Overlapping Generations Model

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Description: Model of Overlapping Generations as presented in chapter 3 of Blanchard & Fisher's "*Lectures on Macroeconomics*" (1989).

Abstract. The overlapping generations model is introduced. Agents work only in the first period. In the second period they consume what they saved in the first period. *Steady-state* existence and properties are analyzed. The result of the planned economy is compared to the equilibrium of the decentralized economy. Further results are obtained by a generalization of the behavior of the agents that allows altruism, and by the introduction of "*fully funded*" or "*pay-as-you-go*" social security systems.

A model of perpetual youth is presented as complementary to the two-period life model. A continuum of generations constitutes the population. Agents face a constant probability of dying and are allowed to make negative life insurance.

Using both models, as well as the model of Ramsey, fiscal policy is analysed. Debt finance is compared with deficit finance, and the effect of the interest rates is investigated. The relation between aggregate saving and interest rates is also examined in the light of the three models. Finally some illustrative exercises are proposed and solved.

Keywords: Overlapping generations. Pareto optimality. Golden rule of capital accumulation. Social security. Model of perpetual youth. Fiscal policy. Ricardian equivalence. Aggregate saving.

[1] The model presented here is based on Olivier Blanchard and Stanley Fisher's "*Lectures on Macroeconomics*" (1989, chapter 3, pp. 91-153) and uses similar notation.

1 Introduction

In the 2-period overlapping generations model, agents live for 2 periods, working only in the first period. They consume part of their income in the first period, and save the remaining for consumption in the second period.

Two polar kinds of economies are analyzed and compared. First we set-up a decentralized economy, in which agents seek the maximization of their individual satisfaction, deciding between consuming in the first period and saving for consumption in the second period. Then we examine a command (or planned) economy, where decisions are taken by a benevolent dictator and imposed upon agents. One of the central features of the OLG model is that competitive equilibrium (which prevails in the decentralized economy) is not necessarily Pareto-optimal.

The overlapping generations model is based in the seminal contributions of Allais (1947), Samuelson (1958) and Diamond (1965).

1.1 Setting Up the Model

We start by presenting the notation to be used:

- Agents work only in the first period, receiving the wage w_t;
- c_{1t} designates consumption of the generation born in t in the first period;
- c_{2t+1} designates the second period consumption of the same generation;
- s_t designates the savings of the young generation in period t;
- w_t designates the wages received by the young generation in t.

- r_t designates the interest rate from $t-1$ to t;

The preferences of the agents are described by a utility function that is separable in time and concave in each period's consumption. The generation born in t has the following utility:

- $U(t) = u(c_{1t}) + \frac{1}{(1+\theta)} \cdot u(c_{2t+1}) \qquad , \ \theta \leq 0, \ u'(\cdot) > 0, \ u''(\cdot) < 0.$

Other assumptions:

- Consumption (and income) in the second period is given by: $c_{2t+1} = (1 + r_{t+1}) \cdot s_t$;
- Population grows exogenously, according to: $N_t = N_0 \cdot (1+n)^t$;
- The production function is neoclassical, $Y = F(K, N)$, having constant returns to scale (can be expressed as $y = f(k)$) but decreasing returns in each factor. It is also in accordance with the conditions of Inada;
- Firms seek to maximize profits, taking w_t and r_t as given.

1.2 Decentralized Equilibrium

Problem of the individuals: constrained maximization of utility.

$$max \ U_t = u(c_{1t}) + \frac{1}{(1+\theta)} \cdot u(c_{2t+1})$$

$$s.t. \begin{cases} c_{1t} + s_t = w_t \\ c_{2t+1} = (1 + r_{t+1}) \cdot s_t \end{cases}$$

Considering s_t as the only decision variable:

$$max\ U_t = u(w_t - s_t) + \tfrac{1}{(1+\theta)} \cdot u((1+r_{t+1}) \cdot s_t)$$

$$\Rightarrow\ \tfrac{dU_t}{ds_t} = 0\ \Rightarrow\ u(w_t - s_t) \cdot (-1) + \tfrac{1}{(1+\theta)} \cdot u'((1+r_{t+1}) \cdot s_t) \cdot (1+r_{t+1}) = 0.$$

