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Minimal coupled cell networks

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Abstract

It is known that non-isomorphic coupled cell networks can have equivalent dynamics. Such networks are said to be ODE-equivalent and are related by a linear algebra condition involving their graph adjacency matrices. A network in an ODE-equivalence class is said to be minimal if it has a minimum number of edges. When studying a given network in an ODE-class it can be of great value to study instead a minimal network in that class. Here we characterize the minimal networks of an ODE-equivalence class—the canonical normal forms of the ODE-class. Moreover, we present an algorithm that computes the canonical normal forms for a given ODE-class. This goes through the calculation of vectors with minimum length contained in a cone of a lattice described in terms of the adjacency matrices of any network in the ODE-class.

Mathematics Subject Classification: 34C15, 05C99, 15A36

1. Introduction

Many important real-world networks can be modelled by dynamical systems on a graph and therefore coupled cell networks. For example, many biological systems can be modelled by networks of nonlinear dynamical systems. See, for example, Stewart [8] and references therein. The nodes of the graph represent the cells and the edges represent the couplings, that is, the interactions between the cells (see figures 1–4).

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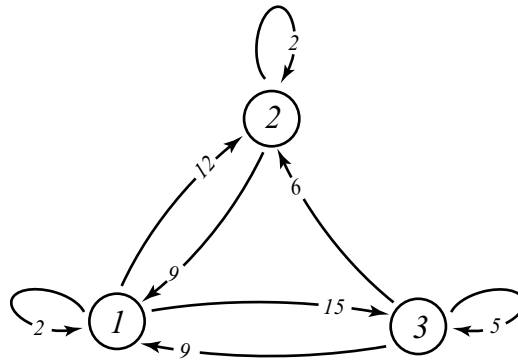


Figure 1. Coupled cell network G_1 with one cell-type and one edge-type.

Following Stewart *et al* [9] and Golubitsky *et al* [7], we associate to each coupled cell network a class of ordinary differential equations (ODEs) compatible with the structure of the network—the class of *coupled cell systems*. A vector field corresponding to the coupled cell system is called *admissible* and respects the topology of the network. For a survey of this formalism see Golubitsky and Stewart [6]. See also an alternative approach of Field [5].

It is possible for two non-isomorphic coupled cell networks to generate the same space of admissible vector fields, that is to be *ODE-equivalent*, see [7]. Thus the corresponding coupled cell systems have equivalent dynamics.

Dias and Stewart [4] prove that two coupled cell networks are ODE-equivalent if and only if they determine the same space of linear admissible vector fields—they are *linearly equivalent*.

As every coupled cell network in a given ODE-equivalence class determines the same dynamical behaviour we look for the set of coupled cell networks in the ODE-class that are more amenable to treat. We aim to find a kind of *canonical normal forms*—a set of networks such that the number of edges is minimal among all the networks of the ODE-equivalence class, which we call the *minimal subclass*.

Using the results of Dias and Stewart [4] on ODE and linear equivalence of networks, this problem can be posed in terms of the networks adjacency matrices. That is, the problem of finding all the minimal networks in a given ODE-equivalence class corresponds to find all the *minimal bases* (with non-negative integer entries) of the real vector space generated by the adjacency matrices of the networks in the ODE-class. The minimal bases define the adjacency matrices of the minimal networks.

For the coupled cell network represented in figure 1 it follows, from theorem 9.3 (see section 9), that the coupled cell network in figure 2 is the unique minimal network in its ODE-equivalence class.

The problem of finding the minimal subclass of an ODE-class of networks with more than one edge-type gets much more complicated and, in general, there is more than one minimal network. For example, using algorithm 7.1 based on theorem 6.7, the minimal subclass of the coupled cell network in figure 3 is given by the minimal networks represented in figure 4. Here, cells representing identical dynamical systems appear with the same symbol in the graph and identical edges symbolize identical couplings between the cells. In this case, the network has more than one edge-type and the cells are all isomorphic. Two cells are *isomorphic* if there is a bijection between the two sets of edges directed to them that preserves the edge-type. An adjacency matrix of a network with isomorphic cells is a \mathbb{R} -linear combination of the adjacency matrices of any network in its ODE-class. A graph where all cells are isomorphic is said to

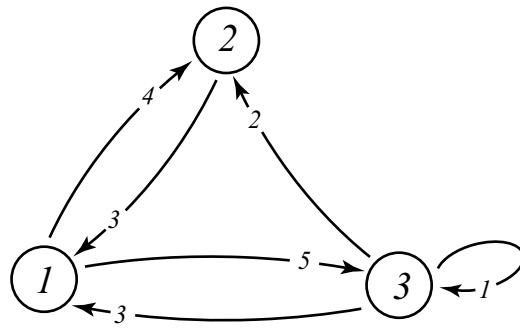


Figure 2. Minimal network G_2 of the ODE-equivalence class of the coupled cell network in figure 1.

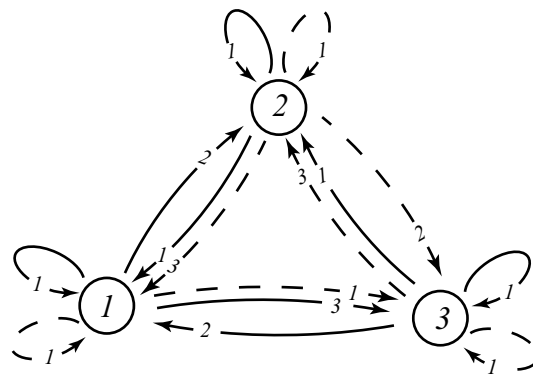


Figure 3. Coupled cell network with more than one edge type.

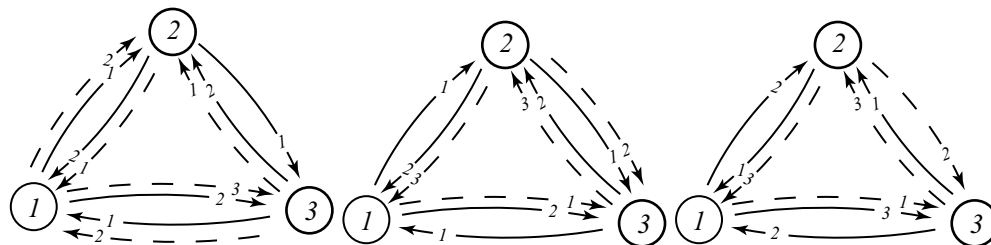


Figure 4. Minimal networks of the ODE-equivalence class of the coupled cell network of figure 3.

be *homogeneous*. Given an homogeneous graph in an ODE-class, we consider the lattice consisting of the vectors with integer entries in the real subspace generated by the adjacency matrices of the graph. The minimal networks are obtained by finding vectors (adjacency matrices) with shortest length in a cone of that lattice (whose rank grows with the number of edge-types).

In this paper, we describe the minimal subclass of a given coupled cell network and present an algorithm that computes it. The minimality of coupled cell networks basically reduces to the minimality of homogeneous networks, i.e. networks whose cells are all isomorphic. We address the case of homogeneous networks in theorem 6.7 and algorithm 7.1.

The rest of the paper is organized in the following way. Section 2 reviews the basic concepts on coupled cell networks and coupled cell systems. Section 3 relates ODE and linear equivalence of networks. The definition of minimality of networks appears in section 4. The minimality of homogeneous networks is addressed in section 5 and answered in section 6. An algorithm is presented in section 7 and illustrated in section 8. The special case of identical-edge homogeneous networks is treated in section 9. Finally, in section 10 we solve the general case using the results of section 6.

2. Coupled cell networks and systems

In this section we review a few definitions and results on the theory of coupled cell networks introduced by Stewart *et al* [9] and Golubitsky *et al* [7]. A coupled cell network can be represented schematically by a directed graph (see, e.g. Tutte [10]) whose nodes correspond to cells and whose edges represent couplings. We employ the definition which permits multiple arrows and self-coupling [7].

Definition 2.1 ([7]). A *coupled cell network* G consists of:

- (a) A finite set $\mathcal{C} = \{1, \dots, n\}$ of *nodes* or *cells*.
- (b) An equivalence relation $\sim_{\mathcal{C}}$ on cells in \mathcal{C} . The *type* or *cell label* of cell c is the $\sim_{\mathcal{C}}$ -equivalence class $[c]_{\mathcal{C}}$ of c .
- (c) A finite set \mathcal{E} of *edges* or *arrows*.
- (d) An equivalence relation $\sim_{\mathcal{E}}$ on edges in \mathcal{E} . The *type* or *coupling label* of edge e is the $\sim_{\mathcal{E}}$ -equivalence class $[e]_{\mathcal{E}}$ of e .
- (e) Two maps $\mathcal{H} : \mathcal{E} \rightarrow \mathcal{C}$ and $\mathcal{T} : \mathcal{E} \rightarrow \mathcal{C}$. For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e .
- (f) Equivalent edges have equivalent tails and heads. That is, if $e_1, e_2 \in \mathcal{E}$ and $e_1 \sim_{\mathcal{E}} e_2$, then

$$\mathcal{H}(e_1) \sim_{\mathcal{C}} \mathcal{H}(e_2) \quad \mathcal{T}(e_1) \sim_{\mathcal{C}} \mathcal{T}(e_2).$$

We write $G = (\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}})$. ◇

In a graphical representation of a coupled cell network, identical cells and identical edges are represented, respectively, by the same symbol. Multiple couplings of the same type between two cells are represented by just one arrow with the number of couplings attached to it. Figures 1–4 show examples of coupled cell networks of three identical cells, and so with one $\sim_{\mathcal{C}}$ -equivalence class, having self-coupling and multiarrows. The networks of figures 1 and 2 have only one type of edge, and so one $\sim_{\mathcal{E}}$ -equivalence class, whereas the networks in figures 3 and 4 have two types of edges, and so two $\sim_{\mathcal{E}}$ -equivalence classes.

Let $G = (\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}})$ be a coupled cell network as in definition 2.1. To each cell c , we can associate the set of edges directed to c , called the *input set* of c and denoted by $I(c)$. An element of $I(c)$ is called an *input edge* of c . Two cells c and d in a network are said to be *isomorphic* if there is an edge-type preserving isomorphism, $\beta : I(c) \rightarrow I(d)$, between their input sets. Any such bijection β is called an *input isomorphism* from cell c to cell d . Denote the set of all those isomorphisms between any two isomorphic cells in the network, $\bigcup_{c,d \in \mathcal{C}} B(c, d)$, where \bigcup indicates disjoint union, by \mathcal{B}_G . Observe that a natural product operation can be defined on \mathcal{B}_G in the following way: $\beta_2 \in B(c, d)$ can be multiplied by $\beta_1 \in B(a, b)$ only when $b = c$, and in this case $\beta_2\beta_1 \in B(a, d)$ is the usual composition of functions. Because the product of two elements in \mathcal{B}_G is not always defined, it follows that \mathcal{B}_G is a groupoid, which is called the *symmetry groupoid* of the network, and in general it is not a group. The structure of a network can thus be described in terms of its symmetry groupoid.

