Is stochastic volatility relevant for dynamic portfolio choice under ambiguity?

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July 18, 2014

Abstract: Literature on dynamic portfolio choice has been finding that volatility risk has low impact on portfolio choice. For example, using long-run U.S. data, Chacko and Viceira (2005) found that intertemporal hedging demand (required by investors for protection against adverse changes in volatility) is empirically small even for highly risk-averse investors. We want to assess if this continues to be true in the presence of ambiguity. Adopting robust control and perturbation theory techniques, we study the problem of a long-horizon investor with recursive preferences that faces ambiguity about the stochastic processes that generate the investment opportunity set. We find that ambiguity impacts portfolio choice, with the relevant channel being the return process. Ambiguity about the volatility process is only relevant if, through a specific correlation structure, it also induces ambiguity about the return process. Using the same long-run U.S. data, we find that ambiguity about the return process may be empirically relevant, much more than ambiguity about the volatility process. Anyway, intertemporal hedging demand is still very low: investors are essentially focused in the short-term risk-return characteristics of the risky asset.

Keywords: Dynamic Portfolio Choice, Stochastic Volatility, Ambiguity, Robust Control.

JEL Classification: C61 · D81 · E21 · G11.

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*We are grateful to António Barbosa, Fabio Trojani, Nikolay Ryabkov, Wolfgang Drobetz and two anonymous referees for extremely useful comments and suggestions. We also thank Alejandro Balbás, Claudia Ribeiro and Frank Riedel for early conversations on the subject, and Carlos Hervés-Beloso and Robert Kosowski for encouragement and hospitality. Previous versions of this work have been presented at the 6th Annual Meeting of the Portuguese Economic Journal, the 7th Finance Conference of the Portuguese Finance Network, the 6th Economic Theory Workshop in Vigo, the 2012 ASSET Meeting in Limassol, the 20th Finance Forum of the Spanish Finance Association in Oviedo, the XXI International Conference on Money, Banking and Finance in Rome, the XXXVII Symposium of the Spanish Economic Association in Vigo, the 7th Luso-Brazilian Finance Meeting in Búzios, the 2013 Royal Economic Society Conference in London, the 2013 European Meeting of the Financial Management Association in Luxembourg. Gonçalo Faria acknowledges support from FCT - Fundação para a Ciência e a Tecnologia (SFRH/BPD/74020/2010), CEF.UP and RGEA, Universidad de Vigo. João Correia-da-Silva acknowledges support from CEF.UP, FCT and FEDER (project PTDC/ IIM-ECO/5294/2012). This research has been financed by Portuguese Public Funds through FCT in the framework of the project PEst-OE/EGE/UI4105/2014.

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1 Introduction

We study optimal dynamic portfolio choice under a stochastic investment opportunity set, of an investor that is averse both to risk and ambiguity. We want to understand if and how ambiguity about the stochastic processes that generate the return and volatility of the risky asset impacts portfolio choice. More particularly, we want to assess if stochastic volatility continues to have a low impact on portfolio choice, as it has been found in the literature, in the presence of ambiguity about the stochastic investment opportunity set.\footnote{Throughout this paper, by “volatility” of the risky asset we mean the variance of the risky asset’s return. For mathematical convenience, we work with precision (the reciprocal of variance).}

There is a large literature on portfolio choice (see, e.g., Kogan and Uppal (2001) and Campbell and Viceira (2002) for a survey), but relatively few works study optimal dynamic portfolio choice with stochastic variance of the risky asset’s return. Some examples are Kim and Omberg (1996) and Chacko and Viceira (2005), with incomplete markets, and Schroder and Skiadas (1999), with complete markets. Schroder and Skiadas (2003) gave a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. Other papers consider multiple risky assets, as Liu (2007) and Buraschi et al. (2010). Potentially adverse changes in the investment opportunity set are associated with stochastic variance of the risky asset’s return, which therefore represents a source of risk to investors. This implies, from Merton (1973), that stochastic variance originates an intertemporal hedging demand.\footnote{In the multivariate setting of Buraschi et al. (2010), with a stochastic variance-covariance matrix, there is an intertemporal hedging demand associated with the stochastic variance and another associated with the stochastic correlation between the returns of the risky assets.}

Chacko and Viceira (2005) concluded, using long-run U.S. data, that this intertemporal hedging demand is empirically small even for highly risk-averse investors.

In all the papers mentioned above, there is only risk, and no ambiguity. Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a huge literature (surveyed in Camerer and Weber (1992) and Epstein and Schneider (2010)).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics ignored ambiguity for several decades, remaining based on the expected utility theory of von Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where, although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

Gradually, ambiguity is being incorporated in decision theory. Two main approaches are being used: (i) the multiple priors (MP) approach, where the single probability measure of the expected utility
models is replaced by a set of probabilities or priors; (ii) the robust control (RC) approach, associated to an assumption of model uncertainty. The relationship between the MP and RC approaches has been widely discussed in the literature. For example, in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

Ahn et al. (2013) found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also concluded that ambiguity aversion can be observed in competitive markets and that it influences portfolio choice and asset prices.

In studies of portfolio choice with ambiguity, Garlappi et al. (2007) and Gollier (2011) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models. The same conclusion was reached in a dynamic MP setting (e.g., Chen et al. (2013)) and in a dynamic RC model (e.g., Maenhout (2004, 2006) and Xu et al. (2011)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, with the exception of Xu et al. (2011), the source of ambiguity is exclusively the expected risky asset’s return or the risky asset’s return process.

In this paper, we extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice, by introducing ambiguity about the data generating process of the stochastic investment opportunity set. The motivation for this is provided by Chacko and Viceira themselves:

“An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.”

Literature on dynamic portfolio choice with stochastic variance has been finding that variance risk has low impact on portfolio decisions (e.g., Chacko and Viceira (2005) and Liu (2007)). We want to understand if this continues to be true if uncertainty is considered in a broader perspective, by taking into account an “ambiguity dimension” alongside the standard “risk dimension”.

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2011)) that it is reasonable to assume that investors estimate the variance of the risky asset’s return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity also about the variance process of the risky asset’s return because

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3 Although the result of Gollier (2011) requires some restrictions on the set of priors and on the investor’s attitude towards risk.
(i) there is no *a priori* reason to assume that investors are not ambiguous about it, and because (ii) we are able to find an asymptotic analytical solution and test it empirically.

In Faria et al. (2009), the setting of Chacko and Viceira (2005) was extended by considering a representative investor that is ambiguous about one specific parameter of the stochastic variance process (the expected value). A MP approach was adopted, and the conclusion was that ambiguity does not impact the instantaneous optimal portfolio choice rule. The ambiguity effect would only exist if the investor were not able to continuously update his portfolio. In Faria and Correia-da-Silva (2010), the optimal portfolio rule was obtained in a dynamic setting, with stochastic variance and ambiguity about its process. There, it was assumed that the representative investor derives utility exclusively through terminal wealth, implying that the intertemporal consumption-savings decision is ignored, and ambiguity is treated through a RC approach. The optimal portfolio rule that was derived showed that ambiguity aversion has an additive impact to risk aversion.

The closest paper to the present work is that of Xu et al. (2011), although our paper is a partial equilibrium model focusing on portfolio and consumption choice, while Xu et al. (2011) is a general equilibrium model focusing on stock pricing. In Xu et al. (2011), preferences of the representative investor are given by the SDU function introduced by Duffie and Epstein (1992b) and ambiguity about the data generating process with stochastic variance is also considered and studied through a RC approach. Compared with the contribution of Xu et al. (2011), our paper brings three major novelties. The first results from the fact that we adopt a different RC methodology: “constraint preferences” instead of “multiplier preferences”. Under “constraint preferences”, there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under “multiplier preferences”, preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations being more penalized than smaller ones. A relevant implication is that under “constraint preferences” the impact of ambiguity on the optimal portfolio choice is more than simply an enhanced risk aversion. Moreover, in order to derive optimal policies under ambiguity, we use perturbation theory, as, for example, in Trojani and Vanini (2002, 2004). The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): “…formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems.” In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity holds in neighborhoods of a model with a log-utility investor and no ambiguity.

The second difference relatively to Xu et al. (2011) is that we want to understand the relevant channels (return process, variance process, or both) through which ambiguity impacts dynamic portfolio choice. For that, we study optimal dynamic portfolio choice when ambiguity is simultaneously about the return and volatility processes, as in Xu et al. (2011), and when it is exclusively about the return process or the variance process.

The third difference versus Xu et al. (2011) is that we simulate our model using long-run U.S. data

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4We adopt this terminology from Hansen and Sargent (2006).
to measure the empirical significance of the impact of ambiguity on optimal portfolio choice. This is relevant, because, ultimately, we are addressing the question of whether stochastic variance is relevant for portfolio choice.

