Nesting vertical and horizontal differentiation in two-sided markets*

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Abstract. We merge the two-sided markets duopoly model of Armstrong (2006) with the nested vertical and horizontal differentiation model of Gabszewicz and Wauthy (2012), which consists of a linear city with different consumer densities on the left and on the right side of the city. In equilibrium, the high-quality platform sells at a higher price and captures a greater market share than the low-quality platform, despite the indifferent consumer being closer to the high-quality platform. The difference between market shares is lower than socially optimal. A perturbation that introduces a negligible difference between the consumer density on the left and on the right side of the city may disrupt the existence of equilibrium in the model of Armstrong (2006).

Keywords: Two-sided markets, Horizontal differentiation, Vertical differentiation.

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1 Introduction

A two-sided market is a market where firms are platforms that allow two distinct groups of consumers to interact in order to engage in an activity that would not be possible without a platform. Examples in the literature include clubs, social networks and dating sites (men and women), online trading sites (buyers and sellers), newspapers (consumers and advertisers), video games (software developers and end users), yellow pages and credit cards (consumers and merchants), and others.

For example, in the clubs market, we can conceive that men and women constitute the two sides of the market. A given man chooses a certain club (platform) taking into consideration, among other aspects: the inter-group externality (number of women attending each of the clubs), the type of music (rock or latin), the environment (air conditioned, cleanliness), the access price and the travel time. There are, evidently, elements of both vertical and horizontal differentiation involved.

We study price competition between two platforms that sell horizontally and vertically differentiated products (whose quality is perfectly observed by both sides of the market) by developing a model that synthesizes the two-sided market model of Armstrong (2006) and the product differentiation model of Gabszewicz and Wauthy (2012).[1]

In the model of Gabszewicz and Wauthy (2012), the competing platforms are located at opposite extremes of a linear city. Consumers, who are distributed along the city, face linear transportation costs to travel to the platform of their choice. The difference with respect to the standard model of horizontal differentiation (Hotelling, 1929) is the fact that consumer density is different on the left side and on the right side of the city. This asymmetry of consumer density introduces vertical differentiation in the model.[2] To this

[1]In most of the literature, competition with differentiated products is studied alternatively using models of horizontal or vertical differentiation. However, there are some contributions where horizontal and vertical differentiation are simultaneously considered, as Neven and Thisse (1990), Economides (1993), Irmen and Thisse (1998), Daughety and Reinganum (2007, 2008), Argenziano (2008), Levin, Peck and Ye (2009), Griva and Vettas (2011), Gabszewicz and Wauthy (2012) and Amir, Gabszewicz and Resende (2013).

[2]Gabszewicz and Wauthy (2012) define the natural market of a firm as the group of consumers who, at equal prices, prefer the variant offered by this firm relatively to the variant offered by its competitor. In the light of this, they generalize the model of Hotelling (1929) to allow for natural markets of different sizes. In their own words: "*In the symmetric linear model with firms located at the extremities of the*

environment, the two-sided market elements of the model of Armstrong (2006) are added. There are two groups of consumers, which have the same distribution along the city and have the same benefit of interacting with agents of the other group.[3]

Considering weak network effects, we provide necessary and sufficient conditions for existence and uniqueness of an interior equilibrium and explain how prices, market shares and profits depend on the strength of inter-group externalities and on the degree of vertical differentiation. The equilibrium is unique and characterized by the fact that the high-quality platform charges a higher price and has a greater market share than the low-quality platform, although the indifferent consumer is located on the right side of the city.

The effect of the inter-group externality on prices is the same as in the model of Armstrong (2006): prices decrease in a way that exactly reflects the benefit of attracting agents on one side to agents on the other side. A stronger inter-group externality increases the market share of the high-quality platform, which is the more populated one, but not sufficiently to offset the negative impact of lower prices on its profit.

Vertical differentiation as defined by Gabszewicz and Wauthy (2012) implies that the low-quality platform competes more fiercely, leading to lower prices at both platforms. In spite of facing a more fierce competition, the high-quality platform sees its market share increase. However, its profit decreases unless the inter-group externality is sufficiently strong.

We also study the existence of tipping equilibria, only to confirm that the more complicated consumer distribution of the model of Gabszewicz and Wauthy (2012) does not interfere with the conditions for existence nor with the characteristics of tipping equilibria, which are exactly as in the model of Armstrong (2006).

Considering the problem of a benevolent planner that allocates consumers to platforms in order to promote social welfare efficiency, we find that it is more efficient to have two active platforms than only one (under the assumption of weak network effects). Moreover,

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unit interval, natural markets are defined by the $\left[0, \frac{1}{2}\right]$ *and* $\left[\frac{1}{2}, 1\right]$ *intervals, respectively. In order to allow for natural markets with different sizes, we then assume that the density differs from one interval to the other. Notice that in this model, a vertical configuration appears as a limiting case where the density of one of the intervals tends to zero while the density of the other tends to 1.*"

[3] Armstrong (2006) does not restrict inter-group externalities to be symmetric in this sense.

we show that, in the market equilibrium, there are too few consumers on the high-quality platform. This is due to the inter-group externality (which is not taken into account by individual agents) and reinforced by the fact that the low-quality platform charges lower prices.[4]

Finally, we investigate two particular cases of possible entry as in Gabszewicz and Wauthy (2012). Our conclusions are that, in the presence of cross-network externalities, the deterrence of an inferior-quality entrant is easier and the conquest of the market by a superior-quality entrant can be achieved with a higher limit price.

The paper is organized as follows. Section 2 presents a description of the model. Section 3 provides the corresponding analysis. Section 4 studies and defines the region where the (interior) equilibrium exists as well the tipping region and Section 5 presents the socially optimal outcome promoted by a benevolent planner. Finally, Section 6 covers entry and Section 7 concludes. The appendix contains the proofs of most propositions.

2 The model

Consider a two-sided market with two platforms, A and B, that are horizontally and vertically differentiated. The platforms are exogenously located at opposite extremes of a linear city: platform A is located at $x = 0$ while platform B is located at $x = 1$. There are no production costs.

There is a continuum of consumers on each of the two sides of the market, 1 and 2. Consumers inelastically demand one unit of the service that is provided by the platform. We follow Armstrong (2006) in considering linear inter-group externalities and Serfes and Zacharias (2009) in assuming symmetry between the two sides of the market.

Horizontal differentiation is captured by transportation costs that are linear in distance, as in the model of Hotelling (1929). For technical convenience, the transportation cost parameter is set to 1, following Neven and Thisse (1990) and Gabszewicz and Wauthy (2012), among others.

[4] This is also the conclusion of Argenziano (2008).

As in Gabszewicz and Wauthy (2012), vertical differentiation is captured by the fact that there are more consumers on the right half of the city than on the left half. On each side of the market, the density of consumers is μ at any $x \in \left[0, \frac{1}{2}\right]$ and $1-\mu$ at any $x \in \left(\frac{1}{2}, 1\right]$. Without loss of generality, we assume that $\mu \leq \frac{1}{2}$. Pure horizontal differentiation corresponds to the particular case in which $\mu = \frac{1}{2}$, while pure vertical differentiation corresponds to $\mu = 0$.

The two platforms, A and B, simultaneously choose their access prices, p^A and p^B, which apply to both sides of the market (price discrimination between sides is not allowed).

The utility of an agent of side j that is located at $x \in [0,1]$ and chooses platform $i \in \{A, B\}$ is given by:

$$u_j^A(x) = V + \alpha D_k^A - p^A - x, \tag{1}$$
$$u_j^B(x) = V + \alpha D_k^B - p^B - (1-x), \tag{2}$$

where V is the stand-alone benefit, α measures the strength of the inter-group externality and D_k^i is the number of consumers of side $k \neq j$ that join platform i.

Notice that, at equal prices and platform sizes, all consumers in $[0, \frac{1}{2})$ prefer to buy from platform A while all consumers in $(\frac{1}{2}, 1]$ prefer to buy from platform B. Since the density of consumers is greater at the right side of the line, this means that the majority of consumers prefers platform B. However, of course, agents also take into account the prices charged by the platforms and the platform choice by the agents of the other side.

The timing of the game is the following: in the first stage, platforms simultaneously set access prices for both sides. In the second stage, agents simultaneously choose which platform to join and their payoffs are determined.