Note that for any $c \in \mathcal{C}$, the subset $B(c, c)$ is always non-empty. Moreover, it is a group which is called the *vertex group* corresponding to c .

We make now precise the connection between coupled cell systems and coupled cell networks. To each coupled cell c we associate a choice of *cell phase space* P_c which we assume is a finite-dimensional real vector space. If cells c and d are in the same $\sim_{\mathcal{C}}$ -equivalence class, we require that $P_c = P_d$ and we identify the two spaces canonically. The *total phase space* P of the coupled cell system is the direct product of the cell phase spaces, $\prod_{c \in \mathcal{C}} P_c$, and we employ the coordinate system $x = (x_c)_{c \in \mathcal{C}}$ on P .

Given a network G and a fixed choice of the total phase space P , we describe now the coupled cell systems that correspond to the class of ODEs $\dot{X} = F(X)$, $X \in P$ compatible with the structure of the network. The vector fields F described in terms of the symmetry groupoid of G are called *admissible*.

Let $\mathcal{D} = (d_1, \dots, d_s)$ be any finite ordered subset of s cells in \mathcal{C} where the same cell can appear more than once in \mathcal{D} . Define $P_{\mathcal{D}} = P_{d_1} \times \dots \times P_{d_s}$ and write $x_{\mathcal{D}} = (x_{d_1}, \dots, x_{d_s})$ where $x_{d_j} \in P_{d_j}$.

Given $c \in \mathcal{C}$, denote by $\mathcal{T}(I(c))$ the ordered set of cells $(\mathcal{T}(i_1), \dots, \mathcal{T}(i_s))$ where the arrows i_k run through $I(c)$.

Suppose that c, d are isomorphic and let β be an input isomorphism between $I(c)$ and $I(d)$. Then for all $i \in I(c)$ we have $i \sim_E \beta(i)$, and so $\mathcal{T}(i) \sim_{\mathcal{C}} \mathcal{T}(\beta(i))$. Consider now the ordered sets $\mathcal{D}_1 = \mathcal{T}(I(c)) = (\mathcal{T}(i_1), \dots, \mathcal{T}(i_s))$ and $\mathcal{D}_2 = \mathcal{T}(I(d)) = (\mathcal{T}(\beta(i_1)), \dots, \mathcal{T}(\beta(i_s)))$ of \mathcal{C} . We can define the *pullback map*

$$\beta^* : P_{\mathcal{D}_2} \rightarrow P_{\mathcal{D}_1}$$

by

$$(\beta^*(z))_{\mathcal{T}(j)} = z_{\mathcal{T}(\beta(j))}$$

for all $\mathcal{T}(j) \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$. Thus $x_{\mathcal{T}(I(c))} = (x_{\mathcal{T}(i_1)}, \dots, x_{\mathcal{T}(i_s)})$ and $\beta^*(x_{\mathcal{T}(I(d))}) = (x_{\mathcal{T}(\beta(i_1))}, \dots, x_{\mathcal{T}(\beta(i_s))})$.

For a given cell c the *internal phase space* is P_c and the *coupling phase space* is $P_{\mathcal{T}(I(c))} = P_{\mathcal{T}(i_1)} \times \dots \times P_{\mathcal{T}(i_s)}$ where as before $\mathcal{T}(I(c))$ denotes the ordered set of cells $(\mathcal{T}(i_1), \dots, \mathcal{T}(i_s))$.

Definition 2.2 ([7]). A vector field $f : P \rightarrow P$ is \mathcal{B}_G -equivariant or G -admissible if:

- (a) For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variables x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$; that is, there exists a smooth function $\hat{f}_c : P_c \times P_{\mathcal{T}(I(c))} \rightarrow P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{\mathcal{T}(I(c))}).$$

- (b) For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\mathcal{T}(I(d))}))$$

for all $x \in P$. ◇

Define the relation \sim_I of *input-equivalence* on \mathcal{C} by: $c \sim_I d$ if and only if c and d are isomorphic cells. Observe that \sim_I is an equivalence relation.

Theorem 2.3. Let $G = (\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_E)$ be a coupled cell network and \mathcal{B}_G the corresponding symmetry groupoid. A vector field $f : P \rightarrow P$ for a given choice of the P_c is \mathcal{B}_G -equivariant

if and only if for each \sim_I -equivalence class \mathcal{Q} of \mathcal{C} , given (any) $c \in \mathcal{Q}$:

(a) \hat{f}_c is $B(c, c)$ -invariant.

(b) For $d \in \mathcal{Q}$ such that $d \neq c$, given (any) $\beta \in B(c, d)$, we have

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\mathcal{T}(I(d))})).$$

Proof. This is a generalization of [9, lemma 4.5] and is proved the same way. \square

Definition 2.4. Let G be a coupled cell network. For a given choice of the P_c , define \mathcal{F}_G^P to consist of all smooth G -admissible vector fields on P which is a vector space over \mathbb{R} . Let \mathcal{P}_G^P be the subspace of \mathcal{F}_G^P consisting of the G -admissible polynomial vector fields on P , and let \mathcal{L}_G^P be the subspace of \mathcal{P}_G^P consisting of the G -admissible linear vector fields on P . \diamond

Remark 2.5. Let G be a coupled cell network and P a given choice of the total phase space consistent with G . By theorem 2.3, every smooth equivariant vector field $f \in \mathcal{F}_G^P$ is determined uniquely by its components f_c where c runs through a set of representatives for the \sim_I -equivalence classes. The only constraints on f_c are that it depends only on $x_c, x_{\mathcal{T}(I(c))}$ and is invariant under the vertex group $B(c, c)$. Thus every smooth equivariant vector field f is determined uniquely by a finite set of $B(c, c)$ -invariant functions, where c runs through a set of representatives of the \sim_I -equivalence classes. Moreover, if $d \sim_I c$ then f_d is related to f_c by a pullback map β^* for $\beta \in B(c, d)$. In particular, if there is only one \sim_I -equivalence class for \mathcal{B}_G then each G -admissible vector field is uniquely determined by a single mapping f_c at some node c , which has to be invariant under the vertex group $B(c, c)$. \diamond

Example 2.6. We return to the coupled cell network G_1 of figure 1. Since all cells $c \in \mathcal{C}_1$ are identical, they have the same cell phase space. Suppose that the cell phase space is \mathbb{R}^k . Then the total phase space for G_1 is $P = \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k$. Moreover, since all cells are isomorphic and so input-equivalent, any $f = (f_1, f_2, f_3) \in \mathcal{F}_{G_1}^P$ is uniquely determined by one of the components, say f_1 , which has to be $B(1, 1)$ -invariant, and then we obtain f_i , for $i = 2, 3$, using a bijection $\beta_i \in B(1, i)$. Throughout, denote by $(x_i^{(j)})$ the vector (x_i, \dots, x_i) where the variable x_i appears j times. Set

$$x_{\mathcal{T}(I(1))} = (x_1^{(2)}, x_2^{(9)}, x_3^{(9)}).$$

The \mathcal{B}_{G_1} -equivariance condition implies the $B(1, 1)$ -invariance of $\hat{f}_1 : P_1 \times P_{\mathcal{T}(I(1))} \rightarrow P_1$. Thus

$$f_1(x) = \hat{f}_1(x_1, x_1^{(2)}, x_2^{(9)}, x_3^{(9)}) = \hat{f}_1(x_1, \beta^*(x_1^{(2)}, x_2^{(9)}, x_3^{(9)}))$$

for all $\beta \in B(1, 1)$, and so \hat{f}_1 is symmetric in the set of the last 20 variables. We show that by putting a bar over that set:

$$f_1(x) = \hat{f}_1(x_1, \overline{x_1^{(2)}, x_2^{(9)}, x_3^{(9)}}).$$

By theorem 2.3 any \mathcal{B}_{G_1} -equivariant vector field $f : P \rightarrow P$ takes the form

$$f(x_1, x_2, x_3) = \left(\hat{f}_1(x_1, \overline{x_1^{(2)}, x_2^{(9)}, x_3^{(9)}}), \hat{f}_1(x_2, \overline{x_1^{(12)}, x_2^{(2)}, x_3^{(6)}}), \hat{f}_1(x_3, \overline{x_1^{(15)}, x_3^{(5)}}) \right). \quad \diamond$$

3. ODE-equivalence and linear equivalence

As pointed by Golubitsky *et al* [7], in the class of coupled cell networks that permits self-coupling and multiarrows, it is possible for two non-isomorphic coupled cell networks G_1 and G_2 to generate the same space of admissible vector fields. This comparison of admissible vector fields for two coupled cell networks involves identifying cells in the two networks, a step that it is formalized in general in terms of a bijection between the two sets of cells. We follow the definition of ODE-equivalent coupled cell networks given by Dias and Stewart [4].

In what follows, given a coupled cell network G_i and a choice of total phase space P_i for G_i , we denote by $P_{i,c}$ the cell phase space corresponding to cell c of \mathcal{C}_i .

Definition 3.1 ([4]). Two coupled cell networks G_1 and G_2 are γ -ODE-equivalent if:

- (1) There is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:
- (2) If we choose cell phase spaces $P_c \neq 0$ for G_1 , and define the corresponding choice of cell phase spaces for G_2 by

$$P_{2,\gamma(c)} = P_{1,c}$$

so that the corresponding total phase spaces are

$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \quad P_2 = \prod_{c \in \mathcal{C}_1} P_{2,\gamma(c)}$$

then:

- (3) The condition

$$\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2} \quad (3.1)$$

is satisfied.

Two coupled cell networks G_1 and G_2 are *ODE-equivalent* if they are γ -ODE-equivalent for some bijection γ . \diamond

We define now the notion of ‘linear equivalence’ between two networks. In [4] it is shown that two coupled cell networks are ODE-equivalent if and only if they are linearly equivalent. Basically, the ODE-equivalence reduces to ‘linear equivalence’, where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of *linear* admissible vector fields. Moreover, when deciding linear equivalence, it can without loss of generality be assumed that each cell phase space is one-dimensional since if (3.1) holds for some choice of nonzero cell phase spaces P_c , then it holds for all such choices (that are consistent with the structure of G).

Definition 3.2. Two coupled cell networks G_1 and G_2 are γ -linearly equivalent if there is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence such that if item (2) of definition 3.1 is satisfied then:

- (3) The condition

$$\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2} \quad (3.2)$$

is satisfied.

