrium.

Using the linear flow near ξ_{p-1}, the local transition maps $\Psi_{i_k,p-1,p} : \Sigma^{\text{in}}_{p-1,i_k} \to \Sigma^{\text{out}}_{p-1,p}$, $k = 1, \dots, s$, are given by,

$$\begin{aligned}
&\Psi_{i_k,p-1,p}(w_1, w_2, \dots, w_{k-1}, w_{k+1}, \dots, w_s, \dots, w_r) =\\
&\quad = \left(w_1 w_r^{\frac{c_1}{e}}, \dots, w_{k-1} w_r^{\frac{c_{k-1}}{e}}, w_r^{\frac{c_k}{e}}, w_{k+1} x_r^{\frac{c_{k+1}}{e}}, \dots, w_s w_r^{\frac{c_s}{e}}, w_{s+1} w_r^{-\frac{e_1}{e}}, \dots, w_{r-1} w_r^{-\frac{e_{u-1}}{e}} \right),
\end{aligned}$$

Thus, the points in $\Sigma^{\text{out}}_{p-1,p}$ that come from $\Sigma^{\text{in}}_{p-1,i_k}$, $k=1,\dots,s$, satisfy

$$\begin{cases} z_1 < z_k^{\frac{c_1}{c_k}} \\ \dots \\ z_{k-1} < z_k^{\frac{c_{k-1}}{c_k}} \\ z_{k+1} < z_k^{\frac{c_{k+1}}{c_k}} \\ \dots. \\ z_s < z_k^{\frac{c_s}{c_k}} \end{cases}$$

In the neighbourhood of ξ_p, using the linearized flow, we define the maps Ψ_{p-1,p,j_l} : $\Sigma^{\text{in}}_{p,p-1} \longrightarrow \Sigma^{\text{out}}_{p,j_l}$, $l=1,\dots,\overline{u}$ by

$$\Psi_{p-1,p,j_1}(w_1,w_2,\dots,w_{r-1}) = \left(w_1 w_{\overline{s}}^{\frac{\overline{c}_1}{\overline{e}_1}}, \dots, w_{\overline{s}-1} w_{\overline{s}}^{\frac{\overline{c}_{\overline{s}-1}}{\overline{e}_1}}, w_{\overline{s}+1} w_{\overline{s}}^{-\frac{\overline{e}_2}{\overline{e}_1}}, \dots, w_{r-1} w_{\overline{s}}^{-\frac{\overline{e}_{\overline{u}}}{\overline{e}_1}}, w_{\overline{s}}^{\frac{c}{\overline{e}_1}} \right),$$

....

$$\Psi_{p-1,p,j_{\overline{u}}}(w_1,w_2,\dots,w_{r-1}) = \left(w_1 w_{r-1}^{\frac{\overline{c}_1}{\overline{e}_{\overline{u}}}}, \dots, w_{\overline{s}-1} w_{r-1}^{\frac{\overline{c}_{\overline{s}-1}}{\overline{e}_{\overline{u}}}}, w_{\overline{s}} w_{r-1}^{-\frac{\overline{e}_1}{\overline{e}_{\overline{u}}}}, \dots, w_{r-2} w_{r-1}^{-\frac{\overline{e}_{\overline{u}-1}}{\overline{e}_{\overline{u}}}}, w_{r-1}^{\frac{c}{\overline{e}_{\overline{u}}}} \right),$$

Thus, the points in $\Sigma^{\text{in}}_{p,p-1}$ that go to $\Sigma^{\text{out}}_{p,j_1}$ satisfy

$$\begin{cases} w_{\overline{s}+1} < w_{\overline{s}}^{\frac{\overline{e}_2}{\overline{e}_1}} \\ \dots. \\ w_{r-1} < w_{\overline{s}}^{\frac{\overline{e}_{\overline{u}}}{\overline{e}_1}} \end{cases}$$

and the points that go to $\Sigma^{\text{out}}_{p,j_{\overline{u}}}$ satisfy

$$\begin{cases} w_{\overline{s}} < w_{r-1}^{\frac{\overline{e}_1}{\overline{e}_{\overline{u}}}} \\ \dots. \\ w_{r-2} < w_{r-1}^{\frac{\overline{e}_{\overline{u}-1}}{\overline{e}_{\overline{u}}}} \end{cases}$$

We can consider the transition map $\Phi_{p-1,p}$ from $\Sigma^{\text{out}}_{p-1,p}$ to $\Sigma^{\text{in}}_{p,p-1}$ as

$$\Phi_{p-1,p}(z_1,\dots,z_{r-1}) = (a_1 z_1, \dots, a_{r-1} z_{r-1})$$

with a_i, $i=1,\dots,n-1$ constants.

As a simplification, and without loss of generality, for the proposes of the proof we can approximate the transition map $\Phi_{p-1,p}$ by the identity map.

So, in $\Sigma^{\text{in}}_{p,p-1}$, we get the conditions

$$\begin{cases} w_2 < w_1^{\frac{c_2}{c_1}} \\ w_3 < w_1^{\frac{c_3}{c_1}} \\ \dots. \\ w_s < w_1^{\frac{c_s}{c_1}} \end{cases} \quad \begin{cases} w_1 < w_2^{\frac{c_1}{c_2}} \\ w_3 < w_2^{\frac{c_3}{c_2}} \\ \dots. \\ w_s < w_2^{\frac{c_s}{c_2}} \end{cases} \quad \dots. \quad \begin{cases} w_1 < w_s^{\frac{c_1}{c_s}} \\ w_2 < w_s^{\frac{c_2}{c_s}} \\ \dots. \\ w_{s-1} < w_s^{\frac{c_{s-1}}{c_s}} \end{cases}$$