The main conclusions of this paper concern the impact of ambiguity on optimal dynamic policies, both when ambiguity is simultaneously about the return and variance processes and when it is exclusively about one of these stochastic processes. In all scenarios, we find that ambiguity does not impact the optimal consumption rule (instantaneous consumption as a function of current wealth). The effect of ambiguity is a reduction of the demand for the risky asset. The relevant channel is the return process, as when ambiguity is exclusively about the variance process there is no impact on the optimal portfolio rule. Ambiguity about the variance process is only relevant if, through a specific correlation structure, it also induces ambiguity about the stochastic process that generates the return of the risky asset.

Using long-run U.S. data, we find that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. Our simulation suggests that ambiguity about the return process is empirically much more relevant than ambiguity about the variance process. We also conclude that, even accounting for ambiguity about the variance process, the intertemporal hedging demand (required by investors for protection against adverse changes in variance) is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset. This had been found under settings where uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting where uncertainty also comprises ambiguity.

The paper is organized as follows. In section 2, we present the model and state the problem to be solved. In section 3, we present the analytical solution to that problem and the key results. In section 4, we analyze alternative scenarios for the sources of ambiguity, deriving analytical solutions and comparing its results with those of section 3. In section 5, we present simulation results. In section 6, we conclude the paper with some remarks.

2 Consumption and Portfolio Choice Problem

In section 2.1, the investment opportunity set is described. In section 2.2, the preferences of the representative investor are presented. In section 2.3, the dynamic optimization problem to be solved is disclosed.

2.1 Investment Opportunity Set

In this section, we describe the investment opportunity set that is faced by the representative investor. We follow closely Chacko and Viceira (2005).

All wealth must be allocated between a riskless asset with price $B_t$ and a risky asset with price $S_t$. 
The instantaneous return of the riskless asset is described by:

\[ \frac{dB_t}{B_t} = r dt, \]  

where \( r \) stands for the risk-free interest rate.

The instantaneous return of the risky asset is given by:

\[ \frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} \left( \rho dW_y + \sqrt{1 - \rho^2} dW_\varepsilon \right), \]  

where \( \mu \) is the expected return of the risky asset and \( y_t \) is the instantaneous precision of the risky asset’s return process (the instantaneous variance is \( v_t = \frac{1}{y_t} \)), while \( W_\varepsilon \) and \( W_y \) are two independent standard Brownian motions.

The precision, \( y_t \), follows a mean-reverting, square-root process as used by Cox et al. (1985):

\[ dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t} dW_y, \]  

where the expected value of precision is \( E[y_t] = \theta \), the reversion parameter is \( \kappa > 0 \), and, thus, \( Var[y_t] = \frac{\sigma^2 \theta}{2\kappa} \). To guarantee standard integrability conditions, it is assumed that \( 2\kappa \theta > \sigma^2 \), as in Cox et al. (1985).

Applying Itô’s Lemma to (3), a mean-reverting, square-root process for proportional changes in variance is obtained:

\[ \frac{dv_t}{v_t} = \kappa_v (\theta_v - v_t) dt - \sigma \sqrt{v_t} dW_y, \]  

where \( \theta_v = \left( \theta - \frac{\sigma^2}{\kappa} \right)^{-1} \) and \( \kappa_v = \kappa \left( \theta - \frac{\sigma^2}{\kappa} \right) = \frac{\kappa}{\theta} \).

An approximation of the unconditional mean of instantaneous variance is:\footnote{Obtained by taking expectations of the second-order Taylor expansion of \( v_t \) around \( \theta \) (Chacko and Viceira (2005)).}

\[ E[v_t] \approx \frac{1}{\theta} + \frac{\sigma^2}{2\kappa \theta^2}. \]  

As the expected return of the risky asset, \( \mu \), is assumed to be constant, (5) is also the unconditional variance of the risky asset’s return. Chacko and Viceira (2005) performed a Monte Carlo simulation to validate this statement and the accuracy of the approximation, having concluded that (5) understates the true variance by 0.27%.

It is implicit in (2)-(3) that shocks in precision \( W_y \) are correlated with shocks in the return of the risky asset, with correlation given by \( \rho \geq 0 \). From (4), this implies that the instantaneous correlation between proportional changes in the risky asset’s return and variance is given by:

\[ Corr_t \left( \frac{dv_t}{v_t}, \frac{dS_t}{S_t} \right) = -Corr_t \left( dy_t, \frac{dS_t}{S_t} \right) = -\rho dt. \]
This investment opportunity set incorporates three of the main stylized facts about the variance of the return of risky assets: the mean-reversion property, the leverage-effect property (given by the negative correlation between return and its variance), and the fact that proportional changes in variance are higher when variance is high.

### 2.2 Investor Preferences

It is assumed that the representative investor is not totally sure about the stochastic processes (2)-(3) that generate the dynamic investment opportunity set. In other words, the uncertainty faced by the representative investor has two dimensions: risk and ambiguity.

Additionally, it is assumed that the preferences of the representative investor are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory in Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman’s characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution.

The utility process that defines the SDU function is represented by:

\[
J = E_t \left[ \int_t^\infty f(C_s, J_s) \, ds \right], 
\]

where \(C_s\) represents current consumption and \(J_s\) is the continuation utility for the consumption flow \(C\), at time \(t = s\), with infinite time horizon. In our setting, the function \(f(C_s, J_s)\) is the normalized aggregator that generates \(J\), defining a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available.

We assume a unitary elasticity of intertemporal substitution (\(\psi = 1\)), because: (i) with \(\psi = 1\) there is an exact solution of the Bellman equation that we will obtain; and (ii) we will study asymptotic solutions of the model holding in neighborhoods of a model with a log-utility investor (\(\psi = \gamma = 1\)), where \(\gamma > 0\) is the coefficient of relative risk aversion, and no ambiguity. With \(\psi = 1\), the normalized aggregator \(f(C, J)\) takes the form (e.g., Duffie and Epstein (1992a,b)):

\[
f(C, J) = \beta (1 - \gamma) J \left\{ \ln(C) - \frac{1}{1 - \gamma} \ln[(1 - \gamma) J] \right\}, 
\]

where \(\beta > 0\) is the rate of time preference. If \(\gamma = 1\), (8) can be replaced by the standard log-utility representation.

A remark regarding the preference for the timing of the resolution of risk. With the preference

\footnote{Chacko and Viceira (2005) present an approximate solution for the Bellman equation that is obtained if \(\psi \neq 1\). That solution converges to the exact solution when \(\psi = 1\).}
structure of Kreps and Porteus (1978), investors can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertemporal utility function implies that investors are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between $\psi$ and $\gamma$: if $\gamma > \frac{1}{\psi}$ ($<, =$) investors have preference for early (late, indifferent) resolution of risk. Our specification (7) from Duffie and Epstein (1992a), being the continuous-time limit of Epstein and Zin (1989), inherits this property. However, contrarily to other streams of literature with Epstein-Zin preferences, for example, the “long-run risk” literature (from the seminal work of Bansal and Yaron (2004)), we do not restrict the investor to have preference for early resolution of risk.$^7$

### 2.3 Dynamic Optimization Problem

Ambiguity about the investment opportunity set is studied with robust control (RC) techniques, firstly introduced in economics by Hansen and Sargent (1995). The representative investor has a reference model, but, facing ambiguity about the true model, considers a family of alternative models that are statistically difficult to distinguish from his benchmark.

Under the RC approach, two main formulations have been used in the ambiguity related literature: the “constraint preferences” and the “multiplier preferences”. Under “constraint preferences” (e.g., in Hansen et al. (2006)), there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Under “multiplier preferences” (e.g., in Maaenhout (2006)), preferences for robustness are constructed by penalizing deviations from the benchmark model, with higher deviations being more penalized than smaller ones. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different.

One advantage of the “constraint preferences” RC approach is that the specification of ambiguity aversion can be based on a rectangular set of priors, which guarantees a dynamically consistent preference ordering. In those cases, preferences can be represented by the recursive multiple priors utility (RMPU) specification (Chen and Epstein (2002) and Epstein and Schneider (2003)).$^8$

Additionally, the “constraint preferences” approach enables ambiguity to expand the range of qualitative behavior that can be rationalized versus the standard expected utility theory (as pointed out by Epstein and Schneider (2010)). This contrasts with the “multiplier preferences” approach, which is observationally equivalent to expected utility theory: it enables reinterpretations of some results obtained under the expected utility theory, that can be quantitatively more appealing,$^9$ but does not enlarge the

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$^7$We avoid this restriction because: (i) there is empirical evidence suggesting that investors have preference for late resolution of risk (Epstein and Zin (1991)); and (ii) the psychological cost implied by early resolution of risk increases with the planning horizon (Arai, 1997), making it more likely that this cost exceeds the planning advantage brought by the early resolution of risk if agents are long-lived (as in our model).