Simplifying the first order condition, we obtain:

$$u'(c_{1t}) - \frac{1+r_{t+1}}{1+\theta} \cdot u'(c_{2t+1}) = 0\ \Rightarrow\ \frac{u'(c_{1t})}{u'(c_{2t+1})} = \frac{1+r_{t+1}}{1+\theta}$$

Savings clearly vary positively with w_t, but its relation with r_{t+1} is ambiguous.

Problem of the firms: profit maximization taking w_t and r_t as given.

We consider competitive markets, with wages equal to the marginal productivity of labor, and interest rates equal to the marginal productivity of capital.

- $MPL_t = w_t \Leftrightarrow \tfrac{\partial F(K_t,N_t)}{\partial N_t} = \tfrac{\partial [N_t \cdot f(k_t)]}{\partial N_t} = f(k_t) + N_t \cdot \tfrac{\partial f(k_t)}{\partial N_t} = w_t \Leftrightarrow$
  $\Leftrightarrow f(k_t) + N_t \cdot \tfrac{\partial f(k_t)}{\partial k_t}\tfrac{\partial k_t}{\partial N_t} = f(k_t) + N_t \cdot \tfrac{\partial f(k_t)}{\partial k_t} \cdot \tfrac{\partial (K_t/N_t)}{\partial N_t} = w_t \Leftrightarrow$
  $\Leftrightarrow f(k_t) + N_t \cdot f'(k_t) \cdot K_t \cdot (-\tfrac{1}{N_t^2}) = w_t$
  $\Leftrightarrow f(k_t) - k_t \cdot f'(k_t) = w_t.$

- $MPK_t = r_t \Leftrightarrow \tfrac{\partial F(K_t,N_t)}{\partial K_t} = \tfrac{\partial [N_t \cdot f(k_t)]}{\partial K_t} = N_t \cdot \tfrac{\partial f(k_t)}{\partial k_t} \cdot \tfrac{\partial k_t}{\partial K_t} = r_t \Leftrightarrow$
  $\Leftrightarrow N_t \cdot f'(k_t) \cdot \tfrac{\partial (K_t/N_t)}{\partial K_t} = N_t \cdot f'(k_t) \cdot \tfrac{1}{N_t} \cdot 1 = r_t \Leftrightarrow$
  $\Leftrightarrow f'(k_t) = r_t.$

Goods market equilibrium: demand equals supply and (equivalently) investment equals saving.

The equality between net investment and net saving is simple, because all the capital stock consists of the savings of the young generation:

$$K_{t+1} - K_t = N_t \cdot s(w_t, r_{t+1}) - K_t\ \Leftrightarrow\ k_{t+1} = \frac{s(w_t, r_{t+1})}{1+n}.$$

1.3 Dynamics and Steady States

The capital intensity evolves according to:

$$k_{t+1} = \frac{s[w(k_t), r(k_{t+1})]}{1+n} \Rightarrow \; k_{t+1} = \frac{s[f(k_t) - k_t \cdot f'(k_t), f'(k_{t+1})]}{1+n}.$$

This relation between k_t and k_{t+1} is called the "*saving locus*". To study its properties, we evaluate the sign of the derivative:

$$(1+n) \cdot \frac{\partial k_{t+1}}{\partial k_t} = \frac{\partial s}{\partial w_t} \cdot \frac{\partial w_t}{\partial k_t} + \frac{\partial s}{\partial r_{t+1}} \cdot \frac{\partial r_{t+1}}{\partial k_t} \;\Rightarrow$$
$$\Rightarrow\; (1+n) \cdot \frac{\partial k_{t+1}}{\partial k_t} = s_w \cdot [f'(k_t) - f'(k_t) - k_t \cdot f''(k_t)] + s_r \cdot f''(k_{t+1}) \cdot \frac{\partial k_{t+1}}{\partial k_t} \;\Rightarrow$$
$$\Rightarrow\; \frac{\partial k_{t+1}}{\partial k_t} \cdot [1 + n - s_r \cdot f''(k_{t+1}] = s_w \cdot [-k_t \cdot f''(k_t)] \;\Rightarrow$$
$$\Rightarrow\; \frac{\partial k_{t+1}}{\partial k_t} = \frac{-s_w(k_t) \cdot k_t \cdot f''(k_t)}{1 + n - s_r(k_{t+1}) \cdot f''(k_{t+1})}.$$

The sign of the numerator is positive, while the sign of the denominator is ambiguous (if s_r is positive, then the denominator is unambiguously positive):

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{- \oplus \cdot \oplus \cdot \ominus}{\oplus - (?) \cdot \ominus}.$$

Besides the well behaved situation of a single steady state, there may be none or multiple steady states. See figure 3.1 (Blanchard & Fisher, p. 95).

Local stability of a steady state requires the absolute value of the point derivative to be smaller than one:

$$\left| \frac{\partial k_{t+1}}{\partial k_t} \right| = \left| \frac{-s_w(k^*) \cdot k^* \cdot f''(k^*)}{1 + n - s_r(k^*) \cdot f''(k^*)} \right| < 1.$$

In the figure this corresponds to a "*saving locus*" that crosses the 45 degree line from above.

1.4 Optimality Properties

Here we consider that a central planner maximizes the present discounted value of current and future utilities, using a social discount rate. We show that the market outcome may not be Pareto-optimal. Inefficiency arises when the economy accumulates more capital than that implied by the golden rule.

Planned Equilibrium with Finite Horizon: maximization of social welfare.

The social welfare function, which the planner wishes to maximize includes the social discount rate R. For now we assume that the planner only cares about the utility of the generations born until $T-1$.

$$U = \frac{1}{1+\theta} \cdot u(c_{20}) + \sum_{t=0}^{T-1} \left\{ \frac{1}{(1+R)^{t+1}} \cdot [u(c_{1t}) + \frac{1}{(1+\theta)} \cdot u(c_{2t+1})] \right\}.$$

In extensive form:

$$\begin{aligned}
U = &\tfrac{1}{1+\theta} \cdot u(c_{20}) + \\
&+ \tfrac{1}{(1+R)} \cdot [u(c_{10}) + \tfrac{1}{(1+\theta)} \cdot u(c_{21})] + \\
&+ \tfrac{1}{(1+R)^2} \cdot [u(c_{11}) + \tfrac{1}{(1+\theta)} \cdot u(c_{22})] + \\
&+ \ldots + \\
&+ \tfrac{1}{(1+R)^{t+1}} \cdot [u(c_{1t}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2t+1})] + \\
&+ \ldots + \\
&+ \tfrac{1}{(1+R)^T} \cdot [u(c_{1T-1}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2T})].
\end{aligned}$$

In each period, total production must be allocated between consumption of the young generation, consumption of the old generation, and capital accumulation (savings of the young generation):

$$\begin{aligned}
&K_t + F(K_t, N_t) = K_{t+1} + N_t \cdot c_{1t} + N_{t-1} \cdot c_{2t} \;\Leftrightarrow \\
\Leftrightarrow\;& k_t + f(k_t) = (1+n) \cdot k_{t+1} + c_{1t} + \tfrac{1}{1+n} \cdot c_{2t} \;\Leftrightarrow \\
\Leftrightarrow\;& c_{1t} = k_t + f(k_t) - (1+n) \cdot k_{t+1} - \tfrac{1}{1+n} \cdot c_{2t}.
\end{aligned}$$