3 Demand and profit functions

We assume that agents have self-fulfilled expectations relatively to the platform choices of the other agents. This implies that all consumers to the left of the indifferent consumer choose platform A, while all consumers to the right choose platform B.

The hypothesis that platforms cannot make price discrimination between the two-sides implies that the indifferent consumers of the two sides have the same location, i.e., that $\tilde{x}_1 = \tilde{x}_2$, allowing us to drop the subscripts and use the simpler notation p^A, p^B, $\tilde{x}$, D^A and D^B.

A consumer that is indifferent between platform A and B must be located at:

$$\tilde{x} = \frac{1}{2}\left[1 + \alpha(D^A - D^B) - (p^A - p^B)\right]. \tag{3}$$

If $\tilde{x} < 0$, all consumers choose platform B; while if $\tilde{x} > 1$, everyone chooses platform A.

Writing the demand for platform A from each consumer side as a function of the location of the indifferent consumer, we obtain:

$$D^A(\tilde{x}) = \begin{cases} \frac{1}{2}, & \text{if } \tilde{x} \geq 1 \\ (\tilde{x} - \frac{1}{2})(1-\mu) + \frac{\mu}{2}, & \text{if } \frac{1}{2} \leq \tilde{x} \leq 1 \\ \tilde{x}\mu, & \text{if } 0 \leq \tilde{x} \leq \frac{1}{2} \\ 0, & \text{if } \tilde{x} \leq 0. \end{cases} \tag{4}$$

The following assumption, maintained throughout the paper except in Section 6, implies that the slope of the demand function is negative.

Assumption 1 (Weak inter-group externality).

The intensity of the inter-group externality is relatively weak: $\alpha < \frac{1}{1-\mu}$.

As a function of prices, the demand for platform A is given by (see Appendix 10.1):

$$D^A(p^A, p^B) = \begin{cases} \frac{1}{2}, & \text{if } p^A \leq p^B + \frac{\alpha}{2} - 1 \\ \frac{2(1-\mu)(p^B - p^A) + 2\mu - \alpha(1-\mu)}{4[1-\alpha(1-\mu)]}, & \text{if } p^B + \frac{\alpha}{2} - 1 \leq p^A \leq p^B - \frac{\alpha}{2}(1-2\mu) \\ \frac{2\mu(p^B - p^A) + \mu(2-\alpha)}{4(1-\alpha\mu)}, & \text{if } p^B - \frac{\alpha}{2}(1-2\mu) \leq p^A \leq p^B - \frac{\alpha}{2} + 1 \\ 0, & \text{if } p^A \geq p^B - \frac{\alpha}{2} + 1. \end{cases} \tag{5}$$

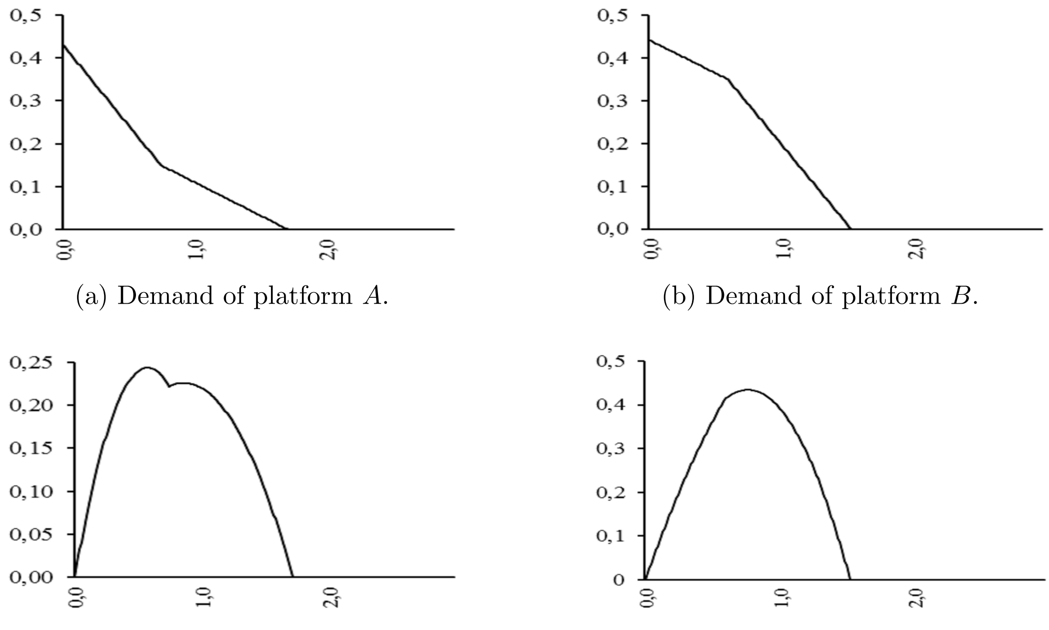
Since total demand is inelastic, the demand for platform B from each consumer side is $D^B(p^A, p^B) = \frac{1}{2} - D^A(p^A, p^B)$.

The profit of each platform, $i \in \{A, B\}$, is given by $\pi^i(p^A, p^B) = 2p^i D^i(p^A, p^B)$.

The demand and profit functions of platforms A and B are defined in four pieces, each corresponding to a different region where the indifferent consumer is located. A problematic characteristic of the function $D^A(\cdot, p^B)$ is the fact that its slope decreases from the second to the third subdomain in (5).[5] This is illustrated in Figure 1a, where it is assumed that $\alpha = 0.3$, $\mu = 0.1$ and $p^B = \frac{2-\mu}{3(1-\mu)} - \frac{\alpha}{2}$ (which, as we will see, is the equilibrium value of p^B for these values of the parameters α and μ).

The profit function of platform A, $\pi^A(\cdot, p^B)$, is obtained by multiplying the demand function by $2p^A$. As illustrated in Figure 1c, it may have two local maxima.

Figure 1b illustrates the behavior of the demand of platform B, given the price set by platform A (again, we consider $\alpha = 0.3$, $\mu = 0.1$ and the equilibrium value of p^A). Since $D^B(p^A, \cdot)$ is concave, the profit function $\pi^B(p^A, \cdot)$ is also concave (see Figure 1d).



(a) Demand of platform A.

(b) Demand of platform B.

(c) Profit of platform A.

(d) Profit of platform B.

Figure 1: Demand and profit functions of both platforms.

[5]This kink in the demand curve has appeared, for example, in the two-sided markets model of Resende (2008), in models with "market inertia" or "switching costs" (Scotchmer, 1986; Farrell and Shapiro, 1989) and in a model with a particular kind of multihoming (Brandão et al., 2013).

4 Interior equilibrium

We start by presenting the benchmark case of pure horizontal differentiation ($\mu = \frac{1}{2}$), which corresponds to the model of Armstrong (2006) in the particular case of symmetric inter-group externalities.

Proposition 1 (Pure horizontal differentiation).

If $\mu = \frac{1}{2}$, there exists an interior equilibrium. It is such that:

$$p^{A*} = p^{B*} = 1 - \frac{\alpha}{2},$$

$$\begin{cases} \tilde{x} = \dfrac{1}{2} \\ D^{A*} = D^{B*} = \dfrac{1}{4}, \end{cases}$$

$$\pi^{A*} = \pi^{B*} = \frac{1}{2} - \frac{\alpha}{4}.$$

Proof. See Appendix 10.2. □

In the presence of vertical differentiation, if the inter-group externality is not too strong, there exists an interior equilibrium with the indifferent consumer located on the right side of the city ($\tilde{x} > \frac{1}{2}$).

Proposition 2 (Horizontal and vertical differentiation).