$^8$Rectangularity is the property that allows updating every prior under the recursive multiple priors utility through a Bayes rule. See Epstein and Schneider (2003) for details about this property. In Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.

$^9$For example, as ambiguity aversion translates into a higher effective risk aversion, contributes to the explanation of the equity premium puzzle.
spectrum of qualitative behavior that can be rationalized. In the “multiplier preferences” RC approach, ambiguity aversion is in practice translated into an enhanced level of risk aversion (as concluded for optimal dynamic portfolio choice in Maenhout (2004, 2006), Faria and Correia-da-Silva (2010) and Xu et al. (2011)).

In this paper, we adopt a “constraint preferences” RC approach as in Faria and Correia-da-Silva (2012). The investor considers contaminations (alternative models), $P^h$, around his reference belief, $P$, under which processes (2)-(3) evolve. The contaminations are assumed to be absolutely continuous with respect to $P$, and, therefore, are equivalently described by contaminating drift processes, $h = [h^y h^\varepsilon]'$, that contaminate the vector of Brownian motions, $W = [W^y W^\varepsilon]'$, associated with the stochastic processes that generate the risky asset’s return and volatility. In an alternative model, $P^h$, the Brownian motion is $W^h(t) = W(t) + \int_0^t h(s) \, ds$.\(^{10}\)

An upper bound is imposed on contaminating drift processes:

$$h' h \leq \vartheta^2, \tag{9}$$

where $\vartheta \geq 0$ is a parameter that can be interpreted as the level of ambiguity. The class of admissible Markovian drift contaminations satisfying this entropy bound (9) is denoted by $\mathcal{H}$.

Alternative models should be statistically close to the reference model. Otherwise, the agent would be able to distinguish them and, consequently, would not face ambiguity. This means that $\vartheta$ must be small. Moreover, the bound (9) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, $P$, and any admissible contaminated belief, $P^h$. Trojani and Vanini (2004) explain that the set \( \{ h : h' h \in [0, \vartheta^2] , \forall t \geq 0 \} \) defines a rectangular set of priors because any process $h$ (and therefore any probability measure $P^h$) in this set corresponds to a selection of transition densities from $t$ to $t + dt$, $t \geq 0$, such that $h' h \in [0, \vartheta^2]$.\(^{11}\)

Under an admissible contamination, $P^h$, the investment opportunity set is described by:

$$\begin{align*}
\frac{dS_t}{S_t} &= \left( \mu + \sqrt{\frac{1}{2t}} \rho h^y + \sqrt{\frac{1}{2t}} \sqrt{1 - \rho^2} h^\varepsilon \right) dt + \sqrt{\frac{1}{2t}} \rho dW^y + \sqrt{\frac{1}{2t}} \sqrt{1 - \rho^2} dW^\varepsilon \\
\frac{dy_t}{y_t} &= \left[ \kappa (\theta - y_t) + \sigma \sqrt{\frac{1}{2t}} h^y \right] dt + \sigma \sqrt{\frac{1}{2t}} dW^y 
\end{align*} \tag{10}$$

Note that in the “contaminated” investment opportunity set (10), the diffusion component continues to be driven by a pair of independent Brownian motions as in (2)-(3).

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\(^{10}\)For tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between $P$ and $P^h$ guarantees the equivalence property between the probability measures and, consequently, the Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

\(^{11}\)In Trojani and Vanini (2004), p.289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (9), and on how this rectangular set of priors can be defined in the $k$-ignorance model of Chen and Epstein (2002).
With $C_t$, $X_t$ and $\pi_t$ representing the instantaneous consumption, wealth and fraction of wealth invested in the risky asset, respectively, the wealth dynamics is given by:

$$dX_t = \pi_t X_t \frac{dS_t}{S_t} + (1 - \pi_t) X_t r dt - C_t dt.$$ 

Considering the dynamics in (10), the intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$dX_t = \left[ \pi_t \left( \mu + \sqrt{\frac{1}{y_t} \rho h^y} + \sqrt{\frac{1}{y_t} \sqrt{1 - \rho^2 h^\varepsilon}} - r \right) X_t + r X_t - C_t \right] dt + \pi_t X_t \sqrt{\frac{1}{y_t}} \left( \rho dW_y + \sqrt{1 - \rho^2} dW_\varepsilon \right).$$

(11)

The intertemporal optimization problem has a max-min structure. The investor chooses the consumption flow, $C : [t_0, +\infty] \rightarrow \mathbb{R}^+$, and the fraction of wealth to invest in the risky asset in each moment, $\pi : [t_0, +\infty] \rightarrow \mathbb{R}$, that maximize his expected utility (7). For each given choice, the ambiguity averse investor considers, from the set of alternative models, the worst-case scenario, i.e., the model that yields the lowest expected utility:

$$\sup_{(\pi, C)} \inf_{(h, \varepsilon)} \int_{t_0}^{\infty} f(C_s, J_s) ds,$$

subject to the contaminated precision and wealth processes in (10) and (11), respectively.

The Bellman equation of this problem is:

$$0 = \sup_{(\pi, C)} \inf_{(h, \varepsilon)} \left\{ f(C, J) + \pi_t \left( \mu + \sqrt{\frac{1}{y_t} \rho h^y} + \sqrt{\frac{1}{y_t} \sqrt{1 - \rho^2 h^\varepsilon}} - r \right) X_t + r X_t - C_t \right] J_X + \\
+ \left[ \kappa (\theta - y_t) + \sigma \sqrt{\frac{1}{y_t}} h^y \right] J_y + \frac{1}{2} \frac{\sigma^2}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{XY} \right\},$$

(13)

where $f(C, J)$ is the normalized aggregator given in (8) and $J_X$, $J_y$, $J_{XX}$, $J_{yy}$ and $J_{XY}$ are partial derivatives of the value function $J(X_t, y_t)$.

Solving for the optimal vector $[h^y, h^\varepsilon]'$, i.e., for the worst-case contamination, and placing the result into (13), the Bellman equation of the problem becomes (Appendix 7.1):

$$0 = \sup_{(\pi, C)} \left\{ f(C, J) + \pi_t (\mu - r) X_t J_X + r X_t J_X - C_t J_X + \kappa (\theta - y_t) J_y + \frac{1}{2} \sigma^2 y_t \frac{1}{y_t} X_t^2 J_{XX} \\
+ \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{XY} - \theta \left[ \sigma^2 y_t J_y^2 + 2 \sigma \rho \pi_t X_t J_y J_X + \pi_t^2 \frac{1}{y_t} X_t^2 J_X^2 \right] \right\}.$$

(14)
3 Optimal Consumption and Portfolio Rules

In general, obtaining closed-form solutions for stochastic investment opportunity sets is difficult. This difficulty is further enhanced by the presence of ambiguity. In this paper, we follow perturbation theory under robust control (e.g., Trojani and Vanini (2002)) to describe the solution of the problem under study. As in Trojani and Vanini (2004), we extend the asymptotic methods in Kogan and Uppal (2001) from models based on standard expected utility to models with ambiguity. This is allowed by the homotheticity of the robust control problem (10)-(12), which implies that the value function that solves the problem and the corresponding optimal consumption and portfolio policies are wealth scale-invariant.\textsuperscript{12}

The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): “[... ] formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems”. In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold in neighborhoods of the model with log utility investor without ambiguity.

The first step is to identify a set of parameters that parametrize the problem under study and specific parameter values for which the solution of the value function is known explicitly. Chacko and Viceira (2005) provided an exact solution for the case in which $\vartheta = 0$ (no ambiguity).