Two coupled cell networks G_1 and G_2 are *linearly equivalent* if they are γ -linearly equivalent for some γ . \diamond

Remark 3.3. Note that this definition is independent of the dimensions of the P_c . Moreover, the cells of G_2 can be renumbered so that γ is the identity. \diamond

Throughout, we denote by $M_{n \times n}(\mathbb{Z}_0^+)$ the set of the square matrices of order n and non-negative integer entries, and by I_n the identity matrix in this set.

Example 3.4. The coupled cell network G_2 of figure 2 is an example of a network linearly equivalent to the network G_1 of figure 1. Both graphs have only one cell-equivalence class and one input-equivalence class. Trivially, the identity function on $\mathcal{C}_1 = \{1, 2, 3\}$ preserves both cell-equivalence and input-equivalence. Moreover, if we take $P_1 = P_2 = \mathbb{R}^3$ then $\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}$ since

$$\mathcal{L}_{G_1}^{P_1} = \mathbb{R}\{\text{Id}_{\mathbb{R}^3}, h_1\} = \mathcal{L}_{G_2}^{P_2} = \mathbb{R}\{\text{Id}_{\mathbb{R}^3}, h_2\}, \quad (3.3)$$

where $\text{Id}_{\mathbb{R}^3}$ denotes the identity function on \mathbb{R}^3 ,

$$h_1(x_1, x_2, x_3) = (2x_1 + 9x_2 + 9x_3, 12x_1 + 2x_2 + 6x_3, 15x_1 + 5x_3)$$

$$h_2(x_1, x_2, x_3) = (3x_2 + 3x_3, 4x_1 + 2x_3, 5x_1 + x_3)$$

and $(x_1, x_2, x_3) \in \mathbb{R}^3$. Note that (3.3) is equivalent to saying

$$\mathbb{R}\left\{\text{Id}_3, \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix}\right\} = \mathbb{R}\left\{\text{Id}_3, \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 2 \\ 5 & 0 & 1 \end{pmatrix}\right\}. \quad (3.4)$$

◇

The next theorem reduces ODE-equivalence to linear equivalence, and says that the cell phase spaces may be assumed one-dimensional in that context.

Theorem 3.5 ([4]). *Let $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a bijection that preserves cell-equivalence and input-equivalence. Then the following conditions on two networks G_1, G_2 are equivalent:*

- (a) G_1 and G_2 are γ -ODE-equivalent.
- (b) G_1 and G_2 are γ -linearly equivalent.
- (c) With the identification $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, the spaces $\mathcal{L}_{G_1}^P$ and $\mathcal{L}_{G_2}^P$ are equal when all cell phase spaces are taken to be \mathbb{R} .

Proof. See theorem 7.1 and corollary 7.9 of [4]. □

Example 3.6. We return to example 3.4. Recall (3.3). From the above theorem it follows that the two coupled cell networks of figures 1 and 2 are ODE-equivalent. ◇

4. Minimality

In this section we compare ODE-equivalent networks in terms of the number of edges.

Definition 4.1. Let G be an n -cell coupled cell network. We denote by $[G]$ the class of all coupled cell networks that are ODE-equivalent to G . By theorem 3.5 we have that $[G]$ coincides with the class of all coupled cell networks that are linearly equivalent to G . ◇

Note that for a given n -cell network $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, the set

$$\{(\mathcal{C}_1, \mathcal{E}_1, \sim_{\mathcal{C}_1}, \sim_{\mathcal{E}_1}) \in [G] : \text{card}(\mathcal{E}_1) \leq \text{card}(\mathcal{E})\} \subseteq [G],$$

where $\text{card}(\mathcal{E})$ denotes the cardinality of the set of edges \mathcal{E} , is finite. In particular, it follows that the set

$$\{\text{card}(\mathcal{E}_1) : (\mathcal{C}_1, \mathcal{E}_1, \sim_{\mathcal{C}_1}, \sim_{\mathcal{E}_1}) \in [G]\}$$

has a minimum. We introduce now some notation:

Definition 4.2. Given an n -cell network G and the corresponding ODE-class $[G]$, let

$$m_{[G]} = \min\{\text{card}(\mathcal{E}_1) : (\mathcal{C}_1, \mathcal{E}_1, \sim_{\mathcal{C}_1}, \sim_{\mathcal{E}_1}) \in [G]\}$$

and

$$\text{Min}_{[G]} = \{(\mathcal{C}_1, \mathcal{E}_1, \sim_{\mathcal{C}_1}, \sim_{\mathcal{E}_1}) \in [G] : \text{card}(\mathcal{E}_1) = m_{[G]}\}$$

We call $\text{Min}_{[G]}$ the *minimal subclass* of $[G]$. \diamond

The minimal subclass $\text{Min}_{[G]}$ of $[G]$ is thus the subclass of networks of $[G]$ such that the number of edges is *minimal* among all the networks of $[G]$ —the *subclass of the canonical normal forms*. In this paper, we describe the minimal subclass $\text{Min}_{[G]}$ of $[G]$, given an n -cell network G . More precisely, we give an algorithm that computes $\text{Min}_{[G]}$. We start by addressing the question for networks with only one \sim_I -equivalence class (sections 5–9). It follows then as a corollary an algorithm that computes $\text{Min}_{[G]}$ for any ODE-class $[G]$ (section 10).

Example 4.3. Consider the graph G_1 of figure 1. For this example we have $m_{[G_1]} = 18$ and $\text{Min}_{[G_1]} = \{G_2\}$, where G_2 is the coupled cell network of figure 2. See section 9 for details. \diamond

5. Homogeneous networks

In this section we address the minimality of networks in terms of the number of edges for ODE-classes associated with networks that have only one \sim_I -equivalence class.

Definition 5.1. An *homogeneous* network is a coupled cell network with only one input-equivalence class. An *identical-edge homogeneous* cell network is an homogeneous network in which all edges in \mathcal{E} are equivalent. A *nonidentical-edge homogeneous* cell network is an homogeneous network with more than one edge-equivalence class. \diamond

Note that homogeneous networks have only one cell-equivalence class since these have only one \sim_I -equivalence class and by definition the equivalence relation \sim_I refines \sim_C .

The graphs of figures 1 and 2 are examples of identical-edge homogeneous coupled cell networks. Figure 5 shows an example of a non-identical-edge homogeneous network with two edge-types.

Definition 5.2. Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be an homogeneous network with n cells, say $\mathcal{C} = \{1, \dots, n\}$, and m edge-types, with $[e_1]_E, \dots, [e_m]_E$, the \sim_E -equivalence classes. We define the *adjacency matrix* of G with respect to $[e_l]_E$, for $l = 1, \dots, m$, to be the $n \times n$ matrix $M_{(G,l)}$ with rows and columns indexed by the cells of G in the following way: the (i, j) -entry of $M_{(G,l)}$ corresponds to the number of edges of type $[e_l]_E$ from cell j to cell i . Thus the sum of the j th entries of row i of the matrices $M_{(G,l)}$, for $l = 1, \dots, m$, gives the number of input edges of cell i with tail cell j . \diamond

Example 5.3. The adjacency matrices of the coupled cell network G_1 of figure 5 with respect to the two edge-types are

$$M_{(G_1,1)} = \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix} \quad \text{and} \quad M_{(G_1,2)} = \begin{pmatrix} 10 & 25 & 10 \\ 10 & 20 & 15 \\ 20 & 10 & 15 \end{pmatrix}. \quad \diamond$$

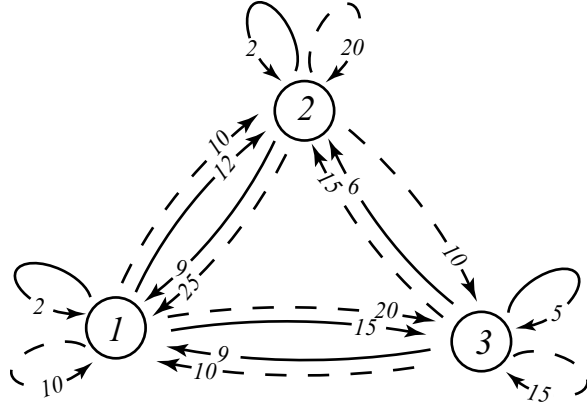


Figure 5. An example of a nonidentical-edge homogeneous network G_1 .

5.1. Linear equivalence of adjacency matrices

We start by relating the adjacency matrices corresponding to linearly equivalent homogeneous networks. Definition 3.2 of linear equivalence is given by:

Proposition 5.4. *Let G_1 be an homogeneous network with n cells and m_1 edge-types, where $\mathcal{C}_1 = \{1, \dots, n\}$. Let G_2 be an homogeneous network with n cells and m_2 edge-types. Denote by $M_{(G_1,l)} = [a_{ij}^l]_{1 \leq i,j \leq n}$ for $l = 1, \dots, m_1$, and $M_{(G_2,k)} = [b_{ij}^k]_{1 \leq i,j \leq n}$ for $k = 1, \dots, m_2$, the adjacency matrices of G_1 and G_2 with respect to $[e_l]_{E_1}$ and $[e_k]_{E_2}$, respectively. We have that G_1 and G_2 are linearly equivalent if and only if there is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that:*

$$\mathbb{R}\{\mathbf{I}_n, [a_{ij}^1]_{1 \leq i,j \leq n}, \dots, [a_{ij}^{m_1}]_{1 \leq i,j \leq n}\} = \mathbb{R}\{\mathbf{I}_n, [b_{\gamma(i)\gamma(j)}^1]_{1 \leq i,j \leq n}, \dots, [b_{\gamma(i)\gamma(j)}^{m_2}]_{1 \leq i,j \leq n}\} \quad (5.5)$$

Proof. Since both networks are homogeneous, trivially any bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ preserves cell-equivalence and input-equivalence. Thus definition 3.2 of linear equivalence translates to equation (5.5). \square

If necessary, we can relabel the cells of G_2 so that $\mathcal{C}_1 = \mathcal{C}_2$ and G_2 is γ -linearly equivalent to G_1 where γ is the identity on $\mathcal{C}_1 = \{1, \dots, n\}$. Moreover, since G_2 is linearly equivalent to G_1 , then by proposition 5.4 it follows that

$$\mathbb{R}\{\mathbf{I}_n, [a_{ij}^1]_{1 \leq i,j \leq n}, \dots, [a_{ij}^{m_1}]_{1 \leq i,j \leq n}\} = \mathbb{R}\{\mathbf{I}_n, [b_{ij}^1]_{1 \leq i,j \leq n}, \dots, [b_{ij}^{m_2}]_{1 \leq i,j \leq n}\}. \quad (5.6)$$

Example 5.5. Using proposition 5.4, the network G_1 of figure 5 is linearly equivalent to the network G_2 of figure 6, where γ is the identity on $\{1, 2, 3\}$. \diamond