If $\mu < \frac{1}{2}$, there exists an interior equilibrium if and only if $\alpha \leq \frac{4-\mu}{6(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6(1-\mu)}$. It is such that:

$$\begin{cases} p^{A*} = \dfrac{1+\mu}{3(1-\mu)} - \dfrac{\alpha}{2} \\ p^{B*} = \dfrac{2-\mu}{3(1-\mu)} - \dfrac{\alpha}{2}, \end{cases} \tag{6}$$

$$\begin{cases} D^{A*} = \dfrac{2(1+\mu) - 3\alpha(1-\mu)}{12\left[1-\alpha(1-\mu)\right]} \\ D^{B*} = \dfrac{2(2-\mu) - 3\alpha(1-\mu)}{12\left[1-\alpha(1-\mu)\right]}, \end{cases} \tag{7}$$

$$\begin{cases} \pi^{A*} = \dfrac{\left[2(1+\mu) - 3\alpha(1-\mu)\right]^2}{36(1-\mu)\left[1-\alpha(1-\mu)\right]}, \\ \pi^{B*} = \dfrac{\left[2(2-\mu) - 3\alpha(1-\mu)\right]^2}{36(1-\mu)\left[1-\alpha(1-\mu)\right]}. \end{cases} \tag{8}$$

Proof. See Appendix 10.2. □

In the presence of vertical differentiation, the high-quality platform sets a higher price ($p^{B*} > p^{A*}$) and captures a greater market share ($D^{B*} > D^{A*}$) than the low-quality platform. Therefore, it also has higher profits ($\pi^{B*} > \pi^{A*}$).

We remark that, for the equilibrium prices (6), the demand is unique, in the sense that there is no other equilibria of the game in which consumers, for given prices, choose which platform to join (see Appendix 10.3).

Observe that, when $\mu \to \frac{1}{2}$, the equilibrium variables converge to their values in the equilibrium with pure horizontal differentiation.

In the case of pure vertical differentiation ($\mu = 0$), which is the other extreme case, we obtain: $p^{A*} = \frac{1}{3} - \frac{\alpha}{2}$ and $p^{B*} = \frac{2}{3} - \frac{\alpha}{2}$; $D^{A*} = \frac{2-3\alpha}{12(1-\alpha)}$ and $D^{B*} = \frac{4-3\alpha}{12(1-\alpha)}$; $\pi^{A*} = \frac{1}{9(1-\alpha)} - \frac{\alpha(4-3\alpha)}{12(1-\alpha)}$ and $\pi^{B*} = \frac{1}{4(1-\alpha)} - \frac{\alpha(8-3\alpha)}{12(1-\alpha)}$.

As a corollary, we conclude that a perturbation that introduces a negligible difference between the consumer density on the left and on the right side of the city may disrupt the existence of equilibrium in the model of Armstrong (2006).

Corollary 1 (No equilibrium).

If $\mu < \frac{1}{2}$ and $\alpha > \frac{4-\mu}{6(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6(1-\mu)}$, there exists no interior equilibrium.

Proof. This result follows from Proposition 2 and Lemmas 1 and 2 in Appendix 10.2. □

5 Comparative statics

We now discuss the impact of the degree of vertical differentiation and of the strength of the inter-group externality on equilibrium prices, market shares and profits. All calculations are presented in Appendix 10.4.

As the degree of vertical differentiation increases (μ becomes lower)[6], both prices decrease, with the price of the low-quality platform decreasing faster ($\frac{\partial p^{A*}}{\partial \mu} > \frac{\partial p^{B*}}{\partial \mu} > 0$). Despite selling at an increasing discount relatively to the high-quality platform, the low-quality platform loses market share ($\frac{\partial D^{A*}}{\partial \mu} > 0$), and, therefore, also loses profits ($\frac{\partial \pi^{A*}}{\partial \mu} > 0$). Since the market share of the high-quality platform increases, the impact of vertical differentiation on its profit is not obvious. We find that the profit of the high-quality platform decreases if and only if the inter-group externality is weak ($\frac{\partial \pi^{B*}}{\partial \mu} > 0 \Leftrightarrow \alpha < \frac{2\mu}{1-\mu}$). It may be surprising that the profit of the high-quality platform may decrease when the degree of vertical differentiation increases. It follows from the fact that the price of the high-quality platform goes down as μ decreases due to an intensification of price competition.[7]

Prices decrease as the inter-group externality becomes stronger ($\frac{\partial p^{A*}}{\partial \alpha} = \frac{\partial p^{B*}}{\partial \alpha} = -\frac{1}{2}$), because attracting consumers on one side generates gains on the other side. More precisely, both prices are adjusted downwards by $\frac{\alpha}{2}$, which corresponds to the marginal benefit on the other side that results from attracting an additional agent on one side. The difference between the two prices does not depend on the strength of the inter-group externality. It is natural, therefore, that market shares diverge as the strength of the inter-group externality increases ($\frac{\partial D^{A*}}{\partial \alpha} < 0$).

Without inter-group externalities ($\alpha = 0$), we recover the setting of Gabszewicz and Wauthy (2012) and, of course, their equilibrium prices ($p^{A*} = \frac{1+\mu}{3(1-\mu)}$ and $p^{B*} = \frac{2-\mu}{3(1-\mu)}$).

As the strength of the inter-group externality increases, the profit of platform A de-

[6]Recall that $\mu \in \left[0, \frac{1}{2}\right]$ is the consumer density on the left side of the city, while $1 - \mu$ is the consumer density on the right side of the city. Therefore, the lower is μ, the greater is the asymmetry between the sizes of the firms' natural markets.

[7]This effect is already present in Gabszewicz and Wauthy (2012).

creases, since price and market share strictly decrease. The profit of platform B also decreases, despite the increase in its market share.

6 Tipping equilibria

Since the utility of accessing a platform is an increasing function of the platforms' membership, platform growth tends to be self-reinforcing and, therefore, one platform may capture the whole market (Katz and Shapiro, 1994).

We conclude that vertical differentiation of the type that we are considering does not interfere with the condition for existence of tipping equilibria.

Proposition 3 (Tipping equilibrium).

There exist tipping equilibria with $\tilde{x} = 0$ and $\tilde{x} = 1$ if and only if $\alpha > 2$.

Proof. See Appendix 10.2. □

7 Social optimum

Suppose that a benevolent planner can allocate consumers to the two platforms. Some questions arise. It is socially optimal to place all consumers in the high-quality platform, so that inter-group externalities are maximized? Or is it preferable to split consumers between the two platforms, to save on transportation costs?

Social welfare is given by:

$$W = \begin{cases} 2\int_0^{\tilde{x}} \left[V + \alpha D^A - x\right] \mu dx + 2\int_{\tilde{x}}^{\frac{1}{2}} \left[V + \alpha D^B - (1-x)\right] \mu dx \\ \quad +2\int_{\frac{1}{2}}^{1} \left[V + \alpha D^B - (1-x)\right] (1-\mu) dx, \quad \text{if } \tilde{x} \leq \dfrac{1}{2}; \\ 2\int_0^{\frac{1}{2}} (V + \alpha D^A - x)\mu dx + 2\int_{\frac{1}{2}}^{\tilde{x}} (V + \alpha D^A - x)(1-\mu) dx \\ \quad +2\int_{\tilde{x}}^{1} (V + \alpha D^B - (1-x))(1-\mu) dx, \quad \text{if } \tilde{x} > \dfrac{1}{2}. \end{cases}$$

Conditionally on $\tilde{x} \in \left[0, \frac{1}{2}\right]$, social welfare as a function of $\tilde{x}$ is given by:

$$W(\tilde{x}) = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \tilde{x}^2 \left[-2\alpha\mu(1 - 2\alpha\mu)\right] + \tilde{x} \left[2\mu(1 - \alpha)\right]. \qquad (9)$$

If $\alpha < 1$, the maximum in $\left[0, \frac{1}{2}\right]$ is attained at:

$$\tilde{x}^{FB} = \frac{1 - \alpha}{2\left(1 - 2\alpha\mu\right)}. \qquad (10)$$

This constrained maximizer is, in fact, the global maximizer (see Appendix 10.5).

Proposition 4.

Let $\alpha < 1$. In the socially optimal outcome, both platforms operate in the market, with the high-quality platform capturing a greater market share than in the market equilibrium.

Proof. See Appendix 10.5. □

Comparing the socially optimal outcome with the market equilibrium outcome, we conclude that there are too many consumers on the low-quality platform at the market equilibrium.

8 Entry

The extension of the model to a triopoly is particularly relevant if we consider the possibility of entry by a third firm. In line with Gabszewicz and Wauthy (2012), we study entry by an inferior-quality platform and entry by a superior-quality platform. We aim at understanding whether inter-group externalities make it easier or harder for the incumbents to deter the entry of an inferior-quality platform and for a superior-quality platform to capture the whole market.