and

$$
\begin{cases}
w_{\overline{s}+1} < w_{\overline{s}}^{\frac{\overline{e}_2}{\overline{e}_1}} \\
w_{\overline{s}+2} < w_{\overline{s}}^{\frac{\overline{e}_3}{\overline{e}_1}} \\
\dots. \\
w_{r-1} < w_{\overline{s}}^{\frac{\overline{e}_{\overline{u}}}{\overline{e}_1}}
\end{cases}
\quad
\begin{cases}
w_{\overline{s}} < w_{\overline{s}+1}^{\frac{\overline{e}_1}{\overline{e}_2}} \\
w_{\overline{s}+2} < w_{\overline{s}+1}^{\frac{\overline{e}_3}{\overline{e}_2}} \\
\dots. \\
w_{r-1} < w_{\overline{s}+1}^{\frac{\overline{e}_{\overline{u}}}{\overline{e}_2}}
\end{cases}
\quad \dots. \quad
\begin{cases}
w_{\overline{s}} < w_{r-1}^{\frac{\overline{e}_1}{\overline{e}_{\overline{u}}}} \\
w_{\overline{s}+1} < w_{r-1}^{\frac{\overline{e}_2}{\overline{e}_{\overline{u}}}} \\
\dots. \\
w_{r-2} < w_{r-1}^{\frac{\overline{e}_{\overline{u}-1}}{\overline{e}_{\overline{u}}}}
\end{cases}
$$

It follows from the proof of Theorem 1 that a necessary and sufficient condition for switching along the connection $[\xi_{p-1} \to \xi_p]$ is that there are no two conditions of the form, for example, $w_k < w_l^{\frac{\alpha}{\beta}}$ and $w_l < w_k^{\frac{\delta}{\gamma}} \Leftrightarrow w_k > w_l^{\frac{\gamma}{\delta}}$, with $\frac{\alpha}{\beta} \neq \frac{\gamma}{\delta}$, involving each pair of variables w_k and w_l, for $k, l \in 1, \dots, r$ and $k \neq l$. As we have seen in the proof of Theorem 1, in the plane (w_k, w_l), the curves $w_k = w_l^{\frac{\alpha}{\beta}}$ and $w_k = w_l^{\frac{\gamma}{\delta}}$, intersect at the origin, and thus divide $\Sigma^{\text{in}}_{p,p-1}$ in at most three regions, when we would need four region to get switching.

Thus, a necessary and sufficient condition for switching along the connection $[\xi_{p-1} \to \xi_p]$ is that $s \leq \overline{s}$. For example, if $s = \overline{s}+1$, then we would have $w_{\overline{s}} < w_{\overline{s}+1}^{\frac{c_{\overline{s}}}{c_{\overline{s}+1}}}$ and $w_{\overline{s}} > w_{\overline{s}+1}^{\frac{\overline{e}_1}{\overline{e}_2}}$.

Until now, we have ignored the existence of 2-dimensional continua of face connections. Now, we are going to analyse what differs in the presence of 2-dimensional continua of connections.

So, assume that there is a 2-dimensional continuum of connections, for example, in the two-dimensional square face containing the connections $[\xi_{i_1} \to \xi_{p-1}]$ and $[\xi_{i_2} \to \xi_{p-1}]$. Let ξ_{i_k} be the other equilibrium vertex at that face. Thus, the other edge heteroclinic connections in the face are $[\xi_{i_k} \to \xi_{i_1}]$ and $[\xi_{i_k} \to \xi_{i_2}]$ and there is a 2-dimensional continuum of connections from ξ_{i_k} to ξ_{p-1}.

The difference between this situation and the situation analysed above without 2-dimensional continua of connections is that not all points in $\Sigma^{\text{out}}_{i_k,i_j}$, for $j = 1, 2$, will follow the connection $[\xi_{i_k} \to \xi_{i_j}]$. Some will follow the continuum of connections $[\xi_{i_k} \to \xi_{p-1}]$.

So, roughly speaking, the situation is as if from the points in $\Sigma^{\text{out}}_{p-1,p}$ that come from $\Sigma^{\text{in}}_{p-1,i_1}$, some follow the connection $[\xi_{i_1} \to \xi_{p-1}]$, while the others follow a connection $[\xi_{i_k} \to \xi_{p-1}]$. Analogously, from the points in $\Sigma^{\text{out}}_{p-1,p}$ that come from $\Sigma^{\text{in}}_{p-1,i_2}$, some follow the connection $[\xi_{i_2} \to \xi_{p-1}]$, while the others follow a connection $[\xi_{i_k} \to \xi_{p-1}]$. See Figure 8.

More specifically, there are $\alpha > \frac{c_{i_2}}{c_{i_1}}$ and $\beta < \frac{c_{i_2}}{c_{i_1}}$ such that from the points in $\Sigma^{\text{out}}_{p-1,p}$ that come from $\Sigma^{\text{in}}_{p-1,i_1}$, the ones satisfying $z_{i_2} < z_{i_1}^{\alpha}$, follow the connection $[\xi_{i_1} \to \xi_{p-1}]$, while the ones satisfying $z_{i_1}^{\alpha} < z_{i_2} < z_{i_1}^{\frac{c_{i_2}}{c_{i_1}}}$ are the points that follow the connection $[\xi_{i_k} \to \xi_{p-1}]$ and the points in $\Sigma^{\text{out}}_{p-1,p}$ that come from $\Sigma^{\text{in}}_{p-1,i_2}$, the ones satisfying $z_{i_2} > z_{i_1}^{\beta}$, follow the connection $[\xi_{i_2} \to \xi_{p-1}]$, while the ones satisfying $z_{i_1}^{\frac{c_{i_2}}{c_{i_1}}} < z_{i_2} < z_{i_1}^{\beta}$ are the points that follow the connection $[\xi_{i_k} \to \xi_{p-1}]$. See Figure 9.

Analogously, if there is a continuum of connections in a two-dimensional square face containing any two connections forward from ξ_p.