Now we substitute this c_{1t} in the social welfare function. Notice that k_0 and k_{T+1} are given (restrictions of the planner):

$$
\begin{aligned}
U = {} & \tfrac{1}{1+\theta} \cdot u(c_{20}) + \\
& + \tfrac{1}{(1+R)} \cdot [u(k_0 + f(k_0) - (1+n) \cdot k_1 - \tfrac{1}{1+n} \cdot c_{20}) + \tfrac{1}{(1+\theta)} \cdot u(c_{21})] + \\
& + \tfrac{1}{(1+R)^2} \cdot [u(k_1 + f(k_1) - (1+n) \cdot k_2 - \tfrac{1}{1+n} \cdot c_{21}) + \tfrac{1}{(1+\theta)} \cdot u(c_{22})] + \\
& + \ldots + \\
& + \tfrac{1}{(1+R)^t} \cdot [u(k_{t-1} + f(k_{t-1}) - (1+n) \cdot k_t - \tfrac{1}{1+n} \cdot c_{2t-1}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2t})] + \\
& + \tfrac{1}{(1+R)^{t+1}} \cdot [u(k_t + f(k_t) - (1+n) \cdot k_{t+1} - \tfrac{1}{1+n} \cdot c_{2t}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2t+1})] + \\
& + \tfrac{1}{(1+R)^{t+2}} \cdot [u(k_{t+1} + f(k_{t+1}) - (1+n) \cdot k_{t+2} - \tfrac{1}{1+n} \cdot c_{2t+1}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2t+2})] + \\
& + \ldots + \\
& + \tfrac{1}{(1+R)^T} \cdot [u(k_{T-1} + f(k_{T-1}) - (1+n) \cdot k_T - \tfrac{1}{1+n} \cdot c_{2T-1}) + \tfrac{1}{(1+\theta)} \cdot u(c_{2T})].
\end{aligned}
$$

Differentiating with respect to c_{2t} and k_t gives us two first order conditions for the finite horizon planned optimum. The first condition optimally distributes the consumption between the two generations that are alive in each period. The second is a condition of inter-temporal optimality:

$$
\begin{cases}
c_{2t}: & \tfrac{1}{1+\theta} \cdot u'(c_{2t}) - \tfrac{1}{1+R} \cdot \tfrac{1}{1+n} \cdot u'(c_{1t}) = 0 \\
k_t: & -(1+n) \cdot u'(c_{1t-1}) + \tfrac{1}{1+R} \cdot [1 + f'(k_t)] \cdot u'(c_{1t}) = 0
\end{cases}
$$

We reach a characterization of steady states:

$$
\begin{cases}
c_2^*: & \tfrac{1}{1+\theta} \cdot u'(c_2^*) - \tfrac{1}{1+R} \cdot \tfrac{1}{1+n} \cdot u'(c_1^*) = 0 \\
k^*: & [1 + f'(k^*)] \cdot u'(c_1^*) = (1+R) \cdot (1+n) \cdot u'(c_1^*)
\end{cases}
\Rightarrow
$$

$$
\Rightarrow
\begin{cases}
c_2^*: & \tfrac{1}{1+\theta} \cdot u'(c_2^*) = \tfrac{1}{1+R} \cdot \tfrac{1}{1+n} \cdot u'(c_1^*) \\
k^*: & 1 + f'(k^*) = (1+R) \cdot (1+n)
\end{cases}
$$

The second relation corresponds to the modified golden rule. With small R and n, that is, with small periods, it is approximately given by $f'(k^*) = R + n$.

A Turnpike Theorem: the golden rule of capital accumulation *versus* the finite horizon's planned solution.

Linearizing the system around the steady state gives us several relations. One between consumption in the first and the second periods, derived from the first order condition on c_{2t}:

$$
\begin{aligned}
&u'(c_{2t}) = \tfrac{1+\theta}{(1+R)\cdot(1+n)} \cdot u'(c_{1t}) \;\Rightarrow \\
\Rightarrow\;& u_2^{'} + u_2^{''} \cdot (c_{2t} - c_2^*) = \tfrac{1+\theta}{(1+R)\cdot(1+n)} \cdot u_1^{'} + u_1^{''} \cdot (c_{1t} - c_1^*) \;\Rightarrow \\
\Rightarrow\;& u_2^{''} \cdot (c_{2t} - c_2^*) = \tfrac{1+\theta}{(1+R)\cdot(1+n)} \cdot u_1^{''} \cdot (c_{1t} - c_1^*) \;\Rightarrow \\
\Rightarrow\;& c_{2t} - c_2^* = \tfrac{1+\theta}{(1+R)\cdot(1+n)} \cdot \tfrac{u_1^{''}}{u_2^{''}} \cdot (c_{1t} - c_1^*) \\
\Rightarrow\;& c_{2t} - c_2^* = \tfrac{u_2^{'}}{u_1^{'}} \cdot \tfrac{u_1^{''}}{u_2^{''}} \cdot (c_{1t} - c_1^*).
\end{aligned}
$$

Which results in:

$$
\begin{cases}
c_{2t} - c_2^* = \tfrac{u_2^{'}}{u_1^{'}} \cdot \tfrac{u_1^{''}}{u_2^{''}} \cdot (c_{1t} - c_1^*) \\
c_{2t-1} - c_2^* = \tfrac{u_2^{'}}{u_1^{'}} \cdot \tfrac{u_1^{''}}{u_2^{''}} \cdot (c_{1t-1} - c_1^*) \\
c_{2t} - c_{2t-1} = \tfrac{u_2^{'}}{u_1^{'}} \cdot \tfrac{u_1^{''}}{u_2^{''}} \cdot (c_{1t} - c_{1t-1})
\end{cases}
$$

Linearizing the first order condition on k_t, we arrive to a relation between consumption in periods $t-1$ and t:

$$
\begin{aligned}
&(1+n) \cdot u'(c_{1t-1}) = \tfrac{1}{1+R} \cdot [1 + f'(k_t)] \cdot u'(c_{1t}) \;\xrightarrow{lin} \\
\xrightarrow{lin}\;& u_1^{'} + u_1^{''} \cdot (c_{1t-1} - c_1^*) = \tfrac{1+f'(k_t)}{(1+R)\cdot(1+n)} \cdot [u_1^{'} + u_1^{''} \cdot (c_{1t} - c_1^*)] \;\Rightarrow \\
\Rightarrow\;& c_{1t-1} - c_1^* = \tfrac{1+f'(k_t)}{(1+R)\cdot(1+n)} \cdot [\tfrac{u_1^{'}}{u_1^{''}} + (c_{1t} - c_1^*)] - \tfrac{u_1^{'}}{u_1^{''}} \;\Rightarrow \\
\Rightarrow\;& c_{1t-1} - c_1^* - \tfrac{1+f'}{(1+R)\cdot(1+n)} \cdot (c_{1t} - c_1^*) = \\
=\;& \tfrac{f''\cdot(k_t-k^*)}{(1+R)\cdot(1+n)} \cdot (c_{1t} - c_1^*) + [\tfrac{1+f'+f''\cdot(k_t-k^*)}{(1+R)\cdot(1+n)} - 1] \cdot \tfrac{u_1^{'}}{u_1^{''}} \;\Rightarrow \\
\Rightarrow\;& (c_{1t-1} - c_1^*) - (c_{1t} - c_1^*) = \tfrac{f''\cdot(k_t-k^*)}{1+f'} \cdot (c_{1t} - c_1^* + \tfrac{u_1^{'}}{u_1^{''}}) \;\Rightarrow \\
\Rightarrow\;& (c_{1t-1} - c_{1t}) = \tfrac{f''\cdot(k_t-k^*)}{1+f'} \cdot (c_{1t} - c_1^* + \tfrac{u_1^{'}}{u_1^{''}}) \;\Rightarrow
\end{aligned}
$$