8.1 Deterrence of an inferior-quality entrant

Suppose that a third platform, C, has the possibility of entering the market, becoming positioned at a distance L from the center of the city (i.e., from $x = \frac{1}{2}$), in a direction that is orthogonal to the linear city.

Any $L > \frac{1}{2}$ implies that no consumer would choose platform C if all platforms charged the same prices and had the same number of customers on the other side of the market.

As in Gabszewicz and Wauthy (2012), suppose that platforms A and B are charging the pre-entry equilibrium prices (6). Can platform C attract any consumers by charging $p^C = 0$, given that $D^C = 0$?[8] It can if and only if it can attract the consumer located at $x \in \frac{1}{2}$:

$$\begin{aligned} V - L \geq V + \alpha D^{A*} - p^{A*} - \frac{1}{2} \;\Leftrightarrow\; L \leq p^{A*} - \alpha D^{A*} + \frac{1}{2} \\ \Leftrightarrow\; L \leq \frac{1+\mu}{3(1-\mu)} + \frac{1}{2} - \frac{\alpha}{2} - \alpha\frac{2(1+\mu) - 3\alpha(1-\mu)}{12\left[1 - \alpha(1-\mu)\right]}. \end{aligned}$$

when $\alpha = 0$, we recover the result of Gabszewicz and Wauthy (2012). The presence of inter-group externalities diminishes the critical value of L for which entry is deterred, i.e., it helps platforms A and B to deter the entrance of platform C.

8.2 Market capture by a superior-quality entrant

Now suppose that the potential entrant offers a service of superior quality, in the sense that its stand-alone value is $\hat{V} > V$.

Assuming that $\hat{V} > V + L + \frac{1}{2}$ implies that all consumers would choose platform C if all platforms charged the same prices and had the same number of customers on the other side of the market.

Can platform C capture the whole market, even with platforms A and B charging a

[8]Notice that we are considering that the *status quo* is the equilibrium of the baseline model, i.e., that consumers are already in the incumbent platforms. The entrant faces, therefore, an additional difficulty in attracting consumers.

null price? The answer is positive if and only if:

$$\hat{V} + \frac{\alpha}{2} - L - \frac{1}{2} \geq V \;\Leftrightarrow\; L \leq \hat{V} - V + \frac{\alpha}{2} - \frac{1}{2}.$$

If the above condition is satisfied, prices $p^{A*} = p^{B*} = 0$ and $p^{C*} = \hat{V} - V + \frac{\alpha}{2} - \frac{1}{2} - L$, together with demand $D^{A*} = D^{B*} = 0$ and $D^{C*} = \frac{1}{2}$, constitute a triopoly equilibrium (unless platform C prefers to charge a higher price in spite of losing some demand).

Notice that the presence of inter-group externalities makes it easier for the superior quality entrant to capture the whole market.

9 Conclusions

We have studied competition between two horizontally and vertically differentiated platforms in a two-sided market, using a model that captures those of Armstrong (2006) and Gabszewicz and Wauthy (2012) as particular cases for pure horizontal differentiation and no inter-group externalities, respectively.

In spite of the technical difficulties that are intrinsic to the model of Gabszewicz and Wauthy (2012), we have been able to fully characterize the existence and uniqueness properties of equilibrium under the assumption that the inter-group externality is relatively weak. One important aspect of our existence results is that a perturbation of the model of Armstrong (2006) that introduces a small jump in the consumer density at the center of the city disrupts the existence of equilibrium.

The equilibrium properties essentially combine the features of the two-sided market model of Armstrong (2006), in particular, the fact that prices are lower due to the benefit on the other side that results from increased membership on one side, with those of the nested horizontal and vertical differentiation model of Gabszewicz and Wauthy (2012), where the low-quality firm becomes a more fierce competitor due to the decrease of the average consumer density coupled with the increase of the marginal consumer density.

Comparing the socially optimal outcome with the market equilibrium, we concluded that a benevolent planner would prefer to increase the market share of the high-quality platform. This was also the conclusion of (2008) in her study of a duopoly with product differentiation and network effects.

Finally, comparing our results with those of Gabszewicz and Wauthy (2012) we concluded that inter-group externalities facilitate the deterrence of an inferior-quality entrant and the capture of the whole market by a superior-quality entrant.

10 Appendix

10.1 Demand

The four consumer utilities assuming the existence of platforms $i \in \{A, B\}$ and sides $j \in \{1, 2\}$, $k \in \{2, 1\}$ and $\alpha = \alpha_1 = \alpha_2$ and $\mu = \mu_1 = \mu_2$ are:

$$\begin{cases} u_1^A(x_1) = V + \alpha D_2^A - p_1^A - x_{1;} \\ u_1^B(x_1) = V + \alpha D_2^B - p_1^B - (1 - x_1); \\ u_2^A(x_2) = V + \alpha D_1^A - p_2^A - x_2; \\ u_2^B(x_2) = V + \alpha D_1^B - p_2^B - (1 - x_2). \end{cases}$$

The indifferent consumer on both sides of the market follows by setting: $u_1^A(x_1) = u_1^B(x_1)$ and $u_2^A(x_2) = u_2^B(x_2)$. Straightforward calculations imply:

$$\begin{cases} \tilde{x}_1 = \frac{1}{2} + \frac{\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)}{2} \\ \tilde{x}_2 = \frac{1}{2} + \frac{\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)}{2}. \end{cases}$$

10.1.1 Assuming that the indifferent consumers are on the right

Demands for $\tilde{x}_i \in \left[\frac{1}{2}, 1\right]$, $i \in \{1, 2\}$, are defined by:

$$\begin{cases} D_i^A(\tilde{x}_i) = \frac{\mu}{2} + (\tilde{x}_i - \frac{1}{2})(1 - \mu); \\ D_i^B(\tilde{x}_i) = (1 - \tilde{x}_i)(1 - \mu). \end{cases}$$

Substituting the expressions for $\tilde{x}_1$ and $\tilde{x}_2$, we obtain:

$$\begin{cases} D_1^A = \frac{\mu}{2} + \frac{1-\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)\right]; \\ D_2^A = \frac{\mu}{2} + \frac{1-\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)\right]; \\ D_1^B = \frac{1-\mu}{2}\left[1 - \alpha(D_2^A - D_2^B) + (p_1^A - p_1^B)\right]; \\ D_2^B = \frac{1-\mu}{2}\left[1 - \alpha(D_1^A - D_1^B) + (p_2^A - p_2^B)\right]. \end{cases}$$

Solving this linear system for $p^A = p_1^A = p_2^A$ and $p^B = p_1^B = p_2^B$, we obtain:

$$\begin{cases} D^A(p^A, p^B) = \frac{(1-\mu)(p^B - p^A)}{2[1-\alpha(1-\mu)]} + \frac{2\mu - \alpha(1-\mu)}{4[1-\alpha(1-\mu)]}; \\ D^B(p^A, p^B) = \frac{(1-\mu)(p^A - p^B)}{2[1-\alpha(1-\mu)]} + \frac{(2-\alpha)(1-\mu)}{4[1-\alpha^2(1-\mu)^2]}. \end{cases} \qquad (11)$$

10.1.2 Assuming that the indifferent consumers are on the left

Demands for $\tilde{x}_i \in \left[0, \frac{1}{2}\right]$, $i \in \{1, 2\}$, are defined by:

$$\begin{cases} D_i^A(\tilde{x}_i) = \tilde{x}_i \mu; \\ D_i^B(\tilde{x}_i) = (\frac{1}{2} - x_1)\mu + \frac{1}{2}(1-\mu). \end{cases}$$

Once we substitute the respective demands by $\tilde{x}_1$ and $\tilde{x}_2$, we obtain:

$$\begin{cases} D_1^A = \frac{\mu}{2} + \frac{\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)\right]; \\ D_2^A = \frac{\mu}{2} + \frac{\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)\right]; \\ D_1^B = \frac{1-\mu}{2} - \frac{\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B\right]; \\ D_2^B = \frac{1-\mu}{2} - \frac{\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B\right]. \end{cases}$$

Solving this linear system for $p^A = p_1^A = p_2^A$ and $p^B = p_1^B = p_2^B$, we obtain:

$$\begin{cases} D^A(p^A, p^B) = \frac{2\mu(p^B - p^A)}{4(1-\alpha\mu)} + \frac{(2-\alpha)\mu}{4(1-\alpha\mu)}; \\ D^B(p^A, p^B) = \frac{2\mu(p^A - p^B)}{4(1-\alpha\mu)} + \frac{2-2\mu-\alpha\mu}{4(1-\alpha\mu)}. \end{cases} \qquad (12)$$

10.2 Equilibrium prices, demand and profits

Proof of Proposition 1

In the particular case of $\mu = \frac{1}{2}$, demand and profits have the same analytical expression on the left and on the right side of the city. Thus, there is no longer a kink in the demand function.