The value function that solves (14) for $\vartheta = 0$ is given by:

$$J (x_t, y_t) = \left[ e^{\psi (v) x_t} \right]^{1-\gamma},$$

where $g (y_t) = Ay_t + B$, with $A$ and $B$ given by (Appendix 7.2):

$$A = \frac{\gamma (\beta + \kappa) - (1 - \gamma) \rho \sigma (\mu - r) - \sqrt{[\gamma (\beta + \kappa) - (1 - \gamma) \rho \sigma (\mu - r)]^2 - (1 - \gamma) \sigma^2 (\mu - r)^2 [\gamma (1 - \rho^2) + \rho^2]}}{(1 - \gamma) \sigma^2 [\gamma (1 - \rho^2) + \rho^2]},$$

$$B = \ln \beta + \frac{r}{\beta} - 1 + \frac{\kappa \psi}{\beta} A.$$

Following the rationale above described, we will perturb a benchmark economy with a log-utility investor with no ambiguity, $(\vartheta = 0, \gamma = 1)$, as the solution for that problem is exactly known.\textsuperscript{14} To obtain the asymptotic expansions of the optimal policies of the problem under study, we consider the

\textsuperscript{12}As explained in Trojani and Vanini (2002), studying non-homothetic robust control settings with perturbation methods is more difficult. Moreover, Maenhout (2004) points out some reasons to support the homotheticity assumption: “Although economies exhibit growth, rates of return are stationary. Second, when the scale of the state variables matters, natural unit invariance of optimal decisions disappears and calibrations have to take this into account. Finally, homotheticity facilitates aggregation and the construction of a representative agent.” As stated by Maenhout (2004), preserving homotheticity guarantees that “[...] robustness will no longer wear off as wealth rises.”

\textsuperscript{13}This expression is valid for $\gamma \neq 1$. If $\gamma = 1$, the value function is of the form $J = \frac{1}{2} \ln (x_t) + g (y_t)$.

\textsuperscript{14}The solution for our problem without ambiguity, with $\psi = 1$ and any level of risk aversion, was obtained by Chacko and Viceira (2005).
value function (15) and the first-order expansion of \( g(y_t) \) around \((\vartheta = 0, \gamma = 1)\):\(^{15}\)

\[
g(y_t) = g_0(y_t) + g_1(y_t) \vartheta + g_2(y_t) (\gamma - 1) + O_g^2 (\vartheta, 1 - \gamma),
\]

where \( O_g^2 (\vartheta, 1 - \gamma) \) represents the residual of the first-order expansion. As it is immediate from (18), \( g_0(y_t) \) is the specification of \( g(y_t) \) that solves the problem when \( \vartheta = 0 \) and \( \gamma = 1 \), i.e., for a log-utility investor and no ambiguity.

**Proposition 1** Optimal consumption and optimal asymptotic portfolio policies under ambiguity about the investment opportunity set dynamics (2)-(3), when \( \gamma \geq \omega \), where \( \omega = 1 - \frac{(\beta + \kappa)^2}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r) (\beta + \kappa)} \), are given by:\(^{16}\)

\[
C_t = \beta X_t, \\
\pi_t = \frac{1}{\gamma + \frac{\vartheta}{\Phi \sqrt{y_t}}} \left[ (\mu - r) y_t + \left( 1 - \gamma - \frac{\vartheta}{\Phi \sqrt{y_t}} \right) \frac{\sigma \rho (\mu - r)^2}{2 (\beta + \kappa) y_t} \right] + O_{\pi}^2 (\vartheta, 1 - \gamma),
\]

with \( \Phi = \left[ (\mu - r)^2 + \frac{\sigma \rho (\mu - r)^3}{(\beta + \kappa)^2} + \frac{\sigma^2 (\mu - r)^4}{4 (\beta + \kappa)^2} \right]^{\frac{1}{2}} \).

**Proof.** See Appendix 7.3.

The first comment on Proposition 1 is that the domain in which the solution is valid depends on the combination of the level of investor’s risk aversion and on the characterization of the investment opportunity set dynamics (represented by \( \omega \)). Note also that regarding the investor’s preferences for the temporal resolution of risk, the domain of analysis (\( \gamma \geq \omega \)) includes scenarios where the investor: has preference for late resolution of risk (\( \omega \leq \gamma < 1 \)); has preference for early resolution of risk (\( \gamma > 1 \)); or is indifferent to that timing (\( \gamma = 1 \)). Only scenarios where the investor has a strong preference for late resolution of risk (\( \gamma < \omega \)) are excluded.

For the log-utility investor with no ambiguity (\( \vartheta = 0, \gamma = 1 \)), the optimal consumption and portfolio rules are given by:

\[
C_t = \beta X_t, \\
\pi_t = (\mu - r) y_t.
\]

This optimal portfolio policy (22) is well known in the literature (e.g. Merton (1969, 1971, 1973)) and

---

\(^{15}\)Considering a first-order expansion only around \( \vartheta = 0 \), while allowing any level of risk-aversion, implies problems of analytical tractability. A closed-form solution can be obtained for the particular case of the investment opportunity set of Heston (1993). However, it involves the Tricomi confluent hypergeometric function and the Laguerre polynomials, which are rather abstruse mathematical expressions, from which it is difficult to obtain any useful economic insights.

\(^{16}\)For \( \gamma < \omega \), the constant \( A \) in (16) is a complex number. Therefore, the value function (15) is only valid if \( \gamma \geq \omega \).
Comparing (19) and (21), it is clear that ambiguity has no effect on the optimal consumption rule (which is to consume a constant fraction \( \beta \) of current wealth). This means that the income and substitution effects on consumption that result from the change in the investment opportunity set exactly cancel out.

On the other hand, from (20) and (22), it is immediate to conclude that ambiguity has a first-order impact on portfolio choice. There are novelties relatively to the existing results in the literature. First, the optimal allocation to the risky asset is instantaneously impacted by ambiguity, which contrasts with results in Faria et al. (2009). Secondly, the optimal portfolio rule is a non-linear function of \( y_t \), which differs from the linear relationship that holds when there is no ambiguity (e.g., Chacko and Viceira (2005)) or when ambiguity is studied using RC with “multiplier preferences” (e.g., Maenhout (2004, 2006), Faria and Correia-da-Silva (2010) and Xu et al. (2011)).

The structure of the optimal portfolio rule under ambiguity (20) continues to be the sum of two well-known components (Merton (1973)): (i) myopic demand, given by \( \pi_{t}^M(0) \gamma \left( \frac{\mu - \sigma}{\gamma + \frac{\sigma}{\sqrt{y_t}}} \right) \), and (ii) intertemporal hedging demand, given by \( \pi_{t}^H(0) \gamma \left( \frac{\rho}{\gamma + \frac{\sigma}{\sqrt{y_t}}} \right) \). Observe that as the coefficient of relative risk aversion tends to 1 (\( \gamma \to 1 \)), in (22), investment opportunities are constant (\( \sigma = 0 \)) or, being time-varying, it is not possible to use the risky asset to hedge against those changes (\( \rho = 0 \)), the intertemporal hedging demand vanishes (and, therefore, the myopic demand becomes optimal). Notice also that the ratio between myopic and intertemporal hedging demand is a function of instantaneous precision (\( y_t \)), contrarily to what happens when there is no ambiguity (\( \vartheta = 0 \)).

From (20), it also results that, without ambiguity, an investor with \( \gamma > 1 \) has a negative intertemporal hedging demand, and the opposite when \( \omega \leq \gamma < 1 \), which is consistent with the findings in Chacko and Viceira (2005). When risk aversion is low (\( \omega \leq \gamma < 1 \)), the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that \( \rho \geq 0 \)). An investor with high risk aversion (\( \gamma > 1 \)) is not willing to accept this trade-off. With the introduction of ambiguity, this relation is not so trivial: investors with low risk aversion (\( \gamma < 1 \)) that face a sufficiently high level of ambiguity have a negative intertemporal hedging demand.\(^{17}\)

Note that the myopic (M) and the intertemporal hedging (H) demand can be written as:

\[
\pi_{t}^M(\vartheta) = \pi_{t}^M(0) \frac{\gamma}{\gamma + \frac{\sigma}{\sqrt{y_t}}} , \tag{23}
\]

\[
\pi_{t}^H(\vartheta) = \pi_{t}^H(0) \frac{\gamma - \frac{\rho}{\gamma + \frac{\sigma}{\sqrt{y_t}}}}{\frac{\sigma}{\sqrt{y_t}}} , \tag{24}
\]

where \( \pi_{t}^M(0) \) and \( \pi_{t}^H(0) \) represent the myopic and intertemporal hedging demand components without ambiguity aversion. From (23) and (24), it results that ambiguity aversion reduces the demand for the

\(^{17}\)For this to happen when \( \omega \leq \gamma < 1 \), it is necessary that \( \gamma > 1 - \frac{\vartheta}{\sqrt{y_t}} \).
risky asset with this reduction being greater if precision is low. The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a model where stochastic precision is one of the sources of ambiguity, in a “constraint preferences” RC setting.

4 Alternative Scenarios

In section 4.1, optimal consumption and portfolio rules are derived for the case in which ambiguity is exclusively about the stochastic process that generates the return of the risky asset. In section 4.2, the same is done in a scenario where ambiguity is only about the precision process.