So, the necessary and sufficient condition for switching, that there are no two conditions of the form $w_k < w_l^{\frac{\alpha}{\beta}}$ and $w_k > w_l^{\frac{\gamma}{\delta}}$, is still valid. $\square$

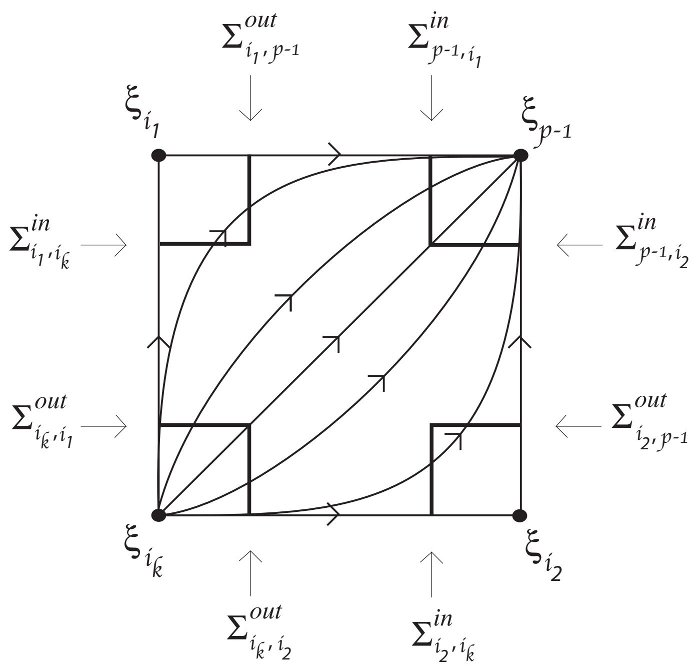

Figure 8: Sketch of the local cross-sections and their intersection with the flow when there is a continuum of connections in a square 2-face.

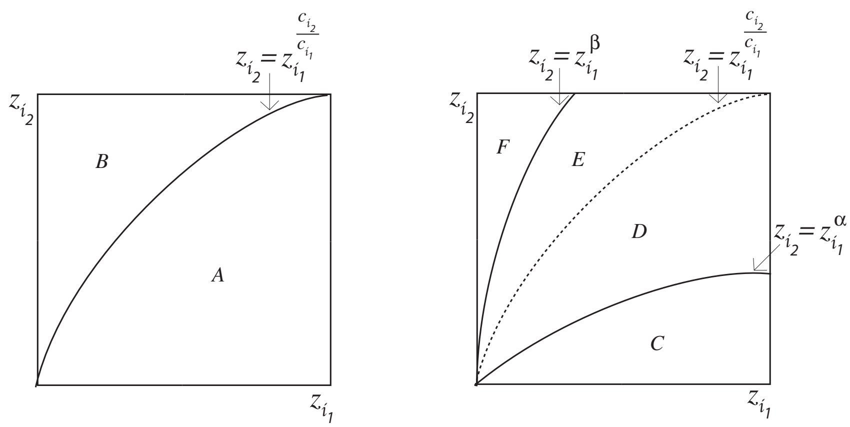

Figure 9: Regions of points in section $\Sigma^{\text{out}}_{p-1,p}$, when there is a 2-dimensional continuum of connections (right) and when there isn't (left). Regions A and B correspond, respectively, to the points that followed the connections $[\xi_{i_1} \to \xi_{p-1}]$ and $[\xi_{i_2} \to \xi_{p-1}]$. Regions C and F correspond, respectively, to the points that followed the connections $[\xi_{i_1} \to \xi_{p-1}]$ and $[\xi_{i_2} \to \xi_{p-1}]$. Regions D and E correspond to the points that followed a connection $[\xi_{i_k} \to \xi_{p-1}]$.

Corollary 14 *There is switching along all the connections of an edge network Σ in $S_n \times S_m$ if and only if $n = m$ and all saddles are of type (n, n).*

Proof By Theorem 13, there is switching along a connection $[\xi_i \to \xi_j]$ of Σ if and only if $s^i \leq s^j$. By a proof analogous to the proof of Lemma 3, this can happen for all connections in the network if and only if $n+m$ is even and all saddles are of type $(\frac{r}{2}, \frac{r}{2})$, with $r = n+m$. We prove that this condition can only occur if and only if $n = m$. Given the conditions in Theorem 10 for the existence of an edge heteroclinic network in $S_n \times S_m$, it follows that for each column $1 \leq j \leq m+1$ of the matrix A there must be a strict ordering of its elements, that is,

$$a_{i_1 j} < a_{i_2 j} < \ldots < a_{i_{n+1} j}, \quad \text{with } \{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n+1\}.$$

Analogously, for each column $1 \leq i \leq n+1$ of the matrix B there must be a strict ordering of its elements, that is,

$$b_{j_1 i} < b_{j_2 i} < \ldots < b_{j_{m+1} i}, \quad \text{with } \{j_1, \ldots, j_{m+1}\} = \{1, \ldots, m+1\}.$$

If $n > m$ then the equilibria $\xi_{i_1 j}$ and $\xi_{i_k j}$ are respectively of type $(s^{i_1 j}, u^{i_1 j})$ and $(s^{i_k j}, u^{i_k j})$ with $s^{i_1 j} \geq n > \frac{r}{2}$ and $u^{i_k j} \geq n > \frac{r}{2}$ and thus fail the condition $(s^{i_1 j}, u^{i_1 j}) = (\frac{r}{2}, \frac{r}{2}) = (s^{i_k j}, u^{i_k j})$.

If $n < m$ then the equilibria $\xi_{i j_1}$ and $\xi_{i j_l}$ are respectively of type $(s^{i j_1}, u^{i j_1})$ and $(s^{i j_l}, u^{i j_l})$ with $s^{i j_1} \geq m > \frac{r}{2}$ and $u^{i j_l} \geq m > \frac{r}{2}$ and thus fail the condition $(s^{i j_1}, u^{i j_1}) = (\frac{r}{2}, \frac{r}{2}) = (s^{i j_l}, u^{i j_l})$.