From (11), the first-order conditions for profit maximization are:

$$\begin{cases} \dfrac{\partial \pi^A}{\partial p^A} = 0 \\ \dfrac{\partial \pi^B}{\partial p^B} = 0 \end{cases} \Leftrightarrow \begin{cases} 2\left(1+\dfrac{\alpha}{2}\right)p^A - \left(1+\dfrac{\alpha}{2}\right)p^B = \left(1-\dfrac{\alpha}{2}\right)\left(1+\dfrac{\alpha}{2}\right) \\ 2\left(1+\dfrac{\alpha}{2}\right)p^B - \left(1+\dfrac{\alpha}{2}\right)p^A = \left(1-\dfrac{\alpha}{2}\right)\left(1+\dfrac{\alpha}{2}\right) \end{cases}$$
$$\Leftrightarrow \; p^A = p^B = 1 - \frac{\alpha}{2}.$$

The equilibrium prices satisfy the second-order condition, because:

$$\frac{\partial^2 \pi^A}{\partial {p^A}^2} = \frac{-\frac{1}{2}}{1-\frac{\alpha^2}{4}},$$

which is strictly negative under Assumption 1. □

Lemma 1.

There exists no equilibrium with $\tilde{x} \in \left(0, \frac{1}{2}\right)$.

Proof. We will use the first-order conditions to find a single candidate equilibrium with the indifferent consumer located at the left of the city center, and, then, we will show that the candidate equilibrium cannot be an actual equilibrium

We will use the superscript "+" to denote the values of prices, demands and profits in that candidate equilibrium.

(i) First-order conditions for profit-maximization

Considering the demand functions for $\tilde{x}_i \in \left(0, \frac{1}{2}\right)$, $i \in \{1, 2\}$, which are given by (12), the first-order conditions for profit maximization yield:

$$\begin{cases} \frac{\partial \pi^A}{\partial p_1^A} = 0 \\ \frac{\partial \pi^A}{\partial p_2^A} = 0 \\ \frac{\partial \pi^B}{\partial p_1^B} = 0 \\ \frac{\partial \pi^B}{\partial p_2^B} = 0 \end{cases} \Leftrightarrow \begin{cases} 4\mu p_1^A + 4\alpha\mu^2 p_2^A - 2\mu p_1^B - 2\alpha\mu^2 p_2^B = -\alpha^2\mu^2 + 2\alpha\mu^2 - \alpha\mu + 2\mu; \\ 4\alpha\mu^2 p_1^A + 4\mu p_2^A - 2\alpha\mu^2 p_1^B - 2\mu p_2^B = -\alpha^2\mu^2 + 2\alpha\mu^2 - \alpha\mu + 2\mu; \\ -2\mu p_1^A - 2\alpha\mu^2 p_2^A + 4\mu p_1^B + 4\alpha\mu^2 p_2^B = -\alpha^2\mu^2 - 2\alpha\mu^2 + \alpha\mu - 2\mu + 2; \\ -2\alpha\mu^2 p_1^A - 2\mu p_2^A + 4\alpha\mu^2 p_1^B + 4\mu p_2^B = -\alpha^2\mu^2 - 2\alpha\mu^2 + \alpha\mu - 2\mu + 2. \end{cases}$$

Solving this system of equations, we obtain the following equilibrium prices:

$$\begin{cases} p^{A+} = \dfrac{1+\mu}{3\mu} - \dfrac{\alpha}{2}, \\ p^{B+} = \dfrac{2-\mu}{3\mu} - \dfrac{\alpha}{2}. \end{cases} \tag{13}$$

The corresponding market shares are obtained by substituting (13) into (5):

$$\begin{cases} D^{A+} = \dfrac{2(1+\mu) - 3\alpha\mu}{12(1-\alpha\mu)}, \\ D^{B+} = \dfrac{2(2-\mu) - 3\alpha\mu}{12(1-\alpha\mu)}. \end{cases} \tag{14}$$

Profits are obtained from (13) and (14):

$$\begin{cases} \pi^{A+} = \dfrac{[2(1+\mu) - 3\alpha\mu]^2}{36\mu(1-\alpha\mu)}, \\ \pi^{B+} = \dfrac{[2(2-\mu) - 3\alpha\mu]^2}{36\mu(1-\alpha\mu)}. \end{cases} \tag{15}$$

(ii) Interiority of the indifferent consumer

The indifferent consumer is located at:

$$\tilde{x}^+ = \frac{1}{2} - \frac{\alpha(1-2\mu)}{12(1-\alpha\mu)} + \frac{1-2\mu}{6\mu}. \tag{16}$$

It must be on the left half of the city, otherwise the candidate equilibrium cannot be an actual equilibrium. This requires that the inter-group externality is relatively strong:

$$\tilde{x}^+ < \frac{1}{2} \;\Leftrightarrow\; \frac{\alpha(1-2\mu)}{12(1-\alpha\mu)} > \frac{1-2\mu}{6\mu} \;\Leftrightarrow\; \alpha\mu > 2 - 2\alpha\mu \;\Leftrightarrow\; \alpha > \frac{2}{3\mu}. \tag{17}$$

The indifferent consumer must also be located inside the city. This requires that the inter-group externality is relatively weak:

$$\begin{aligned}\tilde{x}^+ > 0 \;&\Leftrightarrow\; \frac{1+\mu}{\mu} > \frac{\alpha(1-2\mu)}{2(1-\alpha\mu)} \\ &\Leftrightarrow\; 2(1+\mu-\alpha\mu-\alpha\mu^2) > \alpha\mu - 2\alpha\mu^2 \\ &\Leftrightarrow\; 2+2\mu > 3\alpha\mu \\ &\Leftrightarrow\; \alpha < \frac{2(1+\mu)}{3\mu}. \end{aligned} \tag{18}$$

This is always true under Assumption 1.

Combining (17) with Assumption 1, we obtain $\alpha \in \left(\frac{2}{3\mu}, \frac{1}{1-\mu}\right)$, which is non-empty if and only if $\mu > 0.4$.

(iii) Global profit-maximization

Even if the local second-order conditions are satisfied, since the profit function of platform A is not globally quasi-concave, we must check that the local maximum determined before (candidate equilibrium price p^{A+}) is a global maximum (given p^{B+}).

For this to be true, it is necessary that:

$$\pi^A\left(p^{A+}, p^{B+}\right) \geq \pi^A\left(p^A, p^{B+}\right), \;\forall p^A. \tag{19}$$

A possible deviation is to p^{AR}, which is the local maximum of $\pi^A\left(\cdot, p^{B+}\right)$ in the branch that leads to $\tilde{x} \in \left[\frac{1}{2}, 1\right]$. From (5), in this branch, the profit function is given by:

$$\pi^A\left(p^A, p^{B+}\right) = 2p^A \tfrac{1-\mu}{2[1-\alpha(1-\mu)]}\left(p^{B+} - p^A + \tfrac{\mu}{1-\mu} - \tfrac{\alpha}{2}\right). \tag{20}$$

In this branch, the first-order condition for profit maximization yields:

$$\begin{aligned} & \tfrac{\partial \pi^A\left(p^A,p^{B+}\right)}{\partial p^A} = 0 \\ \Leftrightarrow\ & \tfrac{1-\mu}{2[1-\alpha(1-\mu)]}\left(p^{B+} - p^{AR} + \tfrac{\mu}{1-\mu} - \tfrac{\alpha}{2}\right) = p^{AR}\tfrac{1-\mu}{2[1-\alpha(1-\mu)]} \\ \Leftrightarrow\ & 2p^{AR} = p^{B+} + \tfrac{\mu}{1-\mu} - \tfrac{\alpha}{2}. \end{aligned}$$