Now let's recover and linearize the constraints on capital accumulation in $t-1$ and t:

$$\begin{cases} c_{1t-1} = k_{t-1} + f(k_{t-1}) - (1+n) \cdot k_t - \frac{1}{1+n} \cdot c_{2t-1} \\ c_{1t} = k_t + f(k_t) - (1+n) \cdot k_{t+1} - \frac{1}{1+n} \cdot c_{2t} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{1+R}{1+f'} \cdot c_{1t-1} = \frac{1+R}{1+f'} \cdot [k_{t-1} + f(k_{t-1})] - \frac{(1+R)\cdot(1+n)}{1+f'} \cdot k_t - \frac{1+R}{(1+f'\cdot(1+n)} \cdot c_{2t-1} \\ -\frac{1}{1+n} \cdot c_{1t} = -\frac{1}{1+n} \cdot [k_t + f(k_t)] + k_{t+1} - \frac{1}{(1+n)^2} \cdot c_{2t} \end{cases}$$

With further manipulation, it is possible to arrive at a second-order difference equation in $(k - k^*)$:

$$(k_{t+1} - k^*) - (2 + R + a) \cdot (k_t - k^*) + (1 + R) \cdot (k_{t-1} - k^*) = 0.$$

Where: $a \equiv \left[\frac{f'' \cdot u_1'}{(1+n)\cdot(1+f')\cdot u_1''}\right] \cdot \left\{1 + \frac{(1+\theta)\cdot u_1''}{(1+n)\cdot(1+f')\cdot u_2''}\right\} \geq 0.$

The characteristic equation associated with the difference equation is:

$$G(x) = x^2 - (2 + R + a) \cdot x + (1 + R) = 0$$
$$r = \frac{(2+R+a)\pm\sqrt{(2+R+a)^2-4\cdot(1+R)}}{2}.$$

Is is easy to see that the argument of the square root is positive:

$$(2 + R)^2 = 4 + 4R + R^2 > 4 \cdot (1 + R) \Longrightarrow (2 + R + a)^2 - 4 > (1 + R) \geq 0.$$

Therefore, the characteristic equation has two real roots. Since the roots lie where the sign of $G(x)$ changes, r_1 is between 0 and 1, and r_2 is greater than 1:

$$G(0) = 1 + R > 0 \quad ; \quad G(1) = 1 - 2 - R - a + 1 + R = -a < 0 \quad ; \quad G(\infty) > 0.$$

With two real roots, the solutions of the difference equation are given by:

$$x_t = (k_t - k^*) = \mu_1 \cdot x_1^t + \mu_2 \cdot x_2^t.$$

The initial and terminal conditions are:

$$\begin{cases} x_0 = (k_0 - k^*) = \mu_1 \cdot x_1^0 + \mu_2 \cdot x_2^0 = \mu_1 + \mu_2 \\ x_{T+1} = (k_{T+1} - k^*) = \mu_1 \cdot x_1^{T+1} + \mu_2 \cdot x_2^{T+1} \end{cases}$$

With $T+1$ large, x_1^{T+1} gets close to 1 and x_2^{T+1} becomes very large. So, μ_1 must be close to $k_0 - k^*$, and μ_2 must be close to zero.

With $T+1$ large, figure 3.3 shows the path of capital accumulation. Capital is close to the steady state value for most of the time. This is the turnpike property: the best way to go from any k_0 to any k_T is to stay close to k^* for a long time. So, even in a finite horizon program, the modified golden rule is very significant. The infinite horizon problem is the limit case when T tends to infinity.

The Special Significance of the Golden Rule:

The planned economy converges to a steady state where:

$1 + f'(k^*) = (1+R) \cdot (1+n)$.