4.1 Ambiguity exclusively about the return process

Consider the investment opportunity set described in section 2.1, and the existence of ambiguity exclusively about the return process. Formally, restrict the possible perturbations to be of the form

\[ h^\varepsilon = [0 \ h^\varepsilon]' \]

From (13), the corresponding Bellman equation is given by:

\[
0 = \sup_{(\pi,C)} \inf_{h^\varepsilon} \left\{ f(C,J) + \left[ \pi_t \left( \mu + \frac{1}{\sqrt{y_t}} \sqrt{1 - \rho^2 \varepsilon} - r \right) X_t + r X_t - C_t \right] J_X + \right. \\
\left. + \kappa (\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}.
\]

(25)

Observe that the objective function is monotonically increasing in \( h^\varepsilon \). This implies, from (9), that the worst-case contamination is the corner solution \( h^\varepsilon = -\vartheta \). Introducing this result into (25), the Bellman equation becomes:

\[
0 = \sup_{(\pi,C)} \left\{ f(C,J) + \left[ \pi_t \left( \mu - \frac{1}{\sqrt{y_t}} \sqrt{1 - \rho^2 \vartheta} - r \right) X_t + r X_t - C_t \right] J_X + \right. \\
\left. + \kappa (\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\}.
\]

(26)

Following the same reasoning as in section 3, the optimal portfolio and consumption rules are deducted for the particular case of ambiguity exclusively about the return process.

**Proposition 2** Optimal consumption and optimal asymptotic portfolio policies under ambiguity about the risky asset return process (2), when \( \gamma \geq \omega \), where \( \omega = 1 - \frac{(\beta + \kappa)^2}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r)(\beta + \kappa)} \), are:

\[
C_t = \beta X_t,
\]

(27)

\[
\pi_t = \frac{1}{\gamma} (\mu - r) y_t - \frac{\sqrt{1 - \rho^2 \vartheta}}{\gamma} \sqrt{y_t} + \frac{\sigma \rho (1 - \gamma) (\mu - r)^2}{2 \gamma (\beta + \kappa)} y_t + O_\pi (\vartheta, 1 - \gamma).
\]

(28)
Proof. See Appendix 7.3.

The main conclusions from Proposition 1 extend to Proposition 2. The difference is that now the effect of ambiguity on portfolio choice only concerns the myopic demand, as the intertemporal hedging demand remains equal to \( \frac{\sigma \rho (1-\gamma)(\mu-r)}{2\gamma(1+\gamma)} y_t \). This is natural because the intertemporal hedging demand is driven by the dynamics of stochastic variance. It is, therefore, unaffected by ambiguity about the return process.

4.2 Ambiguity exclusively about the precision process

To investigate the case in which ambiguity is only about the precision process, it is not appropriate to consider the investment opportunity set described by (2)-(3). In that case, a contamination of the precision process would be transmitted to the return process through the assumed correlation between return and precision. The appropriate setting is the following:

\[
\frac{dS}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_S, \tag{29}
\]
\[
dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t} \left( \rho dW_S + \sqrt{1-\rho^2} dW_\epsilon \right). \tag{30}
\]

Ambiguity is again introduced through Markovian contaminating drift processes. If we allowed any contamination \( h = [h^s \ h^\epsilon]^\prime \) satisfying the entropy bound (9), we would obtain exactly the same Bellman equation as in section 2.3 (Appendix 7.4). Therefore, as the Bellman equation would still be given by (14), Proposition 1 would remain valid in this alternative setting. Since now we want to study the case in which ambiguity is exclusively about the precision process, we only contaminate the precision process, i.e., we consider \( h = [0 \ h^\epsilon]^\prime \). The investment opportunity set is now described by:

\[
\begin{align*}
\frac{dS}{S_t} &= \mu dt + \sqrt{\frac{1}{y_t}} dW_S, \\
dy_t &= \left[ \kappa (\theta - y_t) + \sigma \sqrt{y_t} (1 - \rho^2) h^\epsilon \right] dt + \sigma \sqrt{y_t} \left( \rho dW_S + \sqrt{1-\rho^2} dW_\epsilon \right),
\end{align*}
\tag{31}
\]

and the corresponding intertemporal budget constraint faced by the ambiguous representative investor is given by:

\[
dX_t = [\pi_t (\mu - r) X_t + r X_t - C_t] dt + \pi_t \sqrt{\frac{1}{y_t}} X_t dW_S. \tag{32}
\]

The Bellman equation of the optimization problem in this setting is:

\[
0 = \sup_{(r, C)} \inf_{h^\epsilon} \left\{ f(C, J) + \left[ \pi_t (\mu - r) X_t + r X_t - C_t \right] J_{XX} + \frac{1}{2} \sigma^2 \sqrt{y_t} J_{y} + \frac{1}{2} \sigma^2 \sqrt{y_t} J_{yy} + \frac{1}{2} \sigma^2 \sqrt{y_t} \right\}.
\tag{33}
\]
With $J_y \geq 0$, which holds at least for low ambiguity levels, the worst-case contamination is the corner solution $h^c = -\vartheta$. The optimal consumption and portfolio rules are as follows.

**Proposition 3** Optimal consumption and optimal asymptotic portfolio policies under ambiguity about the precision process (30), when $\gamma \geq \omega$, where $\omega = 1 - \frac{(\beta + \kappa)^2}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r) (\beta + \kappa)}$, are given by:

\[
C_t = \beta X_t, \quad (34) \\
\pi_t = \frac{1}{\gamma} (\mu - r) y_t + \frac{\sigma \rho (1 - \gamma) (\mu - r)^2}{2 \gamma (\beta + \kappa)} y_t + \mathcal{O}_\pi^2 (\vartheta, 1 - \gamma). \quad (35)
\]

The main conclusion from Proposition 3 is that ambiguity about the precision process has no impact on optimal consumption choice and no first-order impact on optimal portfolio choice.

Overall, optimal portfolio rules were deducted for the scenarios where there exists ambiguity: about the dynamics of both the return and its precision (Proposition 1), only about the dynamics of the return (Proposition 2); and only about the dynamics of precision (Proposition 3). The conclusion is that, from a theoretical point of view, the relevant channel through which ambiguity impacts portfolio decisions is the return of the risky asset. Ambiguity about the precision process is only relevant if, through the assumed correlation between return shocks and precision shocks, it also induces ambiguity about the return process: this happens under the setting (2)-(3), but not under the setting (29)-(30). In the next section we evaluate the empirical relevance of these findings.

5 Simulation

Chacko and Viceira (2005) found that, estimating their model with long-run U.S. data, the optimal intertemporal hedging demand is empirically small. The same conclusion was reached by Liu (2007). This suggests that variance risk is empirically not very relevant to dynamic portfolio choice. However, Chacko and Viceira (2005), in their concluding remarks, acknowledged that an important caveat of their analysis is that they have assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have generalized their model to account for ambiguity about the stochastic investment opportunity set. As a result, the “myopic demand” and the “intertemporal hedging demand” became ambiguity-adjusted.

Our simulation suggests that the impact of ambiguity on the allocation to the risky asset is empirically relevant. However, this effect is essentially due to ambiguity about the return process. The impact of ambiguity about the variance process is empirically very low.

The reference parameter values used in the simulation are those estimated by Chacko and Viceira
(2005), based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill rate from January 1926 through December 2000:

\[
\mu - r = 0.0811, \\
\kappa = 0.3374, \\
\theta = 27.9345, \\
\sigma = 0.6503, \\
\rho = 0.5241, \\
r = 0.015, \\
\beta = 0.06.
\]

From (5), the expected standard deviation of returns is 19.1314%.

The effect of ambiguity on the optimal allocation to the risky asset (Proposition 1) is illustrated in Table 1. The first column presents results for the scenario without ambiguity. The other three columns represent scenarios with increasing levels of ambiguity: \( \vartheta = 0.05, 0.10, 0.20 \). Recall that alternative models have to be statistically close so that the investor is ambiguous about the reference model. This implies small values of \( \vartheta \). In Trojani and Vanini (2004), two values for \( \vartheta \) are used (0.10, 0.14) while the value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.17.\(^{18}\)

Simulations are run for different levels of risk aversion (\( \gamma = 0.75, 1, 1.5, 2 \)),\(^{19}\) assuming unit elasticity of intertemporal substitution (\( \psi = 1 \)). In panel A, we show the mean allocation to the risky asset (percentage of wealth). In panel B, the intertemporal hedging demand is shown as a percentage of the myopic demand. In panels C and D, the ambiguity effect is explicitly calculated as a percentage of total risky asset demand and myopic demand.