Replacing in expression (20), we obtain:

$$\pi^A\left(p^{AR}, p^{B+}\right) = \tfrac{1-\mu}{4[1-\alpha(1-\mu)]}\left(p^{B+} + \tfrac{\mu}{1-\mu} - \tfrac{\alpha}{2}\right)^2.$$

Using (13), we find:

$$\pi^A\left(p^{AR}, p^{B+}\right) = \tfrac{\left[2-3\mu+4\mu^2-3\mu\alpha(1-\mu)\right]^2}{36\mu^2(1-\mu)[1-\alpha(1-\mu)]}.$$

The equilibrium profit is not lower than this deviation profit if and only if:

$$\begin{aligned} & \tfrac{[2(1+\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)} \geq \tfrac{\left[2-3\mu+4\mu^2-3\mu\alpha(1-\mu)\right]^2}{36\mu^2(1-\mu)[1-\alpha(1-\mu)]} \\ \Leftrightarrow\ & -9(1-\mu)\mu^2\alpha^2 + 3\mu\left(4-2\mu+\mu^2\right)\alpha - 4 - 5\mu^2 \geq 0. \end{aligned}$$

The roots of this quadratic expression in α are:

$$\alpha_c = \frac{1}{2\mu\,(1-\mu)} + \frac{1-\mu}{6\mu} \mp \frac{\sqrt{-8+16\mu+\mu^2}}{6\,(1-\mu)}.$$

For $\mu < 8-6\sqrt{2} \approx 0.485$, the roots are complex, which means that the inequality never holds. For $\mu \geq 8-6\sqrt{2}$, the inequality holds for α between the two real roots. In that case, a necessary condition for (19) to hold is:

$$\alpha \geq \frac{1}{2\mu\,(1-\mu)} + \frac{1-\mu}{6\mu} - \frac{\sqrt{-8+16\mu+\mu^2}}{6\,(1-\mu)}.$$

However, the above condition is incompatible with condition (18):

$$\begin{aligned} &\frac{1}{2\mu(1-\mu)} + \frac{1-\mu}{6\mu} - \frac{\sqrt{-8+16\mu+\mu^2}}{6(1-\mu)} < \frac{2(1+\mu)}{3\mu} \\ \Leftrightarrow\ & 3+1-2\mu+\mu^2-\mu\sqrt{-8+16\mu+\mu^2}-4(1-\mu^2)<0 \\ \Leftrightarrow\ & -2+5\mu<\sqrt{-8+16\mu+\mu^2}. \end{aligned}$$

Since $\mu > 0.4$, this condition is equivalent to: $2\mu^2 - 3\mu + 1 < 0 \ \Leftrightarrow\ \mu > \frac{1}{2}$.

But this deviation profit is only attainable if p^{AR} induces an indifferent consumer $\tilde{x} \in \left(\frac{1}{2}, 1\right)$. This is the case if and only if:

$$\begin{cases} p^{AR} > p^{B+} + \frac{\alpha}{2} - 1 \\ p^{AR} < p^{B+} - \frac{\alpha}{2} + \alpha\mu \end{cases} \Leftrightarrow \begin{cases} \alpha < \frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)} \\ \alpha < \frac{2+\mu}{3\mu(1-\mu)}. \end{cases}$$

The second condition is not restrictive. We can exclude, therefore, the existence of an equilibrium when $\alpha < \frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}$.

When, $\alpha > \max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}$, the most profitable deviation is to choose the price, p^{A1}, that induces $\tilde{x} = 1$. From (5), $p^{A1} = p^{B+} + \frac{\alpha}{2} - 1 = \frac{2-4\mu}{3\mu}$. The corresponding profit is $\pi^{A1} = \frac{2-4\mu}{3\mu}$.

A necessary condition for profit maximization when $\alpha > \max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}$ is:

$$\begin{aligned} \frac{[2(1+\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)} \geq \frac{2-4\mu}{3\mu} \ &\Leftrightarrow\ [2(1+\mu)-3\alpha\mu]^2 \geq 24(1-2\mu)(1-\alpha\mu) \\ &\Leftrightarrow\ 9\mu^2\alpha^2 + (12\mu - 60\mu^2)\alpha + 4\mu^2 + 56\mu - 20 \geq 0. \end{aligned}$$

Combining this condition with Assumption 1, we obtain:

$$\alpha \in \left(\max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}, \frac{10\mu - 2\sqrt{6}(1-2\mu) - 2}{3\mu}\right],$$

which is empty. □

Lemma 2.

If $\mu < \frac{1}{2}$*, there exists no equilibrium with* $\tilde{x} = \frac{1}{2}$*.*

Proof. The strategy of the proof is to show that $\tilde{x} = \frac{1}{2}$ is not compatible with profit-maximization by platform A.

From (3), an equilibrium with $\tilde{x} = \frac{1}{2}$ would have to be such that:

$$\begin{aligned}\frac{1}{2} = \frac{1}{2}\left[1 + \alpha(D^A - D^B) - (p^A - p^B)\right] \\ \Leftrightarrow\ \alpha\left(\frac{\mu}{2} - \frac{1-\mu}{2}\right) - p^A + p^B = 0 \\ \Leftrightarrow\ p^B = p^A + \alpha\left(\frac{1-2\mu}{2}\right).\end{aligned}$$

From (5), if $\tilde{x} \geq \frac{1}{2}$, the profit of platform A is given by:

$$\pi^A\Big|_{\tilde{x}\geq\frac{1}{2}} = 2p^A \frac{2(1-\mu)(p^B - p^A) + 2\mu - \alpha(1-\mu)}{4\left[1 - \alpha(1-\mu)\right]},$$

therefore, the derivative with respect to p^A is:

$$\frac{\partial \pi^A}{\partial p^A}\Bigg|_{\tilde{x}\geq\frac{1}{2}} = \frac{4(1-\mu)(p^B - p^A) + 4\mu - 2\alpha(1-\mu) - 4p^A(1-\mu)}{4\left[1 - \alpha(1-\mu)\right]}.$$

For the profit-maximizing price, p^A, to imply $\tilde{x} = \frac{1}{2}$, it is necessary that the derivative above is non-negative, i.e., that:

$$\begin{aligned}4(1-\mu)(p^B - 2p^A) + 4\mu - 2\alpha(1-\mu) \geq 0 \\ \Leftrightarrow\ 4(1-\mu)\left[\alpha\left(\frac{1-2\mu}{2}\right) - p^A\right] + 4\mu - 2\alpha(1-\mu) \geq 0 \\ \Leftrightarrow\ -4\alpha\mu(1-\mu) - 4(1-\mu)p^A + 4\mu \geq 0 \\ \Leftrightarrow\ p^A \leq \frac{\mu}{1-\mu}\left[1 - \alpha(1-\mu)\right].\end{aligned} \tag{21}$$

From (5), if $\tilde{x} \leq \frac{1}{2}$, the profit of platform A is given by:

$$\pi^A\Big|_{\tilde{x}\leq\frac{1}{2}} = 2p^A \frac{2\mu(p^B - p^A) + \mu(2-\alpha)}{4\,(1-\alpha\mu)},$$

therefore, the derivative with respect to p^A is:

$$\left.\frac{\partial \pi^A}{\partial p^A}\right|_{\tilde{x}\leq\frac{1}{2}} = \frac{4\mu(p^B - p^A) + 2\mu\,(2-\alpha) - 4\mu p^A}{4\,(1-\alpha\mu)}.$$

This derivative must be non-positive for the profit-maximizing price, p^A, to imply $\tilde{x} = \frac{1}{2}$. This occurs if and only if:

$$\begin{aligned} 4(p^B - 2p^A) + 2\,(2-\alpha) &\leq 0 \\ \Leftrightarrow\;\; 4\left[\alpha\left(\frac{1-2\mu}{2}\right) - p^A\right] + 4 - 2\alpha &\leq 0 \\ \Leftrightarrow\;\; 2\alpha\,(1-2\mu) - 4p^A + 4 - 2\alpha &\leq 0 \\ \Leftrightarrow\;\; p^A \geq 1 - \alpha\mu. \end{aligned} \tag{22}$$

But notice that (22) is incompatible with (21), because:

$$\begin{aligned} 1 - \alpha\mu &\leq \frac{\mu}{1-\mu}\left[1 - \alpha(1-\mu)\right] \\ \Leftrightarrow\;\; 1 - \alpha\mu &\leq \frac{\mu}{1-\mu} - \alpha\mu \\ \Leftrightarrow\;\; 1 - \mu &\leq \mu, \end{aligned}$$

which is only true for $\mu = \frac{1}{2}$. □

Proof of Proposition 2

From Lemmas 1 and 2, we know that an interior equilibrium must be such that $\tilde{x} \in \left(\frac{1}{2}, 1\right)$.