If $n = m$ then all the equilibria, including equilibria $\xi_{i_1 j}$, $\xi_{i_k j}$, $\xi_{i j_1}$ and $\xi_{i j_l}$, can be of type (n, n). □

Example 15 Consider as an example the family of bimatrix games in $S_2 \times S_2$ defined in Example 11. Another example is the family of bimatrix games in $S_3 \times S_3$ with matrices A and B satisfying, respectively,

$$\begin{array}{lcl} a_{21} < a_{11} < a_{41} < a_{31} & & b_{21} < b_{41} < b_{11} < b_{31} \\ a_{42} < a_{32} < a_{22} < a_{12} & \text{and} & b_{32} < b_{22} < b_{42} < b_{12} \\ a_{13} < a_{43} < a_{33} < a_{23} & & b_{13} < b_{33} < b_{23} < b_{43} \\ a_{34} < a_{24} < a_{14} < a_{44} & & b_{44} < b_{14} < b_{34} < b_{24}. \end{array}$$

with dynamics at the square 2-faces conjugated to the dynamics shown in Figures 6, 10, 11 and 12. ◊

Denote by $\lambda_{\overline{il}j}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{lj} and by $\lambda_{i\overline{jk}}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{ik}. Analysing the Jacobian matrix of (6) at ξ_{ij} (see the proof of Proposition 7), we conclude that the eigenvalues at ξ_{ij} satisfy $\lambda_{\overline{il}j} = -\lambda_{\overline{li}j}$, for $i \neq l$ and $\lambda_{i\overline{jk}} = -\lambda_{i\overline{kj}}$, for $k \neq j$ for all $i, l \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, m\}$. It follows from this resonance that, for each hyperplane the sum of the expanding eigenvalues of the equilibria of the network in that hyperplane equals the sum of the expanding eigenvalues This means that, on average the contraction around the network equals the expansion, and so, we can expect some *neutral stability* of the network in the sense that trajectories orbit around the network in closed invariant sets which contain no equilibria without leading towards neither away from the network in average. See Appendix A for more considerations on this.

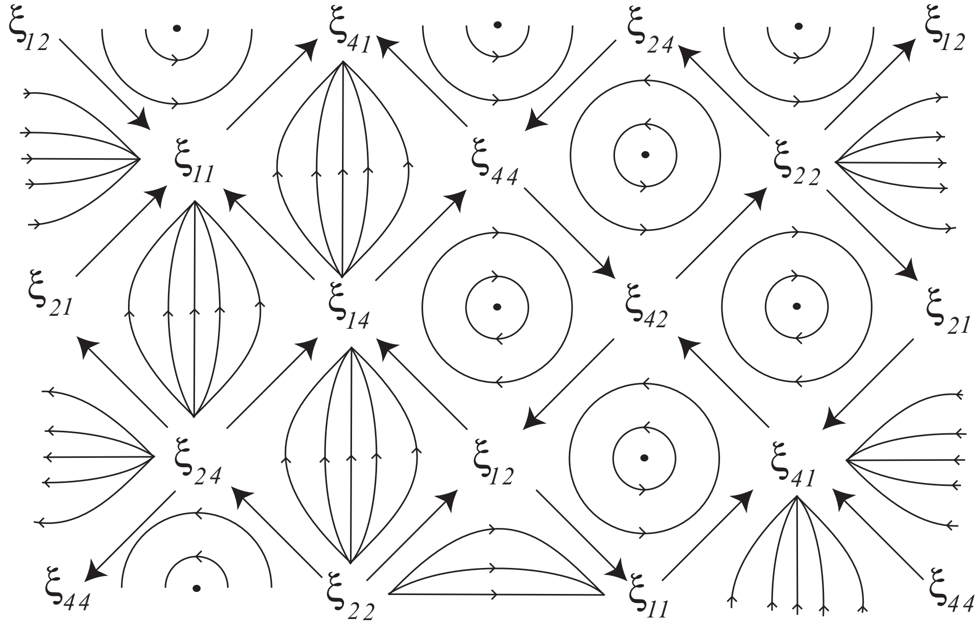

Figure 10: Dynamics, up to conjugacy, at square 2-faces of $S_3 \times S_3$ for Example 15.

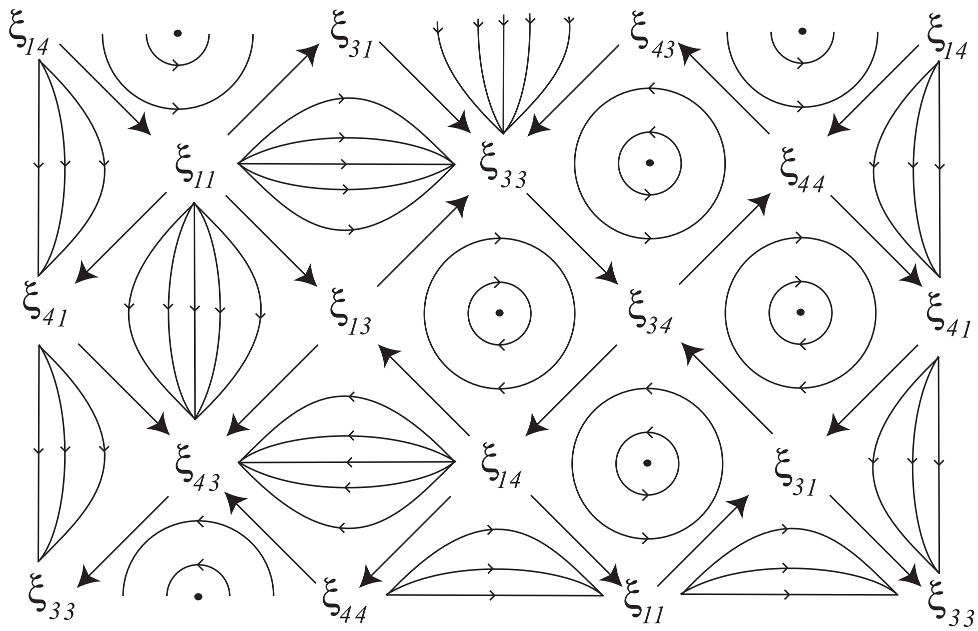

Figure 11: Dynamics, up to conjugacy, a square 2-faces of $S_3 \times S_3$ for Example 15.

In fact, we conjecture the existence of a flow invariant *tubular neighbourhood in* $intS_n \times S_m$ of the edge network. By a *tubular neighbourhood in* $intS_n \times S_m$ we mean the intersection of a tubular neighbourhood of the edge network with the interior of $S_n \times S_m$. That is, the distribution of all solutions in $intS_n \times S_m$ traveling around the network correspond to the intersection of a 'tube' centered on the network with $intS_n \times S_m$.