---

\(^{18}\) Those values for \( \vartheta \) can be taken as a reference without introducing any kind of bias in our analysis. This is because, as explained in section 2.3, the diffusion dimension of “contaminated” processes is unchanged vs. non-contaminated processes (only the drift functions are affected).

\(^{19}\) We run simulations with levels of risk aversion around \( \gamma = 1 \) because, as explained in section 3, we deduct asymptotically the optimal portfolio policy by perturbing a model with no ambiguity and log-utility (\( \gamma = 1 \)).
Table 1: Ambiguity impact on optimal risky asset demand (Proposition 1).

<table>
<thead>
<tr>
<th>( \vartheta )</th>
<th>0</th>
<th>0.05</th>
<th>0.10</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>304.69</td>
<td>263.90</td>
<td>232.53</td>
<td>187.44</td>
</tr>
<tr>
<td>1.00</td>
<td>226.55</td>
<td>202.83</td>
<td>183.47</td>
<td>153.77</td>
</tr>
<tr>
<td>1.50</td>
<td>148.41</td>
<td>137.50</td>
<td>128.01</td>
<td>112.33</td>
</tr>
<tr>
<td>2.00</td>
<td>109.34</td>
<td>103.09</td>
<td>97.48</td>
<td>87.80</td>
</tr>
</tbody>
</table>

Panel A - Mean allocation to risky asset (%)

Panel B - Ratio of hedging demand over myopic demand (%)

Panel C - Variation of the allocation to risky asset due to ambiguity (%)

Panel D - Variation of myopic demand due to ambiguity (%)

Note 1 - \( \omega = 0.14 < 0.75 \), which means that we are in the domain of validity of Proposition 1.

The results presented in Table 1 are consistent with the comments in section 3 and show that ambiguity is empirically relevant: even for a very low level of ambiguity (second column in Table 1), ambiguity implies a decrease of the mean optimal demand of the risky asset that ranges between 13.4% (\( \gamma = 0.75 \)) and 5.7% (\( \gamma = 2 \)) (panel C). Consider, for example, a risk-averse investor, with \( \gamma = 1.5 \), who is ambiguity-neutral. His mean optimal allocation to the risky asset corresponds to 148.4% of his wealth. If this investor becomes ambiguity averse, for example, with \( \vartheta = 0.1 \), his mean optimal allocation to the risky asset declines to 128% of his wealth. These findings can contribute to the explanation of the so-called “flight to quality” effect (stylized fact in financial markets): when investors, for some reason, become more “nervous” and uncertain about market conditions, they reduce exposure to risky assets and invest in less risky or riskless assets.

Panel A in Table 1 shows that the demand for the risky asset is decreasing with risk aversion, \( \gamma \), and with the level of ambiguity, \( \vartheta \). Since the long-term expected return on wealth is \( \pi_\vartheta (\mu - r) + r \), it is a
decreasing function of both risk aversion and the level of ambiguity.

Figure 1: Long-term expected return on wealth as a function of ambiguity and risk aversion.

![Figure 1: Long-term expected return on wealth as a function of ambiguity and risk aversion.](image)

Moreover, the higher the level of ambiguity and the level of instantaneous volatility (inverse of precision), the higher is the impact of ambiguity. This is represented in Figure 2, where it is also evident that the effect of ambiguity is a non-linear function of the volatility level.

Figure 2: Impact of ambiguity as a function of instantaneous volatility

![Figure 2: Impact of ambiguity as a function of instantaneous volatility](image)

These findings suggest that the intensity, i.e., the speed and depth, of the asset reallocation implied in the “flight to quality” phenomena should increase with the level of ambiguity and of instantaneous volatility. This is intuitive: in an anxious market environment as the one following the collapse of
Lehman Brothers in September 2008 (which Blanchard (2009) suggestively named as “Knight time”), during which the VIX index\textsuperscript{20} reached its historical maximum of 80.86\% (November 20th, 2008), the speed and volumes of risky asset “sell-off” trades were much higher than in stable market conditions.

Additionally, as pointed out in section 3, the ratio between intertemporal hedging demand and myopic demand becomes a function of precision when ambiguity is considered. This has an intuitive interpretation: the higher the level of instantaneous volatility, everything else constant, the more the investor is concerned about the impact from volatility changes in his intertemporal utility and, being ambiguous about the process that drives volatility, the higher is the optimal hedging demand. This is graphically highlighted in Figure 3, where it is also clear that the dimension of this adjustment is empirically small.

Figure 3: Hedging demand versus myopic demand as a function of instantaneous volatility

Panel B in Table 1 reports estimates of the intertemporal hedging demand, as a percentage of myopic demand. Again, results show that ambiguity reinforces the effect of risk aversion: the higher the ambiguity and risk aversion, the higher the relative importance of the intertemporal hedging demand. However, the intertemporal hedging demand is always small. The novelty with ambiguity is that for investors with low risk-aversion ($\gamma = 0.75$) that face a sufficiently high level of ambiguity (e.g., $\vartheta = 0.2$), the intertemporal hedging demand becomes negative. This confirms the predictions highlighted in section 3. With no ambiguity or moderate levels of ambiguity, the intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma \geq 1$.

The fact that intertemporal hedging demand is empirically small, even for higher levels of ambiguity, means that ambiguity impacts optimal portfolio decisions essentially through the myopic component of demand. This is confirmed by the results disclosed in Panels C and D in Table 1. This has a clear economic

\textsuperscript{20}The VIX Index from CBOE is probably the most used volatility index, both in the literature and by practitioners. It measures the one-month implied volatility in the S&P 500 Index option prices. For full details on the VIX Index construction methodology please see http://www.cboe.com/micro/vix/.
meaning and provides an answer to the research question addressed in this paper: even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in the short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for optimal intertemporal portfolio decisions: this has been found under settings where uncertainty is exclusively risk (for example in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion to a setting where uncertainty also has an ambiguity dimension.

Moreover, recalling conclusions from section 4.2, this setting (Table 1) is the only one where ambiguity about stochastic volatility process impacts optimal portfolio decisions.

Panels C and D in Table 1 also show that the impact from ambiguity decreases with the level of risk aversion. The economic interpretation may be that as investors with low risk-aversion are highly exposed to the risky asset (Panel A in Table 1), the relative impact of becoming ambiguous is higher, for low risk averse investors, triggering more aggressive reductions in the optimal allocation to the risky asset.

Finally, as a “cross-check” test, the scenario with ambiguity exclusively about the risky asset return process (Proposition 2) was simulated, confirming its empirical relevance.\(^{21}\)

Table 2: Impact of ambiguity on the demand for the risky asset (Proposition 2).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>0</th>
<th>0.05</th>
<th>0.10</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>A - Mean allocation to risky asset (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>384.69</td>
<td>274.68</td>
<td>244.67</td>
<td>184.66</td>
</tr>
<tr>
<td>1.00</td>
<td>226.56</td>
<td>204.04</td>
<td>181.54</td>
<td>136.52</td>
</tr>
<tr>
<td>1.50</td>
<td>148.41</td>
<td>133.40</td>
<td>118.40</td>
<td>88.39</td>
</tr>
<tr>
<td>2.00</td>
<td>108.34</td>
<td>98.08</td>
<td>86.83</td>
<td>64.32</td>
</tr>
<tr>
<td>B - Variation of the allocation to risky asset due to ambiguity (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.00</td>
<td>-9.85</td>
<td>-19.70</td>
<td>-39.40</td>
</tr>
<tr>
<td>1.00</td>
<td>0.00</td>
<td>-9.93</td>
<td>-19.87</td>
<td>-39.74</td>
</tr>
<tr>
<td>1.50</td>
<td>0.00</td>
<td>-10.11</td>
<td>-20.22</td>
<td>-40.44</td>
</tr>
<tr>
<td>2.00</td>
<td>0.00</td>
<td>-10.29</td>
<td>-20.58</td>
<td>-41.17</td>
</tr>
</tbody>
</table>

Note 1 - Panel A: \( \pi(\theta) = \pi(y_t = \theta) \times 100 \); Panel B: \( \left[ \frac{\pi(y_t = \theta, \vartheta) - \pi(y_t = \theta, \vartheta = 0)}{\pi(y_t = \theta, \vartheta = 0)} \right] \times 100 \).

Note 2 - \( \omega = 0.14 < 0.75 \), which means that we are in the domain of validity of Proposition 2.

The comparison of results in Panel B of Table 2 with those in Panel C of Table 1 leads to the conclusion that when ambiguity is solely about the return process (Table 2), its impact on the optimal demand for the risky asset is lower for low risk averse investors (\( \gamma = 0.75 \)) and higher for high risk averse investors (\( \gamma = 1.5 \), 2) and, for a given level of ambiguity, is not sensitive to the level of risk aversion.