(i) First-order conditions for profit-maximization

Considering the demand functions restricted to $\tilde{x} \in \left[\frac{1}{2}, 1\right]$, which are given by (11), the first-order conditions for profit maximization are:

$$\begin{cases} \dfrac{\partial \pi^A}{\partial p^A} = 0 \\ \dfrac{\partial \pi^B}{\partial p^B} = 0. \end{cases}$$

Solving this system of equations, we obtain the equilibrium prices (6).

(ii) Local second-order conditions for profit-maximization

To check that the equilibrium prices satisfy the local second-order condition, calculate:

$$|H_1| = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{-1+\mu}{1-\alpha^2(1-\mu)^2}.$$

For the local second-order condition to be satisfied, it is necessary that:

$$|H_1| < 0 \;\Leftrightarrow\; \alpha < \frac{1}{1-\mu},$$

which is Assumption 1.

Computing the second-order cross partial derivatives we obtain:

$$\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A} = \frac{\partial^2 \pi^A}{\partial p_2^A \partial p_1^A} = \frac{-\alpha(1-\mu)^2}{1-\alpha^2(1-\mu)^2}.$$

Given symmetry:

$$\frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2}.$$

Therefore:

$$|H_2| = \left(\frac{\partial^2 \pi^A}{\partial {p_1^A}^2}\right)^2 - \left(\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A}\right)^2 = \frac{(1-\mu)^2 - \alpha^2(1-\mu)^4}{\left[1-\alpha^2(1-\mu)^2\right]^2}.$$

The local second-order condition is satisfied if:

$$|H_2| > 0 \;\Leftrightarrow\; 1-\alpha^2(1-\mu)^2 > 0 \;\Leftrightarrow\; \alpha < \frac{1}{1-\mu},$$

which is Assumption 1.

(iii) Interiority of the indifferent consumer

The indifferent consumer is located at:

$$\tilde{x}^* = \frac{1}{2} - \frac{\alpha(1-2\mu)}{12\left[1-\alpha(1-\mu)\right]} + \frac{1-2\mu}{6(1-\mu)}. \tag{23}$$

We must have $\tilde{x}^* \geq \frac{1}{2}$ (otherwise, this would not be an equilibrium):

$$\tilde{x}^* \geq \frac{1}{2} \;\Leftrightarrow\; \frac{1}{1-\mu} \geq \frac{\alpha}{2\left[1-\alpha(1-\mu)\right]} \;\Leftrightarrow\; \alpha \leq \frac{2}{3(1-\mu)}. \tag{24}$$

We also check that $\tilde{x}^* \leq 1$ (otherwise platform A would benefit from slightly increasing its price):

$$\tilde{x}^* \leq 1 \;\Leftrightarrow\; \frac{-2+\mu}{6(1-\mu)} \leq \frac{\alpha(1-2\mu)}{12\left[1-\alpha(1-\mu)\right]},$$

which is obviously true.

(iv) Global profit-maximization

Since the profit function of platform A is not globally quasi-concave, we must check that (given p^{B*}), the local maximizer of its profit function, p^{A*}, is a global maximum.

For this to be true, it is necessary that:

$$\pi^A\left(p^{A*}, p^{B*}\right) \geq \pi^A\left(p^A, p^{B*}\right),\; \forall p^A. \tag{25}$$

Denote by p^{AL} the local maximum of $\pi^A\left(\cdot, p^{B*}\right)$ in the branch that leads to $\tilde{x} \in \left[0, \frac{1}{2}\right]$.

From (5), in this branch, the profit function is given by:

$$\pi^A\left(p^A, p^{B*}\right) = 2p^A\left[\tfrac{\mu}{2(1-\alpha\mu)}\left(p^{B*} - p^A + 1 - \tfrac{\alpha}{2}\right)\right]. \tag{26}$$

The first-order condition for profit maximization in this branch yields:

$$\begin{aligned} &\tfrac{\partial \pi^A\left(p^A, p^{B*}\right)}{\partial p^A} = 0 \\ \Leftrightarrow\ &\tfrac{\mu}{2(1-\alpha\mu)}\left(p^{B*} - p^{AL} + 1 - \tfrac{\alpha}{2}\right) = p^{AL}\tfrac{\mu}{2(1-\alpha\mu)} \\ \Leftrightarrow\ &2p^{AL} = p^{B*} + 1 - \tfrac{\alpha}{2}. \end{aligned}$$

From (6), we obtain:

$$p^{AL} = \tfrac{5-4\mu}{6(1-\mu)} - \tfrac{\alpha}{2}.$$

This price is in the domain that leads to $\tilde{x} \in \left(0, \frac{1}{2}\right)$ if and only if:

$$\begin{cases} p^{AL} < p^{B*} - \frac{\alpha}{2} + 1 \\ p^{AL} > p^{B*} - \frac{\alpha}{2} + \alpha\mu \end{cases} \Leftrightarrow\ \alpha < \frac{5-4\mu}{3(1-\mu)},$$

which holds under condition (24).

Substituting p^{AL} in expression (26), we obtain:

$$\pi^A\left(p^{AL}, p^{B*}\right) = \tfrac{\mu}{(1-\alpha\mu)}\left[\tfrac{5-4\mu}{6(1-\mu)} - \tfrac{\alpha}{2}\right]^2. \tag{27}$$

Comparing this deviation profit with the equilibrium profit in (8), we conclude that the equilibrium condition (25) holds if and only if:

$$\frac{\left[2(1+\mu) - 3\alpha(1-\mu)\right]^2}{36(1-\mu)\left[1-\alpha(1-\mu)\right]} \geq \frac{\mu}{(1-\alpha\mu)}\left[\frac{5-4\mu}{6(1-\mu)} - \frac{\alpha}{2}\right]^2.$$

This condition is equivalent to non-negativity of the following polynomial:

$$9(1-\mu)^2\alpha^2 - 3(1-\mu)(4-\mu)\alpha + 4 - 5\mu \geq 0.$$

The roots of the polynomial are:

$$\alpha_c = \frac{4-\mu}{6\,(1-\mu)} \pm \frac{\sqrt{12\mu+\mu^2}}{6\,(1-\mu)}.$$

The equilibrium condition holds for α lower than the inferior root and for α greater than the superior root. However, while any α lower than the inferior root satisfies condition (24), any α greater than the superior root violates condition (24).

We conclude, therefore, that (25) is satisfied if and only if:

$$\alpha \leq \frac{4-\mu}{6\,(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6\,(1-\mu)}.$$

□

Proof of Proposition 3

An equilibrium with tipping in favor of platform B requires that $p^A = 0$ and $\tilde{x} = 0$. From the expression for the indifferent consumer (3), we obtain:

$$0 = 1 - \frac{\alpha}{2} + p^B \;\Leftrightarrow\; p^B = \frac{\alpha}{2} - 1.$$

Analogously, tipping in favor of platform A requires: $p^B = 0$ and $\tilde{x} = 1$. This implies, from (3):

$$2 = 1 + \frac{\alpha}{2} - p^A \;\Leftrightarrow\; p^A = \frac{\alpha}{2} - 1.$$

If $\alpha \leq 2$, prices are null, which implies that a small unilateral increase is profitable. □

10.3 Demand uniqueness at the interior equilibrium

Here, we investigate the existence of multiple possible demands for given prices, i.e., the existence of multiple equilibria in game in which consumers, for given prices, choose which platform to join.