Note that, since the unstable manifolds of all the equilibria in the edge network are all

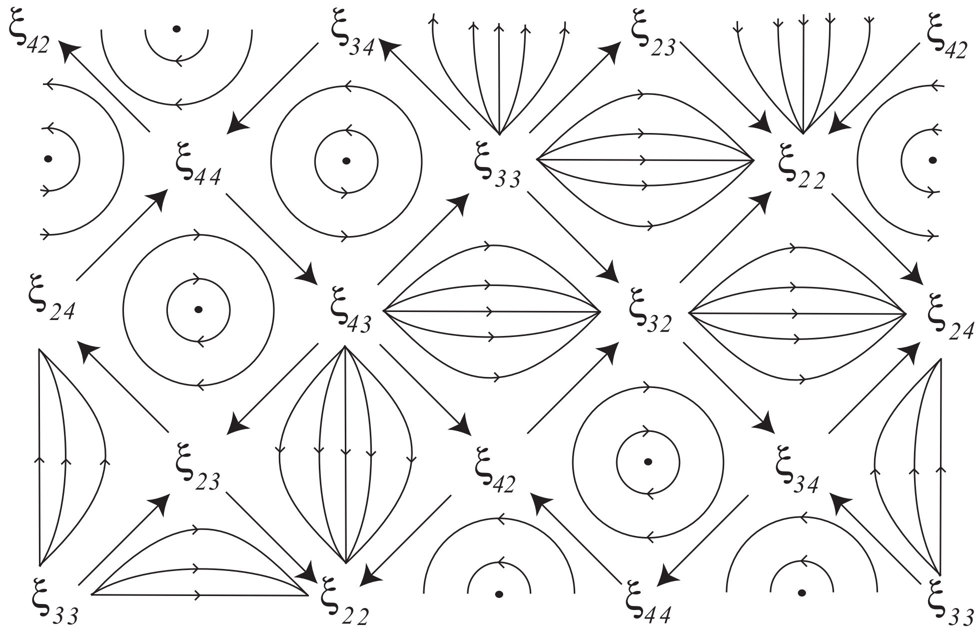

Figure 12: Dynamics, up to conjugacy, at square 2-faces of $S_3 \times S_3$ for Example 15.

contained in the network, we have that a trajectory with initial condition near the edge heteroclinic network stays near the network.

For each equilibrium ξ_j in the edge network and for each connection $[\xi_{i_l} \to \xi_j]$ consider a section $In^{i_l,j}$ in $intS_n \times S_m$ near ξ_j transversal to the connection and to the flow near the connection. For each section $In^{i_l,j}$ consider the maximal section $I^{i_l,j}$ in $intS_n \times S_m$ containing it.

Near each equilibrium ξ_j, iterating the In sections by the flow we get for each connection $[\xi_j \to \xi_{k_l}]$ a section Out^{j,k_l} in $intS_n \times S_m$ near ξ_j transversal to the connection and to the flow near the connection.

This way we get a flow invariant neigbourhood in $intS_n \times S_m$ for each equilibrium in the network.

Roughly speaking, now we want to show that it is possible to choose those neighbourhoods such that we can define for each heteroclinic connection in the network a neighbourhood in $intS_n \times S_m$ using the flow such that we can 'glue' all those neighbourhoods of the equilibria and of the connections to form a flow invariant tubular neighbourhood of the network Σ in $intS_n \times S_m$.

If we choose all the In sections with the same volume v, then all the Out sections also have the same volume v, since the flow is volume-preserving.

For each connection $[\xi_j \to \xi_{k_l}]$ iterating the section Out^{j,k_l} near ξ_j by the flow, following the connection in a flow-box fashion, we have that the intersection with the section I^{j,k_l} is a section in $intS_n \times S_m$ transversal to the connection and to the flow near the connection with volume v but need not coincide with the transversal section $In^{j,k_l} \subset I^{j,k_l}$ (in general it does not coincide). Nevertheless, we can choose the sections $In^{i_l,j}$ such that it happens.

Obviously, in general, it is not possible to do this simultaneously for all the sections near all the equilibria. In this case it is possible since the expanding and the contracting eigenvalues associated to each heteroclinic connection have the same absolute value and thus in each direction the contracting rate equals the expanding rate.

The above arguments support the following conjecture.

Conjecture 16 *Let Σ be an edge network in $S_n \times S_m$ then there is a flow invariant tubular neighbourhood of Σ in $intS_n \times S_m$.*

This conjecture, together with the existence of switching along every connection of the edge network, suggests the existence of infinite switching near the network. We prove switching near the network in Theorem 17, provided there is switching along all the connections in the network and the flow is *topologically mixing*: a continuous flow $X^t : M \to M$ is topologically mixing, if given any two nonempty open subsets A and B of M, there exists $\tau \in \mathbb{R}$ such that $X^t(A) \cap B \neq \emptyset$ for any $t \geq \tau$.

Theorem 17 *Let Σ be an edge network in $S_n \times S_m$ with $n = m$ and all saddles of type (n, n). Assuming the existence of a flow invariant tubular neighbourhood of Σ in $intS_n \times S_m$ we have that, if the the flow is topologically mixing within a neighbourhood of Σ then there is (infinite) switching near Σ.*

Proof Let X^t denote the flow associated to the vector field (6) defined in $S_n \times S_m$. Since in $\text{int}S_n \times S_m$ volume is conserved by the flow X^t, we argue in Conjecture 16 the existence of a tubular neighbourhood U of Σ in $intS_n \times S_m$ that is invariant by the flow.

Near each equilibrium ξ_k consider for each connection $[\xi_i \to \xi_k]$ the intersection Σ^{in}_{ki} of a cross-section, transversal to the connection and to the flow, with the neighbourhood U. For each connection $[\xi_k \to \xi_j]$, let $C^{i,k,j}$ denote the open subset of points in $\Sigma^{\text{in}}_{k,i}$ that come from a neighbourhood of saddle ξ_i are taken close to saddle ξ_k and then follow the connection from saddle ξ_k to saddle ξ_j.