Remark 5.6. Let G_1 be an homogeneous network with n cells. Let $M_{(G_1,1)}, \dots, M_{(G_1,m_1)}$ be the corresponding adjacency matrices. Then the set of matrices

$$\mathcal{L} = M_{n \times n}(\mathbb{Z}) \cap \mathbb{R}\{\mathbf{I}_n, M_{(G_1,1)}, \dots, M_{(G_1,m_1)}\}$$

with the usual sum of matrices is a lattice [1, chapter 3]: trivially, the set \mathcal{L} is a discrete subgroup of $M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. \diamond

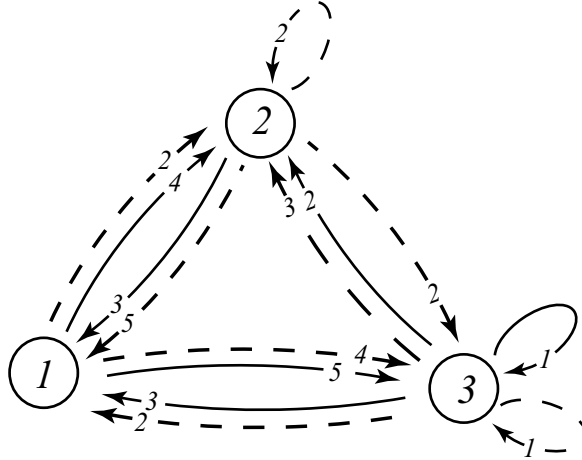


Figure 6. An example of a nonidentical-edge homogeneous network G_2 linearly equivalent to the graph of figure 5.

The next lemma shows that the networks of the ODE-class $[G_1]$ have adjacency matrices lying in the cone $\mathcal{L} \cap M_{n \times n}(\mathbb{Z}_0^+)$.

Lemma 5.7. *Let G_1 be an homogeneous network with n cells, say $\mathcal{C}_1 = \{1, \dots, n\}$, and m_1 edge-types. Let $M_{(G_1, l)} = [a_{ij}^l]_{1 \leq i, j \leq n}$ for $l = 1, \dots, m_1$ be the corresponding adjacency matrices. Then $M = [m_{ij}]_{1 \leq i, j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of a network G (of n cells) linearly equivalent to G_1 if and only if*

$$M \in \mathcal{L} \cap M_{n \times n}(\mathbb{Z}_0^+). \tag{5.7}$$

Proof. Note that each $M_{(G_1, l)}$ has the following properties:

$$M_{(G_1, l)} \in M_{n \times n}(\mathbb{Z}_0^+); \tag{5.8}$$

$$\sum_{j=1}^n a_{ij}^l = \sum_{j=1}^n a_{kj}^l \quad \text{for all } i, k. \tag{5.9}$$

By proposition 5.4 we have that if $M = [m_{ij}]_{1 \leq i, j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of a network G (of n cells) linearly equivalent to G_1 then condition (5.7) follows.

Suppose now that $M = [m_{ij}]_{1 \leq i, j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ satisfies (5.7). In particular, since G is homogeneous, it follows that

$$\sum_{j=1}^n m_{ij} = \sum_{j=1}^n m_{kj} \quad \text{for all } i, k. \quad \square$$

5.2. Minimality

Given an homogeneous network G , we aim to describe the minimal subclass $\text{Min}_{[G]}$ of $[G]$. Recall definition 4.2. Using the above discussion on ODE-equivalence and linear equivalence, we can describe the graphs of $\text{Min}_{[G]}$ by computing the corresponding adjacency matrices.

Moreover, we have:

Lemma 5.8. *Let G_1 be an homogeneous network with n cells. Let $[G_1]$ the corresponding ODE-class and $M_{(G_1,l)}$, for $l = 1, \dots, m_1$, the adjacency matrices of G_1 . Let*

$$m = \dim \mathbb{R}\{I_n, M_{(G_1,1)}, \dots, M_{(G_1,m_1)}\} - 1. \quad (5.10)$$

Then the homogeneous networks of the minimal class $Min_{[G_1]}$ have m edge types.

Proof. Let $G_2 \in Min_{[G_1]}$ and suppose that G_2 has m_2 edge types. Trivially, $m_2 \geq m$ since

$$\mathbb{R}\{I_n, M_{(G_1,1)}, \dots, M_{(G_1,m_1)}\} = \mathbb{R}\{I_n, M_{(G_2,1)}, \dots, M_{(G_2,m_2)}\}$$

and so

$$m_2 + 1 \geq \dim \mathbb{R}\{I_n, M_{(G_2,1)}, \dots, M_{(G_2,m_2)}\} = \dim \mathbb{R}\{I_n, M_{(G_1,1)}, \dots, M_{(G_1,m_1)}\} = m + 1.$$

If $m_2 > m$ then

$$\mathbb{R}\{I_n, M_{(G_2,1)}, \dots, M_{(G_2,m_2)}\} = \mathbb{R}\{I_n, M_{(G_2,i_1)}, \dots, M_{(G_2,i_k)}\}$$

for some set $\{i_1, \dots, i_k\} \subset \{1, \dots, m_2\}$ and $G_2 \notin Min_{[G_1]}$, a contradiction. Thus $m_2 = m$. \square

Remark 5.9. Let m be as defined in (5.10). Since $\dim \mathbb{R}\{I_n, M_{(G,1)}, \dots, M_{(G,m_1)}\}$ equals $m + 1$ we have that \mathcal{L} is a lattice of rank $m + 1$. \diamond

Definition 5.10. Given $M = [m_{ij}]_{1 \leq i, j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ denote by

$$l(M) = \sum_{i=1}^n \sum_{j=1}^n m_{ij}.$$

More generally, for $u = (u_1, \dots, u_k) \in (\mathbb{Z}_0^+)^k$ we call the *length* of u the non-negative integer $l(u)$ defined by

$$l(u) = \sum_{i=1}^k u_i.$$

Note that if $w_1, \dots, w_m \in (\mathbb{Z}_0^+)^k$ then

$$l\left(\sum_{i=1}^m w_i\right) = \sum_{i=1}^m l(w_i).$$

Given a set of vectors $\{w_1, \dots, w_r\} \subset (\mathbb{Z}_0^+)^k$ we denote

$$l(\{w_1, \dots, w_r\}) = \sum_{j=1}^r l(w_j). \quad \diamond$$

We find the minimal subclass $Min_{[G]}$ in the following way:

Proposition 5.11. *Let G be an homogeneous network of n cells and $[G]$ the corresponding ODE-class. Suppose that $M_{(G,1)}, \dots, M_{(G,m_1)}$ are the adjacency matrices of G . Let*

$$m = \dim \mathbb{R}\{I_n, M_{(G,1)}, \dots, M_{(G,m_1)}\} - 1.$$

An homogeneous network of the minimal class $\text{Min}_{[G]}$ has adjacency matrices M_1, \dots, M_m defined by:

- (i) $\{\mathbf{I}_n, M_1, \dots, M_m\} \subset M_{n \times n}(\mathbb{Z}_0^+)$;
- (ii) $\{\mathbf{I}_n, M_1, \dots, M_m\}$ is a basis of the real vector space $\mathbb{R}t\{\mathbf{I}_n, M_{(G,1)}, \dots, M_{(G,m)}\}$;
- (iii) $\sum_{k=1}^m l(M_k) = m_{[G]}$.

Proof. By lemma 5.8 any graph $G_2 \in \text{Min}_{[G]}$ has m edge types. Item (i) follows by the definition of adjacency matrix of a network. Moreover, by lemma 5.7 and proposition 5.4 we have (ii). Condition (iii) follows from definition 4.2 of $\text{Min}_{[G]}$. \square

Remark 5.12. Note that if $M \in M_{n \times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of an homogeneous network, then $l(M) \geq n$ and so $l(M) \geq l(\mathbf{I}_n)$. Denote by $M_0 = b_0 = \mathbf{I}_n$. Assume the conditions of proposition 5.11 and denote $S = \mathbb{R}\{\mathbf{I}_n, M_{(G,1)}, \dots, M_{(G,m)}\}$. A set $\{M_1, \dots, M_m\}$ satisfies (i)–(iii) if and only if it satisfies

- (i) $\{M_0, M_1, \dots, M_m\} \subset M_{n \times n}(\mathbb{Z}_0^+)$;
- (ii) $\{M_0, M_1, \dots, M_m\}$ is a basis of S ;
- (iii) $\sum_{k=0}^m l(M_k) = \min \left\{ \sum_{k=0}^m l(b_k) : \{b_0, b_1, \dots, b_m\} \subset M_{n \times n}(\mathbb{Z}_0^+) \text{ is a basis of } S \right\}$. \diamond

6. Minimal bases

By proposition 5.11 and remark 5.12 we have that the minimal subclass $\text{Min}_{[G]}$ of $[G]$ where G is an homogeneous coupled cell network can be computed by describing the set of all possible adjacency matrices satisfying a minimality condition in terms of the length of those matrices. We abstract the problem in the following way.

Given a set of vectors $w_1, \dots, w_p \in (\mathbb{Z}_0^+)^k$, denote by

$$S = \mathbb{R}\{w_1, \dots, w_p\}$$

the real vector subspace of \mathbb{R}^k generated by the vectors w_1, \dots, w_p . Suppose that

$$\dim S = r,$$

with $r \leq p$. We choose any r linearly independent vectors from w_1, \dots, w_p , say w_1, \dots, w_r . Thus w_1, \dots, w_r form a basis of S . Denote by

$$s = l(\{w_1, \dots, w_r\})$$

Let

$$\mathbf{B} = \{b : b = \{b_1, \dots, b_r\} \subset (\mathbb{Z}_0^+)^k \text{ is a basis of } S\}.$$

Lemma 6.1. *The set $\{l(b) : b \in \mathbf{B}\}$ has a minimum.*

Proof. As $\{w_1, \dots, w_r\} \subset (\mathbb{Z}_0^+)^k$ is a basis of S and $l(\{w_1, \dots, w_r\}) = s$, it follows that the minimum of $\{l(b) : b \in \mathbf{B}\}$ is at most s . Thus,

$$r \leq \min\{l(b) : b \in \mathbf{B}\} \leq s.$$

Moreover,

$$\min\{l(b) : b \in \mathbf{B}\} = \min\{l(b) : b \in \mathbf{B}, r \leq l(b) \leq s\},$$

and $\{b \in \mathbf{B} : r \leq l(b) \leq s\}$ is a finite set. \square

Definition 6.2. We call a basis $b \in \mathbf{B}$ of S such that

$$l(b) = \min\{l(\bar{b}) : \bar{b} \in \mathbf{B}\}$$

a *minimal basis*. ◇

In this section we describe the set of all minimal bases $b \in \mathbf{B}$ of S . Moreover, we present an algorithm that computes this set.