Our conclusion that uncertainty about stochastic volatility has low impact on portfolio decisions may

\(^{21}\)Results for the scenario with ambiguity exclusively about the precision process (Proposition 3) coincide with column for \( \vartheta = 0 \) in Tables 1 and 2 because there is no first-order impact of ambiguity on optimal portfolio policy.
not be robust to the introduction of multiple risky assets (multivariate stochastic variance and covariance setting). In such a setting, Buraschi et al. (2010) showed that joint features of volatility and correlation dynamics play an important role in optimal portfolios. For example, they estimate that, in a univariate stochastic volatility model, total hedging demand for S&P 500 futures of investors with $\gamma = 8$ and investment horizon of 10 years is 4.8% of the myopic demand. This is consistent with our results in Table 1. However, in a model with three risky assets, the estimated total hedging demand for S&P 500 futures jumps to 28% of myopic demand, with 11% and 17% of volatility and correlation hedging demand respectively.

Interestingly, Buraschi et al. (2010) also find that the optimal hedging demand against correlation risk typically dominates hedging against volatility risk. In a study about the relation between correlation risk and the cross-section of hedge fund returns, Buraschi et al. (2014) find evidence that correlation risk is the most significant risk factor for the explanation of hedge fund returns. Those findings suggest that correlation, more than volatility, is the most relevant uncertainty factor driving investors’ intertemporal hedging demand.

6 Concluding Remarks

We study optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the stochastic processes that generate the dynamic investment opportunity set.

Long-horizon investors with recursive preferences, as defined by Duffie and Epstein (1992b) with Kreps and Porteus’ (1978) specification, have two assets to invest in: a risk-free asset and a risky asset. The investor considers a reference model for the data generating processes but, not being totally sure about it, takes into account a set of statistically close models (with the relative entropy between models being bounded). Ambiguity aversion in the spirit of Gilboa and Schmeidler (1989) implies that investor will consider the worst possible alternative model, i.e., the one associated with the lowest expected utility. Optimal dynamic policies under ambiguity are deducted by making use of perturbation theory techniques for robust control problems.

The main conclusions of this paper concern the impact on optimal dynamic policies from ambiguity about the data generating processes, both when ambiguity is simultaneously about the return and volatility processes and when it is exclusively about one of them. In all scenarios, it is found that ambiguity does not impact the optimal consumption-wealth ratio: the optimal consumption policy is to consume a constant fraction of wealth.

However, ambiguity about the data generating processes reduces the optimal demand for the risky asset, with that effect being non-linear with respect to the variance level. The same happens when there is ambiguity only about the return of the risky asset. When ambiguity is exclusively about the stochastic variance process, it is found that there is no impact on the optimal portfolio rule. The conclusion is that
ambiguity about the stochastic variance process is only relevant as long as, through a specific correlation structure, it also induces ambiguity about the return stochastic process.

Making use of long-run US data, we measure the empirical dimensions of those effects. The first conclusion is that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. This can be a contribute for the explanation of the “flight to quality” stylized fact in financial markets. Our simulation suggests that this highly relevant ambiguity effect on the risky asset demand acts mainly through the myopic component. The first implication is the confirmation that, under our setting and simulation, ambiguity about the risky asset return process is empirically much more relevant than ambiguity about stochastic volatility process. The second implication is that, even accounting for ambiguity about the stochastic volatility process, it is found that the optimal hedging demand required by investors for protection against adverse changes in volatility is still very low. Investors are essentially focused in short term risk-return characteristics of the risky asset (myopic dimension), and stochastic volatility has low relevance for intertemporal portfolio decisions: this has been found under settings were uncertainty is exclusively risk (e.g., in Chacko and Viceira (2005) and Liu (2007)) and we extend that conclusion for a setting where uncertainty also has an ambiguity dimension.

Our conclusion that uncertainty about stochastic volatility has low impact on portfolio decisions may change significantly if a multivariate stochastic variance setting (multiple risky assets) is considered, as Buraschi et al. (2010) shows: the authors find that hedging demands are typically four to five times higher than those of models with constant correlations or single-factor stochastic volatility. Moreover, Buraschi et al. (2010) also find that correlation risk hedging demand is typically higher than volatility risk demand suggesting that correlation, more than volatility, is the crucial uncertainty factor to hedge in an intertemporal optimization portfolio problem.

An interesting research topic for future work is, therefore, the consideration of ambiguity about the dynamics of stochastic volatility and stochastic correlation in a model of dynamic portfolio choice with multiple risky assets.
7 Appendices

7.1 Bellman Equation (14)

Define $\Lambda$ as the diffusion matrix of state variables $y_t$ and $X_t$, according to processes in (10) and (11):

$$
\Lambda = \begin{bmatrix}
\sigma \sqrt{y_t} & 0 \\
\rho \pi_t \sqrt{\frac{1}{y_t} X_t} & \pi_t \sqrt{\frac{1}{y_t} X_t \sqrt{1 - \rho^2}}
\end{bmatrix}. 
$$

Minimization of (13) with respect to $h = [h^y \ h^z]'$ yields (see, for example, Anderson et al. (1998), p.22):

$$
\begin{bmatrix}
 h^y \\
 h^z
\end{bmatrix} = -\frac{\vartheta}{\sqrt{[J_y \ J_X] \Lambda \Lambda' \ [J_y \ J_X]}} \begin{bmatrix}
 J_y \\
 J_X
\end{bmatrix}. 
$$

Replacing (36) in (37), the vector of optimal contaminating drifts is obtained and given by:

$$
\begin{bmatrix}
 h^y \\
 h^z
\end{bmatrix} = -\frac{\vartheta}{\sqrt{J_y^2 \sigma^2 y_t + 2 J_y J_X \sigma \rho \pi_t X_t + J_X^2 \pi_t^2 X_t^2}} \begin{bmatrix}
 \sigma \sqrt{y_t} \ J_y + \rho \pi_t \sqrt{\frac{1}{y_t} X_t J_X} \\
 \pi_t \sqrt{\frac{1}{y_t} X_t \sqrt{1 - \rho^2} J_X}
\end{bmatrix}. 
$$

Substituting this result into (13), after some algebra, gives (14).

7.2 Expression (16)

Chacko and Viceira (2005) give the exact optimal portfolio policy, without ambiguity, for an investor with $\psi = 1$ and any level of risk-aversion:

$$
\pi_t = \frac{1}{\gamma} (\mu - r) y_t + \frac{\sigma \rho}{\gamma} \bar{A} y_t, 
$$

with $\bar{A}$ given by:

$$
\bar{A} = \frac{\gamma (\beta + \kappa) - (1 - \gamma) \rho \sigma (\mu - r) - \sqrt{[\gamma (\beta + \kappa) - (1 - \gamma) \rho \sigma (\mu - r)]^2 - (1 - \gamma) \sigma^2 (\mu - r)^2 [\gamma (1 - \rho^2) + \rho^2]}}{\sigma^2 [\gamma (1 - \rho^2) + \rho^2]}. 
$$

Chacko and Viceira (2005) also obtained the following corresponding value function:

$$
J (X_t, y_t) = \frac{e^{\bar{A} y_t + \bar{B} X_t^{1 - \gamma}}}{1 - \gamma}, 
$$

24
where $A$ is given above and $B$ is given by:

$$B = (1 - \gamma) \left( \ln \beta + \frac{r}{\beta} - 1 \right) + \frac{\kappa \theta}{\beta} A. \tag{40}$$

It is equivalent to write (15)-(17).