Platform A can capture all demand if and only if it can attract the consumer that is located at $x = 1$:

$$u^A(1) \geq u^B(1) \;\Leftrightarrow\; V + \frac{\alpha}{2} - p^A - 1 \geq V + 0 - p^B - 0 \;\Leftrightarrow\; p^B - p^A \geq 1 - \frac{\alpha}{2}.$$

Similarly, platform B can capture all demand if and only if it can attract the consumer that is located at $x = 0$:

$$u^B(0) \geq u^A(0) \;\Leftrightarrow\; p^A - p^B \geq 1 - \frac{\alpha}{2}.$$

For the interior equilibrium prices, we have:

$$p^{B*} - p^{A*} = \frac{1 - 2\mu}{3(1 - \mu)}.$$

The interior equilibrium demand is unique if and only if:

$$\frac{1 - 2\mu}{3(1 - \mu)} < 1 - \frac{\alpha}{2} \;\Leftrightarrow\; \alpha < \frac{2(2 - \mu)}{3(1 - \mu)},$$

which is always satisfied. □

10.4 Characterization of the interior equilibrium

10.4.1 Impact of α on market shares ($\frac{\partial D^{A*}}{\partial \alpha} < 0$)

To verify that $\frac{\partial D^{A*}}{\partial \alpha} < 0$, calculate:

$$\frac{\partial D^{A*}}{\partial \alpha} = \frac{-3(1-\mu)12\left[1-\alpha(1-\mu)\right] + 12(1-\mu)\left[2(1+\mu) - 3\alpha(1-\mu)\right]}{144\left[1-\alpha(1-\mu)\right]^2},$$

and notice that $\frac{\partial D^{A*}}{\partial \alpha} < 0$ if and only if:

$$36\left[1-\alpha(1-\mu)\right] > 12\left[2(1+\mu) - 3\alpha(1-\mu)\right] \;\Leftrightarrow\; \mu < \frac{1}{2}.$$

10.4.2 Impact of μ on profits ($\frac{\partial \pi^{B*}}{\partial \mu}$)

From (8), we can calculate:

$$\frac{\partial \pi^{B*}}{\partial \mu} = \frac{\left[4-2\mu-3\alpha(1-\mu)\right]\left[2\mu-\alpha(1-\mu)\right]}{36\left[1-\alpha(1-\mu)\right]^2(1-\mu)^2}.$$

Therefore:

$$\begin{aligned}\frac{\partial \pi^{B*}}{\partial \mu} > 0 \;&\Leftrightarrow\; \left[4-2\mu-3\alpha(1-\mu)\right]\left[2\mu-\alpha(1-\mu)\right] > 0\\ &\Leftrightarrow\; 3(1-\mu)^2\alpha^2 - 4(1-\mu^2)\alpha + 8\mu - 4\mu^2 > 0\end{aligned}$$

The roots of this second-degree polynomial in α are:

$$r_1 = \frac{2\mu}{1-\mu} \quad \text{and} \quad r_2 = \frac{4-2\mu}{3(1-\mu)}.$$

Since the value of the second root is outside the domain of existence of equilibrium, we conclude that:

$$\begin{cases} \dfrac{\partial \pi^{B*}}{\partial \mu} > 0, & \text{if } \alpha \in \left(0, \dfrac{2\mu}{1-\mu}\right) \\ \dfrac{\partial \pi^{B*}}{\partial \mu} < 0, & \text{if } \alpha \in \left(\dfrac{2\mu}{1-\mu}, \dfrac{4-\mu}{6\,(1-\mu)} - \dfrac{\sqrt{12\mu+\mu^2}}{6\,(1-\mu)}.\right). \end{cases}$$

10.4.3 Impact of α on profits ($\frac{\partial \pi^{B*}}{\partial \alpha} < 0$)

From (8), we can calculate:

$$\frac{\partial \pi^{B*}}{\partial \alpha} = \frac{-\left[4-2\mu-3\alpha(1-\mu)\right]\left[2(1+\mu)-3\alpha(1-\mu)\right]}{36\left[1-\alpha(1-\mu)\right]^2}.$$

Therefore:

$$\frac{\partial \pi^{B*}}{\partial \alpha} > 0 \;\Leftrightarrow\; \left[4-2\mu-3\alpha(1-\mu)\right]\left[2(1+\mu)-3\alpha(1-\mu)\right] < 0.$$

Since $\alpha \leq \frac{2}{3(1-\mu)}$, the second parcel is surely positive. Thus:

$$\frac{\partial \pi^{B*}}{\partial \alpha} > 0 \;\Leftrightarrow\; 4-2\mu-3\alpha(1-\mu) < 0 \;\Leftrightarrow\; \alpha > \frac{4-2\mu}{3(1-\mu)},$$

which is always false.

10.5 Social welfare maximization

If $\tilde{x} \in \left[0, \frac{1}{2}\right]$, social welfare is given by:

$$W(\tilde{x}) = 2\left\{\int_0^{\tilde{x}} \left(V + \alpha D^A - x\right)\mu dx + \int_{\tilde{x}}^{\frac{1}{2}} \left[V + \alpha D^B - (1-x)\right]\mu dx \right.$$
$$\left. + \int_{\frac{1}{2}}^{1} \left[V + \alpha D^B - (1-x)\right](1-\mu)dx \right\}$$
$$= V - \frac{1}{4} - \frac{\mu}{2} + 2\mu\tilde{x}(1-\tilde{x}) + 2\alpha\mu\tilde{x}(D^A - D^B) + \alpha D^B.$$

Since $D^A = \mu\tilde{x}$ and $D^B = \frac{1}{2} - \mu\tilde{x}$, we obtain:

$$W(\tilde{x}) = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \tilde{x}^2\left[-2\mu(1-2\alpha\mu)\right] + \tilde{x}\left[2\mu(1-\alpha)\right].$$

If $\alpha < 1$, this is a concave function, with maximum at:

$$\tilde{x}^{FB} = \frac{1-\alpha}{2\left(1-2\alpha\mu\right)}.$$

With $\tilde{x} = \tilde{x}^{FB}$, welfare is given by:

$$W^{FB} = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \frac{\mu(1-\alpha)^2}{2(1-2\alpha\mu)}.$$

Now let us check that welfare cannot be greater for some $\tilde{x} \in \left(\frac{1}{2}, 1\right]$. At $\tilde{x} = 1$, welfare is lower than at $\tilde{x} = 0$, because total transportation costs are higher. Therefore, we only need to check the possible interior maximum.

With $\tilde{x} \in \left(\frac{1}{2}, 1\right]$, welfare is given by:

$$W(\tilde{x}) = 2\left\{\int_0^{\frac{1}{2}} (V + \alpha D^A - x)\mu dx + \int_{\frac{1}{2}}^{\tilde{x}} (V + \alpha D^A - x)(1-\mu)dx \right.$$
$$\left. + \int_{\tilde{x}}^{1} (V + \alpha D^B - (1-x))(1-\mu)dx \right\}.$$

Replacing $D^A = \frac{\mu}{2} + (\tilde{x} - \frac{1}{2})(1-\mu)$ and $D^B = (1-\tilde{x})(1-\mu)$, and simplyfying, we obtain:

$$\begin{aligned} W(\tilde{x}) = V &- \frac{3}{4} + \frac{\mu}{2} + \frac{5\alpha}{2} - 6\alpha\mu + 4\alpha\mu^2 - 2\tilde{x}^2[1 - 2\alpha(1-\mu)](1-\mu) \\ &+ 2\tilde{x}(1-\mu)[1 - \alpha(3 - 4\mu)]. \end{aligned}$$

For $\alpha < \frac{1}{2(1-\mu)}$, this is a concave function, with maximum at:

$$\tilde{x}^+ = \frac{1 - \alpha(3 - 4\mu)}{2 - 4\alpha(1-\mu)}.$$

This potential maximizer is inside the domain if and only if:

$$\alpha(3 - 4\mu) \leq 2\alpha(1-\mu) \;\Leftrightarrow\; 3 - 4\mu \leq 2 - 2\mu \;\Leftrightarrow\; 1 \leq 2\mu,$$

which is false. The constrained maximum is not interior, therefore, it is lower than the constrained maximum at $\tilde{x} \in \left[0, \frac{1}{2}\right]$.

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