6.1. Sketch of the procedure

The method we follow for finding the minimal bases of S relies mostly on two steps:

- (1) Find the set C defined by

$$C = \{w \in S \cap (\mathbb{Z}_0^+)^k : 0 < l(w) \leq s - r + 1\}.$$

- (2) Find all the minimal bases $b \in \mathbf{B}_C$ of S where

$$\mathbf{B}_C = \{b \in \mathbf{B} : b \subseteq C\}.$$

Remarks 6.3.

- (a) If $b = \{b_1, \dots, b_r\} \subset (\mathbb{Z}_0^+)^k$ is a basis of S and $r \leq l(b) \leq s$ then $b \subseteq C$. Moreover, $\mathbf{B}_C \subseteq \mathbf{B}$ and

$$\min\{l(b) : b \in \mathbf{B}\} = \min\{l(b) : b \in \mathbf{B}_C\}.$$

Thus the minimal bases $b = \{b_1, \dots, b_r\}$ of S are contained in C .

- (b) Note that the set C depends on the chosen basis $\{w_1, \dots, w_r\}$ of S which determines the length $s = l(\{w_1, \dots, w_r\})$. We show in theorem 6.7 that the minimal bases do not depend on the chosen set of the vectors w_i that form a basis of S , nor on another choice of vectors (with non-negative integers coordinates) generating S . ◇

6.2. Finding C .

Let W be the $k \times r$ matrix with non-negative integer entries with columns given by the vectors w_1, \dots, w_r that we assume linearly independent and generating S . In notation

$$W = (w_1 | \dots | w_r).$$

Denote by M_W the set of all $r \times r$ matrices formed by r rows of W and with nonzero determinant and take

$$\delta = \gcd\{\det(M) : M \in M_W\}.$$

Lemma 6.4. If $w \in S \cap (\mathbb{Z}_0^+)^k$ then there are rational numbers x_1, \dots, x_r whose denominators divide δ such that

$$w = x_1 w_1 + \dots + x_r w_r.$$

Proof. By hypothesis $\{w_1, \dots, w_r\}$ is a basis of S . Thus given $w \in S \cap (\mathbb{Z}_0^+)^k$, there are real numbers $x_1, \dots, x_r \in \mathbb{R}$ such that $w = x_1 w_1 + \dots + x_r w_r$. Moreover, we have that w has integer components. Given $M \in M_W$, if we consider the linear system in the variables x_1, \dots, x_r obtained from $x_1 w_1 + \dots + x_r w_r = w$ by keeping the rows corresponding to M , by Cramer's rule we have that the unique x_1, \dots, x_r are rational whose denominator divide the (nonzero) determinant of M . □

By the above lemma, we have that C is the set of linear combinations of w_1, \dots, w_r , say $x_1 w_1 + \dots + x_r w_r \in (\mathbb{Z}_0^+)^k$, with rational coefficients x_1, \dots, x_r whose denominators divide δ and such that

$$0 < l(x_1 w_1 + \dots + x_r w_r) \leq s - r + 1.$$

Thus $x_i = p_i/q_i$ where $p_i, q_i \in \mathbb{Z}$, $q_i \neq 0$ and $\delta = n_i q_i$ for some $n_i \in \mathbb{Z}$. It follows then that

$$(n_1 p_1) w_1 + \dots + (n_r p_r) w_r \equiv 0 \pmod{\delta}.$$

We describe C in the following way:

$$C = \left\{ \frac{1}{\delta} \sum_{j=1}^r y_j w_j \in (\mathbb{Z}_0^+)^k : \begin{cases} \sum_{j=1}^r y_j w_j \equiv 0 \pmod{\delta} \\ 0 < \frac{1}{\delta} l \left(\sum_{j=1}^r y_j w_j \right) \leq s - r + 1 \end{cases} \right\}.$$

Remarks 6.5.

- (a) Note that $w_1, \dots, w_r \in C$ since $\delta w_i \equiv 0 \pmod{\delta}$ and for $i = 1, \dots, r$ we have $0 < l(w_i) \leq s - r + 1$.
- (b) The set C is finite since it is a subset of the finite set formed by the vectors $w \in (\mathbb{Z}_0^+)^k$ such that $0 < l(w) \leq s - r + 1$. \diamond

6.3. Finding the minimal bases

In order to find the bases $b = \{b_1, \dots, b_r\} \subseteq C$ of S such that

$$l(b) = \min\{l(b) : b \in \mathbf{B}_C\}$$

we partition the set C by the length of its vectors:

$$C = C_1 \dot{\cup} C_2 \dot{\cup} \dots \dot{\cup} C_t,$$

where given $u, v \in C$

$$\begin{cases} l(u) = l(v) \Leftrightarrow u, v \in C_i & \text{for some } i, \\ l(u) < l(v) \Leftrightarrow u \in C_i, v \in C_j, & \text{with } i < j. \end{cases}$$

As $\{w_1, \dots, w_r\}$ is a basis of S and $w_1, \dots, w_r \in C$, it follows that the real subspace of \mathbb{R}^k generated by C is S .

Definition 6.6. Let S_1, S_2 be real subspaces of \mathbb{R}^k . We denote

$$S_1 + S_2 = \{u + v : u \in S_1, v \in S_2\}. \quad \diamond$$

Denote by $S_i, i = 1, \dots, t$, the subspace of S generated by C_i . Thus

$$S_1 + \dots + S_t = S.$$

Moreover, there is q such that $1 \leq q \leq t$ and

$$\dim(S_1 + \dots + S_q) = r, \quad \dim(S_1 + \dots + S_{q-1}) < r. \quad (6.11)$$

Thus we have

$$S_1 + \dots + S_q = S.$$

We show below that the minimal bases of S are contained in

$$C_1 \dot{\cup} \dots \dot{\cup} C_q.$$

Denote by l_i the common length of any vector $u \in C_i$, for $i = 1, \dots, q$. Let

$$d_1 = \dim S_1,$$

$$d_i = \dim \left(\sum_{j=1}^i S_j \right) - \dim \left(\sum_{j=1}^{i-1} S_j \right), \quad 2 \leq i \leq q.$$

Let $U_1 \subseteq C_1$ be a set of d_1 linearly independent vectors. For $i = 2, \dots, q$, let $U_i \subseteq C_i$ be a set of d_i vectors such that $U_1 \cup \dots \cup U_i$ is free.

Theorem 6.7. *The minimum of the set*

$$\{l(b) : b \in \mathbf{B}\}$$

is

$$m = \sum_{j=1}^q l_j d_j. \quad (6.12)$$

The minimal bases $b \in \mathbf{B}$ of S are the bases obtained by taking all the possible unions $U_1 \cup \dots \cup U_q$.

Proof. By lemma 6.1 the set $\{l(b) : b \in \mathbf{B}\}$ has a minimum, say m .

Any set $U_1 \cup \dots \cup U_q = \{b_1, \dots, b_r\}$ obtained as above is free and has r vectors by construction. (Recall that by assumption $\dim(S_1 + \dots + S_q) = r$.) Thus it is a basis of S and

$$l(\{b_1, \dots, b_r\}) = \sum_{j=1}^q l_j d_j.$$

We prove now that the minimum m of the set $\{l(b) : b \in \mathbf{B}\}$ is

$$\sum_{j=1}^q l_j d_j$$

and that any minimal basis $b \in \mathbf{B}$ is the union $U_1 \cup \dots \cup U_q$ of sets U_i constructed as above. In particular, it follows that $b \subseteq C$.

Starting with a different set of basis vectors w_i of S , the set C varies if the total length of the chosen basis, say s^* , differs from s . However, in this case, the new set, say C^* , would contain C if $s^* > s$, and would be contained in C if $s^* < s$. (Recall that C^* would be formed by the non-null vectors in $S \cap (\mathbb{Z}_0^+)^k$ with length lower or equal to $s^* - r + 1$.) In both cases, we have $(C_1 \cup \dots \cup C_q) \subseteq C^*$ since $(C_1 \cup \dots \cup C_q) \subseteq S$ and generates S . Note that in the second case, where C^* would be contained in C , if $C_1 \cup \dots \cup C_q \not\subseteq C^*$, then we would have $C^* = C_1 \cup \dots \cup C_s$ where $s < q$ and so $S_1 + \dots + S_s = S$, which cannot happen by the definition of q (recall (6.11)).

Suppose by contradiction that there is a basis $b = \{b_1, \dots, b_r\} \in \mathbf{B}_C$ such that b is not a union $U_1 \cup \dots \cup U_q$ of sets U_i constructed as above and that

$$l(b) = m.$$

We have two cases to consider:

- (i) $b \not\subseteq C_1 \cup \dots \cup C_q$. Then b does not contain r linearly independent vectors of $C_1 \cup \dots \cup C_q$. But $C_1 \cup \dots \cup C_q$ contains r linearly independent vectors since by hypothesis $\dim(S_1 + \dots + S_q) = r$. Thus we can choose a vector $u \in C_1 \cup \dots \cup C_q$ such that $u \notin b$ and such that u and the vectors of $b \cap (C_1 \cup \dots \cup C_q)$ are linearly independent.

But $u \in S$ and b is a basis of S . Then we can write u as a linear combination of the vectors b_i and such that at least one of the nonzero coefficients is of a vector in b that do not belong to $C_1 \cup \dots \cup C_q$. Say b_i . It follows then that $l(b_i) > l(u)$ and so the basis obtained from b by substituting b_i by u , $\{b_1, \dots, b_{i-1}, u, b_{i+1}, \dots, b_r\}$ is such that

$$l(\{b_1, \dots, b_{i-1}, u, b_{i+1}, \dots, b_r\}) < m.$$

A contradiction.

- (ii) $b \subseteq C_1 \cup \dots \cup C_q$ and there is j between 1 and q such that the number of vectors of b with length lower or equal to l_j is (strictly) lower than $\dim(S_1 + \dots + S_j)$. Then there is $u \in C_1 \cup \dots \cup C_j$ such that u and the vectors of $b \cap (C_1 \cup \dots \cup C_j)$ are linearly independent. Again, u is a linear combination of the vectors b_i , and at least one nonzero coefficient corresponds to a vector, say b_k , such that $l(b_k) > l_j$. The basis obtained from b by substituting b_k by u has total length strictly lower than m . Again a contradiction.