7.3 Optimal Consumption and Portfolio rules

7.3.1 Domain $\gamma \geq \omega$

The domain of analysis is set so that $A$ in (16) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$[\gamma (\beta + \kappa) - (1 - \gamma) \rho \sigma (\mu - r)]^2 \geq (1 - \gamma) \sigma^2 (\mu - r)^2 \left[ \gamma (1 - \rho^2) + \rho^2 \right]. \tag{41}$$

For $\gamma > 1$, the right-hand side is negative, therefore, (41) is always true. For $\gamma < 1$, (41) is true as long as:

$$\gamma^2 (\beta + \kappa)^2 - 2 \gamma (\beta + \kappa) (1 - \gamma) \rho \sigma (\mu - r) \geq \gamma (1 - \gamma) \sigma^2 (\mu - r)^2$$

$$\iff \gamma (\beta + \kappa)^2 + \gamma \left[ \sigma^2 (\mu - r)^2 + 2 (\beta + \kappa) \rho \sigma (\mu - r) \right] \geq \sigma^2 (\mu - r)^2 + 2 (\beta + \kappa) \rho \sigma (\mu - r)$$

$$\iff \gamma \geq \omega,$$

where $\omega = 1 - \frac{(\beta + \kappa)^2 - \sigma^2 (\mu - r)^2 + 2 \rho \sigma (\mu - r)(\beta + \kappa)}{(\beta + \kappa)^2 + \sigma^2 (\mu - r)^2 + 4 \rho \sigma (\mu - r)(\beta + \kappa)}$. Note that $\omega < 1.$

7.3.2 Limit of $A$ when $\gamma \to 1$

Observe, from (16), that the limit of $A$ when $\gamma \to 1$ is indeterminate $(0/0)$. Denoting the argument of the square root by $\Delta$ and applying l'Hôpital’s rule (i.e., calculating the limit of the ratio between the derivatives of the numerator and denominator with respect to $\gamma$), we obtain:
\[
\lim_{\gamma \to 1} A = \lim_{\gamma \to 1} \left\{ \frac{(\beta + \kappa) + \rho \sigma (\mu - r)}{-\sigma^2 \left[ \gamma (1 - \rho^2) + \rho^2 \right] + (1 - \gamma) \sigma^2 (1 - \rho^2)} - \frac{1}{2} \Delta - \frac{1}{2} \left[ \frac{\sigma^2 (\mu - r)^2}{\gamma (1 - \rho^2) + \rho^2} - (1 - \gamma) \sigma^2 (1 - \rho^2) \right] \right\}
\]

\[
= \frac{(\beta + \kappa) + \rho \sigma (\mu - r)}{\sigma^2 (1 - \gamma) \sigma^2 (1 - \rho^2)} - \frac{1}{2} \left\{ \frac{\sigma^2 (\mu - r)^2}{\gamma (1 - \rho^2) + \rho^2} - (1 - \gamma) \sigma^2 (1 - \rho^2) \right\}
\]

\[
= \frac{(\mu - r)^2}{2(\beta + \kappa)}
\]

\[\square\]

7.3.3 Optimal rules (19) and (20)

Considering the Bellman equation (14) and a value function with the form \( J(X_t, y_t) = \frac{1}{1-\gamma} \left[ e^{y(t)} X_t \right] ^{1-\gamma} \), the FOC with respect to \( \pi_t \) gives:

\[
\pi_t = \frac{1}{\gamma + \frac{\bar{\theta}}{\sqrt{G(\pi_t, y_t)}}} \left[ (\mu - r) y_t + \left( 1 - \gamma - \frac{\bar{\theta}}{\sqrt{G(\pi_t, y_t)}} \right) \sigma \rho y_t \frac{\partial g}{\partial y} \right], \tag{42}
\]

where:

\[
G(\pi_t, y_t) = \frac{\pi_t^2}{y_t} + 2 \sigma \rho \gamma_t \frac{\partial g}{\partial y_t} + \sigma^2 y_t \left( \frac{\partial g}{\partial y_t} \right)^2, \tag{43}
\]

i.e., optimal portfolio rule under ambiguity is the solution of an implicit function in \( \pi_t \). In order to provide an approximate solution for this optimization problem, consider the first-order expansions around \((\bar{\theta} = 0, \gamma = 1)\) of the functions \( g \) and \( \pi_t \). Expansion of \( g \) is given in (18) and expansion of \( \pi_t \) is given by:

\[
\pi(y_t) = \pi_0(y_t) + \pi_1(y_t) \bar{\theta} + \pi_2(y_t) (\gamma - 1) + O_\pi^2(\bar{\theta}, 1 - \gamma), \tag{44}
\]

where \( O_\pi^2(\bar{\theta}, 1 - \gamma) \) represents the residual of the first-order expansion, and \( \pi_0(y_t) \) represents the solution for the log-utility investor with no ambiguity \((\bar{\theta} = 0, \gamma = 1)\), given by (22). Considering that \( \lim_{\gamma \to 1} A = \frac{(\mu - r)^2}{2(\beta + \kappa)} \) (Appendix 7.3.2), for the log-utility investor with no ambiguity, \( G(\pi_t, y_t) \) is:

\[
G_0(y_t) = \left[ (\mu - r)^2 + \rho \sigma (\mu - r)^3 + \sigma^2 (\mu - r)^4 \right] y_t.
\]

Expanding (42) as given by (44), making use of (18) and considering again that \( \lim_{\gamma \to 1} A = \frac{(\mu - r)^2}{2(\beta + \kappa)} \), leads
to the conclusion that the approximate optimal portfolio choice is given by (20).

Regarding the optimal consumption rule (19), computations are more straightforward. Considering the Bellman (14) and the aggregator (8), the FOC with respect to variable \( C_t \) is simply:

\[ f_C = J_X, \]

where \( f_C \) is the gradient of the aggregator (8) with respect to consumption. In fact, under the assumption of SDU function with \( \psi = 1 \), given the value function (15), by the envelope theorem we have \( f_C = J_X \), which means \( C_t \) equals exactly to \( \beta X_t \). □

### 7.3.4 Optimal rules (27) and (28)

Considering the Bellman equation (26) and a value function with the form \( J(X_t, y_t) = \frac{1}{1-\gamma} \left[ e^{\theta(m) X_t} \right]^{1-\gamma} \), the FOC with respect to \( \pi_t \) yields:

\[
\pi_t = \frac{1}{\gamma} (\mu - r) y_t - \vartheta \frac{\sqrt{1 - \rho^2} \sqrt{y_t}}{\gamma} + \frac{\sigma \rho (1 - \gamma)}{\gamma} \frac{\partial g(y)}{\partial y} y_t. \tag{45}
\]

As for the general case in section 3, following perturbation theory, the function \( g \) is expanded around \( (\vartheta = 0, \gamma = 1) \) to first order. Expansion of \( g \) is given in (18). Expanding (45) as given by (44), making use of (18) and considering that \( \lim_{\gamma \to 1} A = \frac{(\mu - r)^2}{2(\beta + \kappa)} \), the asymptotic optimal portfolio rule (28) is immediately obtained.

Regarding the optimal consumption rule (27), considering the Bellman (26) and the aggregator (8), the FOC with respect to variable \( C_t \) is again given by:

\[ f_C = J_X, \]

where \( f_C \) is the gradient of the aggregator (8) with respect to consumption. The optimal consumption rule is exactly given by:

\[ C_t = \beta X_t, \]

which is (27). □

### 7.4 Bellman Equation for general contamination in section 4.2

From (29)-(30), for any admissible contamination, \( h = [h^s, h^e]^t \), following the same steps as in section 2.3 the investment opportunity set and the intertemporal budget constraint faced by the ambiguity investor are deduced.
The corresponding Bellman equation is given by:

\[
0 = \sup_{\theta \in \Theta} \inf_{\pi, C} \left\{ f(C, J) + \left[ \pi_t \left( \mu + \sqrt{\frac{T}{y_t}} h^s - r \right) X_t + r X_t - C_t \right] J_X + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \right.
\]
\[
\left. + \left[ \kappa (\theta - y_t) + \sigma \sqrt{y_t} \rho h^s + \sigma \sqrt{y_t} (1 - \rho^2) h^e \right] J_y + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{XY} \right\}.
\]

(46)

Following the same approach as in appendix 7.1, \( \Xi \) represents the diffusion matrix of state variables \( y_t \) and \( X_t \) in their contaminated processes:

\[
\Xi = \begin{bmatrix}
\sigma \sqrt{y_t} \rho & \sigma \sqrt{y_t} (1 - \rho^2) \\
\pi_t \sqrt{\frac{T}{y_t}} X_t & 0
\end{bmatrix}.
\]

(47)

The minimization of (46) with respect to the vector \( h \) gives:

\[
\begin{bmatrix}
h^s \\
h^e
\end{bmatrix} = - \frac{\vartheta}{\sqrt{\left[ J_y \ J_X \right] \Xi \Xi'} \left[ J_y \ J_X \right]' \left[ J_y \ J_X \right]}.
\]

(48)

Replacing (47) in (48), the vector of optimal contaminating drifts is obtained and given by:

\[
\begin{bmatrix}
h^s \\
h^e
\end{bmatrix} = - \frac{\vartheta}{\sqrt{J_y^2 \sigma^2 y_t + 2 J_y J_X \sigma \rho \pi_t X_t + J_X^2 \pi_t^2 \frac{1}{y_t} X_t^2}} \begin{bmatrix}
\sigma \sqrt{y_t} \rho \vartheta J_y + \pi_t \sqrt{\frac{T}{y_t}} X_t J_X \\
\sigma \sqrt{y_t} \sqrt{1 - \rho^2} J_y
\end{bmatrix}.
\]

Substituting this result into (46), after some algebra, gives (14).

\[ \square \]
References