As follows from Theorem 13 and Corollary 14, all the sets $C^{i,k,j}$ are nonempty if and only if $n = m$ and all saddles are of type (n, n), which proves that, under these conditions, there is switching along all the connections of an edge network Σ in $S_n \times S_m$. Switching along every connection of a network is a necessary condition for the existence of switching near the network, but it is not enough.

Let $\xi_{i_1} \to \xi_{i_2} \to \xi_{i_3} \to \xi_{i_4} \to \ldots \to \xi_{i_n}$ be any finite sequence of connections in the edge network. We shall prove the existence of an open set of initial conditions near the edge network whose trajectories shadow the given sequence.

Since we are assuming the vector field (6) to be topologically mixing in $\text{int}S_n \times S_m$, we have that

$$\exists \tau_1 > 0 \text{ such that } \forall\ t > \tau_1,\ X^t(C^{i_1,i_2,i_3}) \cap C^{i_2,i_3,i_4} \neq \emptyset.$$

Moreover, $\cup_{t>\tau_1} X^t(C^{i_1,i_2,i_3}) \cap C^{i_2,i_3,i_4}$ is an open set (as X^t is a C^∞ flow and the intersection of a finite number of open sets is open).

Now, consider the open set $C^{i_1,i_2,i_3,i_4} = \cup_{t>\tau_1} X^{-t}\left(X^t(C^{i_1,i_2,i_3}) \cap C^{i_2,i_3,i_4}\right) \cap C^{i_1,i_2,i_3}$ which corresponds to the points in $\Sigma^{\text{in}}_{i_2,i_1}$ that come from a neighbourhood of saddle ξ_{i_1} are taken close to saddle ξ_{i_2}, follow the connection from saddle ξ_{i_2} to saddle ξ_{i_3} and then follow the connection from ξ_{i_3} to ξ_{i_4}.

Since the vector field (6) is topologically mixing, we have that

$$\exists \tau_2 > \tau_1 > 0 \text{ such that } \forall\, t > \tau_2,\ X^t(C^{i_1,i_2,i_3,i_4}) \cap C^{i_3,i_4,i_5} \neq \emptyset.$$

Let $C^{i_1,i_2,i_3,i_4,i_5} = \cup_{t>\tau_2} X^{-t}\left(X^t(C^{i_1,i_2,i_3,i_4}) \cap C^{i_3,i_4,i_5}\right) \cap C^{i_1,i_2,i_3,i_4}$ which corresponds to the points in $\Sigma^{\text{in}}_{i_2,i_1}$ that come from a neighbourhood of saddle ξ_{i_1} are taken close to saddle ξ_{i_2}, follow the connection from saddle ξ_{i_2} to saddle ξ_{i_3}, then follow the connection from ξ_{i_3} to ξ_{i_4} and then follow the connection from ξ_{i_4} to ξ_{i_5}.

Continuing iteratively we end up with an open subset $C^{i_1,i_2,i_3,\ldots,i_n} \subset C^{i_1,i_2,i_3}$ of points whose trajectory follows the given sequence of connections $\xi_{i_1} \to \xi_{i_2} \to \xi_{i_3} \to \xi_{i_4} \to \ldots \to \xi_{i_n}$.

If we consider an infinite sequence of connections, that is, if we take $n \to +\infty$, then we get infinite intersections of open sets which need not be open. Nevertheless, by the topological mixing property, we know that it is nonempty and thus contains at least one point. □

We remark that the assumption of topological mixing seems reasonable as it seems likely to happen. For example, in [6], Bessa proves that, generically the flow of a divergence-free vector field defined on a n-dimensional compact connected, boundaryless C^∞ Riemannian manifold is topologically mixing.

Theorem 17 gives conditions for switching dynamics in games with two players and three or more strategies per player.

Example 18 An example is the RSP-game studied in [31] and [2] which is a particular case of the family of bimatrix games in $S_2 \times S_2$ defined in Example 11. In [2], Aguiar and Castro prove switching near the edge heteroclinic network in $S_2 \times S_2$ for the dynamics of the RSP-game. The results proved here add to the dynamical conclusions in [2], showing that there is switching dynamics near a bigger heteroclinic network than the one described there. More concreatly, by Proposition 9, we get the heteroclinic connections $(R, R) \to (S, S)$, $(S, S) \to (P, P)$ and $(S, S) \to (R, R)$. The author thanks this observation to Peter Ashwin.

◊

Note that in the case of bimatrix games with two strategies there can be not switching. This is the case, for example, of the Matching Pennies (see Section 4.1 in [31]).

As in [2], depending on the sign of the eigenvalues of the equilibria, besides switching near an edge network for the dynamics of a bimatrix game, one or more heteroclinic cycles in the network can be relatively asymptotically stable.

4 Conclusions

It seems that the minimal dimension for the existence of switching, without rotating nodes (i.e., equilibria with complex eigenvalues or periodic trajectories) and/or transversal intersections of invariant manifolds, is four; consider an edge network in $S_2 \times S_2$ satifying the hypotheses in Theorem 17.

For edge heteroclinic networks in bimatrix games, the mechanism for switching seems to be the absence of a Kirk&Silber subnetwork, and the conservation of volume by the flow.

As shown here, the absence of a Kirk&Silber subnetwork is a necessary condition for switching dynamics. This condition is satisfied in the examples in Chawanya [8] and Sato *et al.* [31] and, in both cases, switching is observed.

A consequence of volume preservation is the resonance on the eigenvalues, $\lambda_{\overline{il}j} = -\lambda_{\overline{li}j}$, for $i \neq l$ and $\lambda_{i\overline{jk}} = -\lambda_{i\overline{kj}}$, for $k \neq j$ for all $i, j, k, l \in \{1, \ldots, n\}$. This allowed us to argue, in Conjecture 16, the existence of a trapping region containing the heteroclinic network. An analogous resonance occurs for homoclinic orbits to saddle points in conservative flows, $\lambda^s = -\lambda^u$, see Homburg and Knobloch [19] and Vanderbauwhede and Fiedler [34]. We can ask if the same can occur for the replicator dynamics. More specifically, we ask if there can be a resonance on the eigenvalues and, if so, if that happens only in the conservative case. By Theorem 19, in the Appendix B, the answer to both questions is positive. We conjecture that every heteroclinic connection in a volume preserving flow satisfies the resonance relation $\lambda^s = -\lambda^u$, with λ^s and λ^u the stable and unstable eigenvalues associated with the connection.