It follows then that any basis $b = \{b_1, \dots, b_r\} \subseteq ((\mathbb{Z}_0^+)^k \cap C)$ of the space S such that $l(\{b_1, \dots, b_r\}) = m$ is contained in $C_1 \cup \dots \cup C_q$ and the number of vectors of b with length lower or equal to l_j is equal to $\dim(S_1 + \dots + S_j)$, for $j = 1, \dots, q$. Thus $b = U_1 \cup \dots \cup U_q$ where the U_j are any sets as constructed above. Moreover, we have the equality (6.12). \square

7. Algorithm

Let G be an homogeneous network with n cells and $p-1$ edge-types. Let $M_{(G,l)} = [a_{ij}^l]_{1 \leq i,j \leq n}$, for $l = 1, \dots, p-1$, be the adjacency matrices associated with G and let $M_{(G,p)} = I_n$. For $l = 1, \dots, p$, let w_l be the vector $w_l = (a_{11}^l, \dots, a_{1n}^l, a_{21}^l, \dots, a_{2n}^l, \dots, a_{n1}^l, \dots, a_{nn}^l)$. Let $k = n^2$ and S the real vector subspace of \mathbb{R}^k generated by the p vectors w_1, \dots, w_p of $(\mathbb{Z}_0^+)^k$. By theorem 6.7 the following algorithm finds all the minimal bases of S in $(\mathbb{Z}_0^+)^k$.

In what follows, if M_1 and M_2 are matrices of order $m \times n$ and $m \times p$, respectively, we denote by $(M_1|M_2)$ the matrix of order $m \times (n+p)$ whose first n columns are those of M_1 and the last p columns are those of M_2 .

Algorithm 7.1. Given p vectors w_1, \dots, w_p of $(\mathbb{Z}_0^+)^k$ that generate a real vector subspace S of \mathbb{R}^k , this algorithm finds all the minimal bases of S in $(\mathbb{Z}_0^+)^k$.

- (1) [Initial base] Let r be the rank of the matrix $(w_1 | \dots | w_p)$. For each w_i , $i = 1, \dots, p$ do the following: set $s_i = \sum_{j=1}^k (w_i)_j$ and then sort the vectors w_i by ascending order of the s_i . Let v_1, \dots, v_r be the ‘first’ r linearly independent vectors. Set $s = \sum_{j=1}^k (v_r)_j$.
- (2) [Compute gcd δ] Let W be the matrix $(v_1 | \dots | v_r)$. Compute the set M_W of all the $r \times r$ matrices formed by r rows of W and with nonzero determinant. Let δ be the greatest common divisor of the determinants of the matrices in M_W .
- (3) [Solve linear system of congruences] Apply HNF (Hermite Normal Form), algorithm A.3, to the matrix $(W|\delta I_k)$, and let $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ be a unimodular matrix and H the HNF matrix such that $(W|\delta I_k)U = (0|H)$. Here U_1 is a $r \times r$ square matrix. We can discard U_2, U_3 and U_4 . Let H_B be the HNF matrix of the matrix $(U_1|\delta I_r)$ obtained from algorithm A.3. The matrix H_B is invertible because it has rank r .
- (4) [Compute set C] Let W^* be a matrix in M_W . Determine the set

$$A = \left\{ (W^* H_B)^{-1} (\delta c_1, \delta c_2, \dots, \delta c_r)^\top : \begin{cases} c_1, c_2, \dots, c_r \in \mathbb{Z}_0^+ \\ 0 < c_1 + c_2 + \dots + c_r \leq s \end{cases} \right\} \cap \mathbb{Z}^r.$$

Determine

$$C = \left\{ \frac{1}{\delta} W H_B(x_1, \dots, x_r)^\top : (x_1, \dots, x_r)^\top \in A \right\} \cap \\ \{w \in (\mathbb{Z}/(s+1)\mathbb{Z})^k : 0 < l(w) \leq s\}.$$

For $i = 1$ until s , set $C_i = \emptyset$. For each w in C do the following: set $C_{l(w)} = C_{l(w)} \cup \{w\}$.

- (5) [Compute minimum sum] Let j be the least i such that $C_i \neq \emptyset$. Set $m = 0$, $i = j$ and $d = 0$. While $i \leq s$ and $d < r$ do the following: if $C_i \neq \emptyset$, let d_i be the dimension of the real subspace generated by $C_j \cup \dots \cup C_i$ minus the dimension of the real subspace generated by $C_j \cup \dots \cup C_{i-1}$ and set $d = d + d_i$, $m = m + i d_i$, else set $d_i = 0$; set $i = i + 1$.
- (6) [All subsets of C with minimum sum] Set $E = \emptyset$, $q = 0$ and $S_{j-1} = \{0\}$. For $n = j$ until $i - 1$ do the following: if $d_n \neq 0$, set $I_n = \{w \in C_n : w \notin S_j + \dots + S_{n-1}\}$ and let E_n be the set of all the subsets of I_n with d_n linearly independent vectors and set $E = E \times E_n$ and $q = q + 1$.
- (7) [All minimal bases] Set $U = \emptyset$. For each (U_1, \dots, U_q) in E do the following: if the dimension of the subspace generated by $U_1 \cup \dots \cup U_q$ is r (that is, $U_1 \cup \dots \cup U_q$ is a minimal base of S in $(\mathbb{Z}_0^+)^k$) then set $U = U \cup \{(U_1 \cup \dots \cup U_q)\}$.

The set U contains all the minimal bases of S in $(\mathbb{Z}_0^+)^k$. Output U and m and terminate algorithm. \diamond

Item (3) of algorithm 7.1 for solving homogeneous linear systems of congruences is based on Cohen [3, algorithm 4.1.22].

Each basis $B = \{b_1, \dots, b_r\} \in U$, computed by algorithm 7.1, corresponds to a minimal network with $r - 1$ edge-types in the following way: Each $b_i \in (\mathbb{Z}_0^+)^k$ defines an $n \times n$ matrix M_i . Specifically, if $b_i = (b_{i1}, \dots, b_{in})$ where $b_{ij} \in (\mathbb{Z}_0^+)^n$ then the j th row of M_i is b_{ij} . Each matrix $M_i \neq I_n$ corresponds to an adjacency matrix of the minimal network with respect to an edge-type.

8. Examples

In this section we apply algorithm 7.1 to two examples of homogeneous coupled cell networks G obtaining the adjacency matrices corresponding to the minimal networks of the subclasses $Min_{[G]}$ of $[G]$.

Example 8.1. Consider the homogeneous network G of three cells of figure 7. Thus the adjacency matrices are

$$M_{(G,1)} = \begin{pmatrix} 4 & 0 & 6 \\ 8 & 0 & 2 \\ 3 & 2 & 5 \end{pmatrix}, \quad M_{(G,2)} = \begin{pmatrix} 4 & 8 & 4 \\ 8 & 4 & 4 \\ 10 & 4 & 2 \end{pmatrix}.$$

Note that

$$\dim \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\} = 3$$

and recall that the adjacency matrices of the networks in $Min_{[G]}$ satisfy the three properties of remark 5.12. Thus, our aim is to find all the possible matrices $M_1, M_2 \in M_{3 \times 3}(\mathbb{Z}_0^+)$ such that $\mathbb{R}\{M_0 = I_3, M_1, M_2\} = \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\}$ and

$$\sum_{k=0}^2 l(M_k) = \min \left\{ \sum_{k=0}^2 l(b_k) : \{b_0 = I_3, b_1, b_2\} \subset M_{3 \times 3}(\mathbb{Z}_0^+) \right. \\ \left. \text{is a basis of } \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\} \right\}.$$

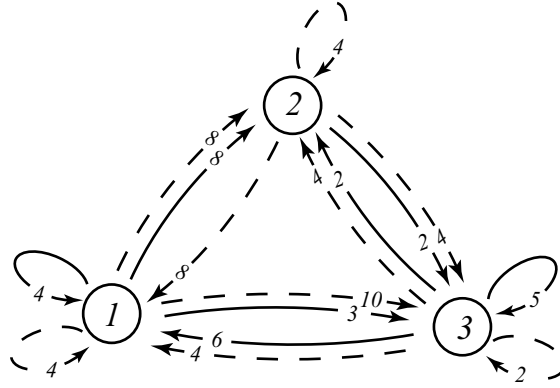


Figure 7. Homogeneous coupled cell network of example 8.1.

For that we apply algorithm 7.1. We consider

$$w_1 = (1, 0, 0, 0, 1, 0, 0, 0, 1)^\top, \quad w_2 = (4, 0, 6, 8, 0, 2, 3, 2, 5)^\top,$$

$$w_3 = (4, 8, 4, 8, 4, 4, 10, 4, 2)^\top$$

in $(\mathbb{Z}_0^+)^9$ and S the real subspace of \mathbb{R}^9 generated by these linearly independent vectors:

$$S = \mathbb{R}\{w_1, w_2, w_3\}.$$

Note that

$$s = \max\{l(w_1), l(w_2), l(w_3)\} = l(w_3) = 48$$

and the matrix

$$W^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 6 & 8 & 0 & 2 & 3 & 2 & 5 \\ 4 & 8 & 4 & 8 & 4 & 4 & 10 & 4 & 2 \end{pmatrix}$$

has rank 3. The greatest common divisor of the determinants of all 3×3 matrices formed by 3 rows of W and with nonzero determinant is $\delta = 8$. Applying the HNF algorithm A.3 to the matrix $(W|8I_9)$ we obtain matrices U and H , the HNF of $(W|8I_9)$, such that $(W|8I_9)U = (0|H)$, where

$$U = \begin{pmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{pmatrix}$$

with

$$U_1 = \begin{pmatrix} -8 & 44 & -16 \\ 0 & -10 & 4 \\ 0 & 3 & -2 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 8 & -5 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$U_4^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -3 & 6 & 7 & -7 & 1 \\ 1 & 2 & -2 & -2 & 3 & 0 \end{pmatrix},$$

$$U_6^\top = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 1 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$U_7^\top = I_3$, $U_5 = I_6$, and U_2, U_8 and U_9 are zero matrices.

Using algorithm A.3 to compute H_B , the HNF of the matrix $(U_1|8I_3)$, we obtain

$$H_B = \begin{pmatrix} 8 & 0 & 4 \\ 0 & 8 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (WH_B)^\top = \begin{pmatrix} 8 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 8 \\ 32 & 0 & 48 & 64 & 0 & 16 & 24 & 16 & 40 \\ 16 & 8 & 16 & 24 & 8 & 8 & 16 & 8 & 16 \end{pmatrix}$$

and so

$$C = \left\{ \begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ x_3 \\ 6x_2 + 2x_3 \\ 8x_2 + 3x_3 \\ x_1 + x_3 \\ 2x_2 + x_3 \\ 3x_2 + 2x_3 \\ 2x_2 + x_3 \\ x_1 + 5x_2 + 2x_3 \end{pmatrix} \in (\mathbb{Z}_0^+)^9 : x_1, x_2, x_3 \in \mathbb{Z}, 0 < x_1 + 10x_2 + 5x_3 \leq 16 \right\}.$$

In particular, it follows that for

$$w = (x_1 + 4x_2 + 2x_3, x_3, 6x_2 + 2x_3, 8x_2 + 3x_3, x_1 + x_3, 2x_2 + x_3, 3x_2 + 2x_3,$$

$$2x_2 + x_3, x_1 + 5x_2 + 2x_3)^\top \in C$$

we have $l(w) \geq 3x_3$ and $x_3 \geq 0$. Easy computations show that

$$C_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad C_6 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}, \quad C_9 = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}, \quad C_{12} = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Moreover, if S_i denotes the real subspace of S generated by C_i , we have that

$$\dim S_3 = \dim(S_3 + S_6) = \dim(S_3 + S_6 + S_9) = 1, \quad \dim(S_3 + S_6 + S_9 + S_{12}) = \dim S = 3$$

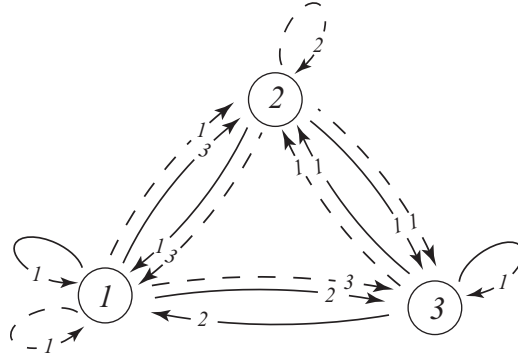


Figure 8. Minimal network of the ODE-equivalence class of the homogeneous coupled cell network of example 8.1.

By theorem 6.7, there is a unique minimal basis of S which is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Thus, by proposition 5.11 and remark 5.12, $\text{Min}_{[G]}$ contains only one minimal network with adjacency matrices given by

$$M_1 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}.$$

See figure 8. ◇

Example 8.2. Consider the homogeneous network G of figure 3 in section 1. The adjacency matrices of G are

$$M_{(G,1)} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}, \quad M_{(G,2)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{and } \dim \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\} = 3.$$

We aim to find all the possible matrices $M_1, M_2 \in M_{3 \times 3}(\mathbb{Z}_0^+)$ such that $\mathbb{R}\{M_0 = I_3, M_1, M_2\} = \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\}$ and such that

$$\sum_{k=0}^2 l(M_k) = \min \left\{ \sum_{k=0}^2 l(b_k) : \{b_0 = I_3, b_1, b_2\} \subset M_{3 \times 3}(\mathbb{Z}_0^+) \right.$$

$$\left. \text{is a basis of } \mathbb{R}\{I_3, M_{(G,1)}, M_{(G,2)}\} \right\}.$$

Again, we apply algorithm 7.1 to the vectors

$$w_1 = (1, 0, 0, 0, 1, 0, 0, 0, 1)^\top, \quad w_2 = (1, 1, 2, 2, 1, 1, 3, 0, 1)^\top, \\ w_3 = (1, 3, 0, 0, 1, 3, 1, 2, 1)^\top$$

in $(\mathbb{Z}_0^+)^9$. Note that $s = l(w_3) = 12$. Easy computations show that

$$C = \left\{ \begin{pmatrix} x_1 + x_2 + x_3 \\ x_2 + 2x_3 \\ 2x_2 + x_3 \\ 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_2 + 2x_3 \\ 3x_2 + 2x_3 \\ x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} \in (\mathbb{Z}_0^+)^9 : x_1, x_2, x_3 \in \mathbb{Z}, 0 < x_1 + 4x_2 + 4x_3 \leq 4 \right\}.$$

Moreover,

$$C_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad C_6 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}, \quad C_9 = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

If S_i denotes the real subspace of S generated by C_i , we have that

$$\dim S_3 = \dim(S_3 + S_6) = 1, \quad \dim(S_3 + S_6 + S_9) = \dim S = 3.$$

Thus, by proposition 5.11 and remark 5.12, $\text{Min}_{[G]}$ contains three minimal networks with adjacency matrices given, respectively, by

$$\left\{ M_1 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \right\},$$

$$\left\{ M_1 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{pmatrix} \right\}$$

and

$$\left\{ M_1 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{pmatrix} \right\}.$$

See figure 4 in section 1. ◇

9. Identical-edge homogeneous networks

In the special class of identical-edge homogeneous cell networks, the application of algorithm 7.1 simplifies. Recall that an identical-edge homogeneous cell network is an homogeneous network in which all edges in \mathcal{E} are equivalent. We prove below (theorem 9.3) that, if G is an identical-edge homogeneous network with n cells, with adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i, j \leq n}$, and $[G]$ the corresponding ODE-class, then $\text{Min}_{[G]}$ has a unique graph with adjacency matrix M defined by

$$M = \frac{1}{\delta}(M_{(G,1)} - m \mathbf{I}_n),$$

where $\delta = \gcd\{a_{ij} : i, j = 1, \dots, n; j \neq i\}$ and $m = \min\{a_{ii} : i = 1, \dots, n\}$. Direct application of algorithm 7.1 gives the same result. See remark 9.5. Here, we consider only networks G such that there is at least one edge $e \in \mathcal{E}$ satisfying $\mathcal{H}(e) \neq \mathcal{T}(e)$, and so $M_{(G,1)}$ is not a scalar multiple of the identity matrix.

Example 9.1. The graphs G_1 and G_2 of figures 1 and 2 in section 1, respectively, are examples of identical-edge homogeneous coupled cell networks. The corresponding adjacency matrices are

$$M_{(G_1,1)} = \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix}, \quad M_{(G_2,1)} = \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

We show below that the minimal subclass of $[G_1]$ is $\text{Min}_{[G_1]} = \{G_2\}$. \diamond

9.1. Linear equivalence and adjacency matrices

Definition 3.2 of linear equivalence for identical-edge homogeneous networks in terms of the adjacency matrices is now a special case of proposition 5.4. More precisely, if G_1 and G_2 are two identical-edge homogeneous networks with n cells, where $\mathcal{C}_1 = \{1, \dots, n\}$, and $M_{(G_1,1)} = [a_{ij}]_{1 \leq i, j \leq n}$ and $M_{(G_2,2)} = [b_{ij}]_{1 \leq i, j \leq n}$ are the corresponding adjacency matrices, then G_1 and G_2 are linearly equivalent if and only if there is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that:

$$\mathbb{R}\{\text{Id}_n, [a_{ij}]_{1 \leq i, j \leq n}\} = \mathbb{R}\{\text{Id}_n, [b_{\gamma(i)\gamma(j)}]_{1 \leq i, j \leq n}\}. \quad (9.13)$$

Example 9.2. Consider the identical-edge homogeneous networks of figures 1 and 2 in section 1, and recall example 3.4. Trivially, the identity function on $\{1, 2, 3\}$ is a bijection that preserves cell-equivalence and input-equivalence (like any bijection on $\{1, 2, 3\}$). Moreover, equation (3.4) of example 3.4 on the corresponding adjacency matrices is equivalent to the linear equivalence of the networks. \diamond

9.2. Minimality

We prove now that the minimal subclass $\text{Min}_{[G]}$ of an ODE-class $[G]$ where G is an identical-edge homogeneous network contains a unique graph. We then describe an algorithm that computes this minimal graph in $\text{Min}_{[G]}$.

Theorem 9.3. *Let G be an identical-edge homogeneous network of n cells, say $\mathcal{C} = \{1, \dots, n\}$, and $[G]$ the corresponding ODE-class. Let $M_{(G,1)} = [a_{ij}]_{1 \leq i, j \leq n}$ be the adjacency matrix of G and assume that it is not a scalar multiple of the identity matrix. Let*

$$\delta = \gcd\{a_{ij} : i, j = 1, \dots, n; j \neq i\},$$

$$m = \min\{a_{ii} : i = 1, \dots, n\}.$$

Then

$$\text{Min}_{[G]} = \{G_2\},$$

where G_2 is the network with adjacency matrix

$$M_{(G_2,1)} = \frac{1}{\delta}(M_{(G,1)} - mI_n).$$

Proof. The proof consists in showing that G_2 as described above is linearly equivalent to G , and that if G_1 is linearly equivalent to G_2 and $G_1 \neq G_2$ then $\text{card}(\mathcal{E}_2) < \text{card}(\mathcal{E}_1)$.

As $M_{(G_2,1)} = (1/\delta)(M_{(G,1)} - mI_n)$ it follows that the identical-edge homogeneous network with adjacency matrix $M_{(G_2,1)}$ is linearly equivalent to G and the entries of $M_{(G_2,1)}$ are non-negative integers as we show now. Let $M_{(G_2,1)} = [b_{ij}]_{1 \leq i, j \leq n}$, then

$$b_{ii} = \frac{a_{ii} - m}{\delta}, \quad i = 1, \dots, n \quad (9.14)$$

$$b_{ij} = \frac{a_{ij}}{\delta}, \quad i, j = 1, \dots, n; \quad j \neq i. \quad (9.15)$$

Since G is an identical-edge homogeneous network it follows that $M_{(G,1)}$ satisfies for any $i, k \in \{1, \dots, n\}$

$$\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{kj}$$

and thus

$$a_{ii} - a_{kk} = \sum_{j \neq k} a_{kj} - \sum_{j \neq i} a_{ij}.$$

Let k be such that $m = a_{kk}$. Then $a_{kk} \leq a_{ii}$, $\forall i$ and $\sum_{j \neq k} a_{kj} \geq \sum_{j \neq i} a_{ij}$. Moreover, δ is a divisor of $\sum_{j \neq k} a_{kj} - \sum_{j \neq i} a_{ij}$ and thus of $a_{ii} - a_{kk}$.

Let G_1 be a network linearly equivalent to G_2 (and so to G) with adjacency matrix $M_{(G_1,1)}$. It follows then that

$$M_{(G_1,1)} = \frac{1}{\beta}(M_{(G_2,1)} - \alpha I_n),$$

for some $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q}^+$. Since there is i such that $b_{ii} = 0$ we have that $\alpha \in \mathbb{Z}_0^-$. If $\text{card}(\mathcal{E}_1) \leq \text{card}(\mathcal{E}_2)$ then $\beta \geq 1$. Since the greatest common divisor of the entries b_{ij} is 1, then $\beta = 1$. If $\alpha = 0$ then $G_1 = G_2$. Otherwise, $\text{card}(\mathcal{E}_1) > \text{card}(\mathcal{E}_2)$. \square

Example 9.4. Recall the graphs G_1 and G_2 of figures 1 and 2 in section 1. By theorem 9.3 it follows that $\text{Min}_{G_1} = \{G_2\}$. Note that $\delta = 3$ and $m = 2$. \diamond

If we denote the rows of a $k \times r$ matrix M by M_1, \dots, M_k , then M^* represents the column vector $(M_1, \dots, M_k)^\top$.