Here we have analysed the existence of complicated behaviour, more specifically switching, in the neighbourhood of an edge network in $S_n \times S_m$, for bimatrix game dynamics, which is conservative. In [19], Homburg and Knobloch show the existence of hyperbolic suspended Smale horseshoes near homoclinic belows (union of two homoclinic orbits to the same saddle point) in conservative systems. We can ask about special behaviour near an edge network in S_n when the flow is conservative.

We have performed some numerical simulations for replicator dynamics in the 4-dimensional simplex S_4 in $\mathbb{R}^5$. In the case of no resonance on the eigenvalues, the trajectories tend to follow one of the heteroclinic cycles, as in the Kirk and Silber case [24]. In the resonance case, the observed behaviour resembles the situation in Postlethwaite and Dawes [28] with the trajectories following the heteroclinic network in a 'random' way. We can identify the edge heteroclinic network as the union of 5 heteroclinic cycles. It seems that trajectories tend to loop around those cycles. For some paremeter values there are initial conditions such that trajectories loop around the cycles in a sequence and it seems that the number of loops around each cycle is always the same. Other trajectories seem to visit the cycles in a more irregular way.

In [28], Postlethwaite and Dawes analyse the behaviour near a heteroclinic network in a topological globally attracting invariant five-dimensional sphere in $\mathbb{R}^6$. The network has 6 equilibria, each with two incoming and two outgoing connections, as in the RSP-game. The difference is that in the heteroclinic network in the RSP-game there are 9 equilibria, and so, no Kirk&Silber subnetwork. In [28], for some parameter values they observe what they call 'regular cycling' and 'irregular cycling'. In both cases, a trajectory follows three cycles in the network in a sequencial way. But, in the case of regular cycling, there is the same number n of turns around each cycle, whereas, in the irregular cycling, that number of turns varies in an irregular way. We note that, each two cycles in the network that are visited sequencially have an heteroclinic connection in common.

The key ingredient for (infinite) switching in bimatrix games seems to be the existence of an edge heteroclinic network without a Kirk&Silber subnetwork in an incompressible flow. We conjecture that this should apply to more general settings.

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A Neutral stability of edge heteroclinic networks in bimatrix games

Let $\lambda_{\overline{i}lj}$ be the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{lj} and $\lambda_{i\overline{j}k}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{ik}. By the proof of Proposition 7, the eigenvalues at ξ_{ij} satisfy $\lambda_{\overline{i}lj} = -\lambda_{\overline{l}ij}$, for $i \neq l$ and $\lambda_{i\overline{j}k} = -\lambda_{i\overline{k}j}$, for $k \neq j$ for all $i, l \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, m\}$. We have then that, for each hyperplane the sum of the expanding eigenvalues of the equilibria in the network contained in it equals the sum of the expanding eigenvalues Thus, on average the contraction around the network equals the expansion, and so, we conjecture the heteroclinic network to be neutrally stable.

To support this conjecture, we rely on the work of Hofbauer in [15] making use of the notion of an average Lyapunov function and of the characteristic matrix C of an edge heteroclinic network. Roughly speaking, an average Lyapunov function is a weaker version of a Lyapunov function in which the time average behaves as a Lyapunov function. For an account of average Lyapunov functions, see Hofbauer [14], Hutson [21], Jansen [22], Law and Blackford [26], and Vance [33].

Following [15], we use the average 'Lyapunov' function

$$P(\mathbf{x}, \mathbf{y}) = \Pi_{i=1}^{n+1} (x_i)^{p_i} \Pi_{j=1}^{m+1} (y_j)^{p_{n+1+j}},$$

with $p_k \in \mathbb{R}$, $k = 1, \ldots, n + m + 2$.

The function P satisfies

$$P(\mathbf{x}, \mathbf{y}) = 0 \text{ if } (\mathbf{x}, \mathbf{y}) \in \text{bd}S_n \times S_m$$

$$P(\mathbf{x}, \mathbf{y}) > 0 \text{ if } (\mathbf{x}, \mathbf{y}) \in \text{int}S_n \times S_m.$$

We want to analise the sign of $\dot{P}$ close to the boundary of $S_n \times S_m$. We have

$$\frac{\dot{P}}{P} = \sum_{i=1}^{n+1} p_i \frac{\dot{x}_i}{x_i} + \sum_{j=1}^{m+1} p_{n+1+j} \frac{\dot{y}_j}{y_j}$$

thus, $\dot{P}(\mathbf{x},\mathbf{y}) = P(\mathbf{x},\mathbf{y})\Psi(\mathbf{x},\mathbf{y})$ with Ψ a continuous function, and so, it is enough to analise the sign of Ψ at bd$S_n \times S_m$. Moreover, we reduce the analysis to the equilibria in the heteroclinic network, as trajectories near the network spend most of the time near the equilibria.

We now make use of the characteristic matrix C of the edge network which is a matrix with rows indexed by the equilibria in the network and the columns indexed by the boundary hyperplanes of $S_n \times S_m$. The entry in row k column j is the eigenvalue of the equilibria corresponding to row k in the eigendirection corresponding to column j. For each equilibrium the 'radial' eigenvalues are zero and so there are two zero entries in each row. Moreover, due to the resonance on the eigenvalues of the equilibria at the edge network, the elements of each column can be associated in pairs such that the two elements of each pair have symmetric value.

We have that $\dot{P}/P$ takes the (average) value $(Cp)_k$ at each equilibrium ξ_k, $k = (n+1) \times (m+1)$. Since the sum of the elements of each column of the characteristic matrix C is zero, it follows that $\sum_{k=1}^{(n+1)\times(m+1)} (Cp)_k = 0$ and so, intuitively it seems plausible to expect P to be constant (in the average) near the heteroclinic network and so the network to be neutrally stable.

B Conservative Replicator Dynamics

Theorem 19 *Up to a change in velocity, the flow of (1) in intS_n is incompressible if and only if $\lambda_{ij} = -\lambda_{ji}$ for $i \neq j$.*

The proof of the theorem is given by the following two lemmas.

Lemma 20 *Up to a change in velocity, the flow of (1) in intS_n is incompressible if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \ldots, n\}$.*

Proof We follow the proof of Theorem 11.3.1 in [17]. The idea of the proof is to use Liouville's formula to determine the conditions for $\dot{V}(t)$ to be zero in the interior of the simplex S_n. This is equivalent to find the conditions for the divergence of the vector field (1) in the interior of the simplex S_n to be zero.

We start by dividing the vector field (1) by the positive function $P = \Pi_{i=1}^{n} x_i$ getting the modified vector field

$$\dot{x}_j = \frac{1}{\Pi_{i=1 \wedge i\neq j}^{n} x_i}\left[\sum_{i=1}^{n} a_{ji}x_i - \mathbf{x}^T A\mathbf{x}\right]. \tag{10}$$

The divergence of (10) in int$\mathbb{R}_n$ is given by

$$\begin{aligned} div &= \textstyle\sum_{j=1}^{n} \frac{\partial \dot{x}_j}{\partial x_j} = \\ &= \textstyle\frac{1}{\Pi_{i=1}^{n} x_i}\left[\sum_{j=1}^{n} a_{jj}x_j - \sum_{j=1}^{n} x_j \sum_{i=1}^{n}(a_{ji}+a_{ij})x_i\right] = \\ &= \textstyle\frac{1}{\Pi_{i=1}^{n} x_i}\left[\sum_{j=1}^{n} a_{jj}x_j - \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ji}x_i - \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij}x_j\right] = \\ &= \textstyle\frac{1}{\Pi_{i=1}^{n} x_i}\left[\sum_{j=1}^{n} a_{jj}x_j - 2\mathbf{x}^T A\mathbf{x}\right] \end{aligned}$$

In order to obtain the divergence div_0 within the state space intS_n, we have to subtract the eigenvalues of the Jacobian which are orthogonal to S_n. Since

$$\left(\sum x_j\right)^{\cdot} = \left(1 - \sum x_j\right) \mathbf{x}^T A\mathbf{x},$$

the superfluous engeivalues are $-\mathbf{x}^T A\mathbf{x}$.

Taking into account the factor $1/P$, we have

$$\begin{aligned} div_0 &= \tfrac{1}{P}\left[\sum_{j=1}^{n} a_{jj}x_j - \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ji}x_i\right] = \\ &= \tfrac{1}{P}\left[\sum_{j=1}^{n-1}(a_{jj} - a_{nn})x_j + a_{nn} - \sum_{j=1}^{n-1}[\sum_{i=1}^{n-1}(a_{ji} - a_{jn})x_i + a_{jn}\right] x_j - \\ &\qquad - \left[\sum_{j=1}^{n-1}(a_{nj} - a_{nn})x_j + a_{nn}\right]\left(1 - \sum_{i=1}^{n-1} x_i\right)\Big] \end{aligned}$$

since $x_n = 1 - \sum_{i=1}^{n-1} x_i$.

Rewritting the expression of div_0, we have

$$\begin{aligned} div_0 &= \tfrac{1}{P}\left[\sum_{j=1}^{n-1}(a_{jj} - a_{jn} - a_{nj} + a_{nn})x_j + \sum_{j=1}^{n-1}[\sum_{i=1}^{n-1}(-a_{ji} + a_{jn} + a_{ni} - a_{nn})x_i]x_j\right] = \\ &= \tfrac{1}{P}[\sum_{j=1}^{n-1}\left((a_{jj} - a_{jn} - a_{nj} + a_{nn})(1 - x_j)\right)x_j + \\ &\qquad + \sum_{j=1}^{n-1}[\sum_{i=j+1}^{n-1}(-a_{ji} + a_{jn} + a_{ni} - a_{nn} - a_{ij} + a_{in} + a_{nj} - a_{nn})]x_ix_j] \end{aligned}$$

Thus, $div_0 \equiv 0$ if and only if

$$\begin{cases} (a_{jj} - a_{jn} - a_{nj} + a_{nn}) = 0 \\ -a_{ji} + a_{jn} + a_{ni} - a_{nn} - a_{ij} + a_{in} + a_{nj} - a_{nn} = 0 \end{cases}$$

for all $j = 1, \ldots, n$ and $i = 1, \ldots, n-1$

That is, if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \ldots, n\}$. □

Lemma 21 *The eigenvalues of the equilibria at the vertices satisfy $\lambda_{ij} = -\lambda_{ji}$, for $i \neq j$, if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \ldots, n\}$.*

Proof The partial derivatives of (1) are given by

$$\begin{aligned} \frac{\partial \dot{x}_j}{\partial x_j} &= \sum_{i=1}^{n} a_{ji}x_i - \mathbf{x}^T A\mathbf{x} + x_j\left(a_{jj} - \frac{\partial}{\partial x_j}\mathbf{x}^T A\mathbf{x}\right) \\ \frac{\partial \dot{x}_j}{\partial x_k} &= a_{jk}x_j - x_j\frac{\partial}{\partial x_j}\mathbf{x}^T A\mathbf{x}, \text{ for } k \neq j \end{aligned}$$

with $\frac{\partial}{\partial x_j}\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}(a_{ji} + a_{ij})x_i$.

The Jacobian matrix at a vertex equilibrium ξ_i is given by

$$\begin{bmatrix} a_{1i} - a_{ii} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & a_{2i} - a_{ii} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & .. \\ 0 & 0 & \ldots & a_{i-1i} - a_{ii} & 0 & 0 & \ldots & 0 \\ -a_{1i} & -a_{2i} & \ldots & -a_{i-1i} & -a_{ii} & -a_{i+1i} & \ldots & -a_{ni} \\ 0 & 0 & \ldots & 0 & 0 & a_{i+1i} - a_{ii} & .. & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & .. \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & a_{ni} - a_{ii} \end{bmatrix}$$

Thus, $\lambda_{ij} = a_{ji} - a_{ii}$ for all $i \neq j$ and so $\lambda_{ij} = -\lambda_{ji} \Leftrightarrow a_{ij} + a_{ji} = a_{ii} + a_{jj}$ □