Remark 9.5. Direct application of algorithm 7.1 gives the result of theorem 9.3 as we now show. Suppose G is an identical-edge homogeneous network with adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i, j \leq n}$. Assume $M_{(G,1)}$ is not a scalar multiple of the identity matrix. Using the same procedure as in section 6 we have the following: we take the $n^2 \times 2$ matrix $W = (w_1 | w_2)$ where $w_1 = I_n^*$ and $w_2 = M_{(G,1)}^*$; the greatest common divisor δ of the set of all 2×2 matrices formed by 2 rows of W and with nonzero determinant is the greatest common divisor of the set $\{a_{ij} : i \neq j\} \cup \{a_{ii} - a_{kk} : i \neq k\}$. From the proof of theorem 9.3, it follows that δ is the

greatest common divisor of the set $\{a_{ij} : i \neq j\}$. We aim to find the finite set C and so to find $y_1, y_2 \in \mathbb{Z}$ such that

$$y_1 w_1 + y_2 w_2 = \delta w$$

for some w with non-negative integer entries and such that $0 < l(w) \leq l(w_2)$. Note that

$$l(w) = \frac{1}{\delta}(y_1 n + y_2 l(w_2))$$

and

$$\delta w_{(k-1)n+k} = y_1 + y_2 a_{kk} \quad (k = 1, \dots, n),$$

$$\delta w_{(k-1)n+i} = y_2 a_{ki} \quad (i, k = 1, \dots, n; \quad i \neq k).$$

Thus $y_2 \geq 1$. Trivially, the minimum $l(w)$ is obtained by taking $y_2 = 1$ and $y_1 = -m$ where

$$m = \min\{a_{kk} : k = 1, \dots, n\}$$

and so there is a unique minimal basis of $\mathbb{R}\{w_1, w_2\}$ corresponding to the minimal network with adjacency matrix

$$M_{(G_2,1)} = \frac{1}{\delta}(M_{(G,1)} - mI_n). \quad \diamond$$

Algorithm 9.6. Given the adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i, j \leq n}$, with $M_{(G,1)} \neq \alpha I_n$, of an identical-edge homogeneous network G , this algorithm computes the adjacency matrix $M_{(G_2,1)} = [b_{ij}]_{1 \leq i, j \leq n}$ corresponding to the unique network in $\text{Min}_{[G]}$.

- (1) [Compute gcd δ and minimum m] Set $m = \min\{a_{ii} : i = 1, \dots, n\}$ and $\delta = \gcd\{a_{ij} : i, j = 1, \dots, n; \quad j \neq i\}$.
- (2) [Compute $M_{(G_2,1)}$] For $i, j = 1, \dots, n$ do the following: if $j \neq i$ then set $b_{ij} = a_{ij}/\delta$ else set $b_{ii} = (a_{ii} - m)/\delta$. Output $M_{(G_2,1)} = [b_{ij}]_{1 \leq i, j \leq n}$. \diamond

10. Minimality for coupled cell networks

Let G be a coupled cell network. From theorem 2.3 (recall also remark 2.5) the space of \mathcal{B}_G -equivariant maps has a natural decomposition according to the connected components (the \sim_I -equivalence classes) of the groupoid \mathcal{B}_G , and this decomposition is inherited by the polynomial and linear vector fields:

Definition 10.1. Let $\mathcal{Q} \subseteq \mathcal{C}$ be an \sim_I -equivalence class. Define

$$\mathcal{F}_G^P(\mathcal{Q}) = \{f \in \mathcal{F}_G^P : f_s(x) = 0, \quad \forall s \notin \mathcal{Q}\},$$

$$\mathcal{P}_G^P(\mathcal{Q}) = \{f \in \mathcal{P}_G^P : f_s(x) = 0, \quad \forall s \notin \mathcal{Q}\},$$

$$\mathcal{L}_G^P(\mathcal{Q}) = \{f \in \mathcal{L}_G^P : f_s(x) = 0, \quad \forall s \notin \mathcal{Q}\}.$$

We say that vector fields in $\mathcal{F}_G^P(\mathcal{Q})$, $\mathcal{P}_G^P(\mathcal{Q})$, and $\mathcal{L}_G^P(\mathcal{Q})$ are *supported on* \mathcal{Q} . \diamond

Remark 10.2. From theorem 2.3, there are direct sum decompositions

$$\mathcal{F}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{F}_G^P(\mathcal{Q}) \quad \mathcal{P}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{P}_G^P(\mathcal{Q}) \quad \mathcal{L}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{L}_G^P(\mathcal{Q}),$$

where \mathcal{Q} runs over the \sim_I -equivalence classes of G . \diamond

For detailed proofs see [9], end of section 4, especially proposition 4.6.

By theorem 3.5, two coupled cell networks G_1 and G_2 are ODE-equivalent if and only if they are linearly equivalent. That is, there exists some bijection γ between the corresponding sets of cells \mathcal{C}_i , preserving cell-equivalence and input-equivalence, such that with the identification $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, the vector spaces $\mathcal{L}_{G_1}^P$ and $\mathcal{L}_{G_2}^P$ are equal. By the above discussion, this is equivalent to have $\mathcal{L}_{G_1}^P(\mathcal{Q}) = \mathcal{L}_{G_2}^P(\mathcal{Q})$, for each connected component \mathcal{Q} . Moreover, all cell phase spaces may be assumed to be \mathbb{R} in this context.

We look now for $\text{Min}_{[G]}$ where G is an n -cell network. Suppose that G has connected components (\sim_I -equivalence classes) $\mathcal{Q}_1, \dots, \mathcal{Q}_l$. For $i = 1, \dots, l$ define the subnetworks G_i in the following way:

$$G_i = \{\mathcal{C}, \mathcal{E}_i, \sim_C, \sim_{E_i}\},$$

where

$$\mathcal{E}_i = \{e \in \mathcal{E} : \mathcal{H}(e) \in \mathcal{Q}_i\} \quad \text{and} \quad \forall e_1, e_2 \in \mathcal{E}_i, e_1 \sim_{E_i} e_2 \Leftrightarrow e_1 \sim_E e_2.$$

We can apply algorithm 7.1 to the graphs G_i , for $i = 1, \dots, l$, computing $\text{Min}_{[G_1]}, \dots, \text{Min}_{[G_l]}$. The minimal class $\text{Min}_{[G]}$ is then obtained by considering all possible graphs that can be formed by junction of one graph of each class $\text{Min}_{[G_1]}, \dots, \text{Min}_{[G_l]}$.

11. Summary

Non-isomorphic coupled cell networks with the same number of cells can generate the same dynamical behaviour and are called ODE-equivalent. Given an arbitrary number k of cells, we can partition the set of the k -cell networks into ODE-equivalence classes.

Given an ODE-class, we can ask for the network, or the networks, in the class with minimal number of edges. We start by addressing the case where the networks are homogeneous, that is, where all the cells are isomorphic. We prove that, in the special case of an ODE-class including an identical-edge homogenous network, there is a unique minimal network (theorem 9.3) which can be obtained using algorithm 9.6.

For general homogeneous networks, there can be more than one minimal network (theorem 6.7) which can be computed using algorithm 7.1. In section 8, we illustrate the application of the algorithm with two examples.

We end by remarking that the minimality of a non-homogeneous network reduces to the minimality of certain homogeneous subnetworks.

Appendix A. Algorithms

In this section we present an algorithm given by Cohen [2] to calculate the Hermite normal form of a matrix with integer coefficients. We use the following definition:

Definition A.1 ([2], definition 2.4.2). We say that an $m \times n$ matrix $M = [m_{ij}]$ with integer coefficients is in *Hermite normal form* (abbreviated HNF) if there exists $r \leq n$ and a strictly increasing map f from $[r + 1, n]$ to $[1, m]$ satisfying the following properties:

- (1) For $r + 1 \leq j \leq n$, $m_{f(j)j} \geq 1$, $m_{ij} = 0$ if $i > f(j)$ and $0 \leq m_{f(k)j} < m_{f(k)k}$ if $k < j$.
- (2) The first r columns of M are equal to 0. ◇

We have the following result:

Theorem A.2 ([2], theorem 2.4.3). Let A be an $m \times n$ matrix with coefficients in \mathbb{Z} . Then there exists a unique $m \times n$ matrix $B = [b_{ij}]$ in HNF of the form $B = AU$ with $U \in GL_n(\mathbb{Z})$, where $GL_n(\mathbb{Z})$ is the group of matrices with integer coefficients which are invertible, that is, whose determinant is equal to ± 1 .

Although B is unique, the matrix U is not unique. The matrix W formed by the nonzero columns of B is called the Hermite normal form of the matrix A .

Algorithm A.3 ([2], algorithm 2.4.4). (Hermite normal form). Given an $m \times n$ matrix A with integer coefficients $[a_{ij}]$ this algorithm finds the Hermite normal form H of A and a $n \times n$ unimodular matrix U such that $AU = (0|H)$. We write h_{ij} for the coefficients of H , A_i (resp. H_i , U_i) for the columns of A (resp. H , U).

- (1) [Initialize] Set $i = m$, $k = n$ and $U = I_n$. If $m \leq n$ then set $l = 1$ else set $l = m - n + 1$.
- (2) [Row finished?] If all the a_{ij} with $j < k$ are zero, then if $a_{ik} < 0$ replace column A_k by $-A_k$, replace column U_k by $-U_k$ and go to step 5.
- (3) [Choose non-zero entry] Pick among the non-zero a_{ij} , with $j \leq k$, one with the smallest absolute value, say a_{ij_0} . Then if $j_0 < k$, exchange column A_k with column A_{j_0} and exchange column U_k with column U_{j_0} . In addition, if $a_{ik} < 0$ replace column A_k by $-A_k$ and replace column U_k by $-U_k$. Set $b = a_{ik}$.
- (4) [Reduce] For $j = 1, \dots, k - 1$ do the following: set $q = \lfloor a_{ik}/b \rfloor$, and $A_j = A_j - qA_k$, $U_j = U_j - qU_k$. Then go to step 2.
- (5) [Final reductions] Set $b = a_{ik}$. If $b = 0$, set $k = k + 1$ and go to step 6. Otherwise, for $j > k$ do the following: set $q = \lfloor a_{ik}/b \rfloor$, and $A_j = A_j - qA_k$, $U_j = U_j - qU_k$.
- (6) [Finished?] If $i = l$ then for $j = 1, \dots, n - k + 1$ set $H_j = A_{j+k-1}$ and terminate the algorithm, else set $i = i - 1$, $k = k - 1$ and go to step 2. \diamond

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