We now wonder if a certain steady state (different, in general, from the planned optimum) is at least a Pareto optimum. Defining $c_t \equiv c_{1t} + \frac{c_{2t}}{1+n}$, and recovering the accumulation equation:

$$k_t + f(k_t) = (1+n) \cdot k_{t+1} + c_{1t} + \tfrac{1}{1+n} \cdot c_{2t} \implies$$
$$\implies k_t + f(k_t) = (1+n) \cdot k_{t+1} + c_t \implies$$
$$\implies f(k^*) - n \cdot k^* = c^* \implies \tfrac{\partial c^*}{\partial k^*} = f'(k^*) - n$$
$$\implies \begin{cases} k^* = k_{GR} \Rightarrow f'(k^*) = n \Rightarrow \frac{\partial c^*}{\partial k^*} = 0 \\ k^* < k_{GR} \Rightarrow f'(k^*) > n \Rightarrow \frac{\partial c^*}{\partial k^*} > 0 \\ k^* > k_{GR} \Rightarrow f'(k^*) < n \Rightarrow \frac{\partial c^*}{\partial k^*} < 0 \end{cases}$$

This is what is meant by the "*special significance of the golden rule*": if the steady state capital exceeds the golden rule level, a decrease in the capital stock increases steady state consumption. This happens because the capital stock became so large that its productivity is insufficient to supply the resources that are necessary to maintain capital intensity. Obviously, steady states with capital above the golden rule level are not Pareto optima. Everyone can be made better off through a marginal decrease in the stock of capital. We call these economies that have excessive capital accumulation "*dynamically inefficient*".

The Market Economy and Altruism:

Now let's assume that each generation cares about the utility of the next generation. Since the generation $t+1$ cares about generation $t+2$, generation t also cares about generation $t+2$. By the same reasoning we find that the intergenerational links imply that generation t cares about all future generations. The utility function becomes similar to the objective function of the planner:

$$V_t = \sum_{i=0}^{\infty} \left\{ \frac{1}{(1+R)^i} \cdot \left[u(c_{1t+i}) + \frac{1}{(1+\theta)} \cdot u(c_{2t+1+i}) \right] \right\}.$$

Receiving the bequest b_t in the first period, the budget constraint of generation t becomes:

$$\begin{cases} c_{1t} + s_t = w_t + b_t \\ c_{2t+1} + (1+n) \cdot b_{t+1} = (1 + r_{t+1}) \cdot s_t \end{cases}$$

Competition in factor markets, and equilibrium in the goods market still implies that:

$$\begin{cases} w_t = f(k_t) - k_t \cdot f'(k_t) \\ r_t = f'(k_t) \\ k_t + f(k_t) = (1+n) \cdot k_{t+1} + c_{1t} + \frac{1}{1+n} \cdot c_{2t} \end{cases}$$

Each generation allocates its income between consumption (in the first and second periods), and bequests. The corrected marginal utility of consumption in the first and the second periods must be equal:

$$u'(c_{1t}) = \tfrac{1+r_{t+1}}{1+\theta} \cdot u'(c_{2t+1}).$$

And, if bequests are positive, equal to the marginal utility of the bequest:

$$\begin{cases} \frac{1}{1+\theta} \cdot (1+n) \cdot u'(c_{2t+1}) \geq \frac{1}{1+R} \cdot u'(c_{1t+1}) & , b_{t+1} = 0 \\ \frac{1}{1+\theta} \cdot (1+n) \cdot u'(c_{2t+1}) = \frac{1}{1+R} \cdot u'(c_{1t+1}) & , b_{t+1} > 0 \end{cases}$$

In steady state:

$$\begin{cases} u'(c_1^*) = \frac{1+r^*}{1+\theta} \cdot u'(c_2^*) \\ \frac{1}{1+\theta} \cdot (1+f'(k^*)) \cdot u'(c_2^*) \geq u'(c_1^*) \quad ,b^* = 0 \\ \frac{1}{1+\theta} \cdot (1+f'(k^*)) \cdot u'(c_2^*) = u'(c_1^*) \quad ,b^* \geq 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} 1+r^* \leq (1+n) \cdot (1+R) \quad , \quad \text{and } b = 0 \\ 1+r^* = (1+n) \cdot (1+R) \quad , \quad \text{and } b \geq 0 \end{cases}$$

Assuming positive bequests, these conditions are identical to the planned optimum, which is an important result. In this case, the interest rate is equal to the modified golden rule. Bequests prevent the interest rate from being higher than the modified golden rule. So, the capital stock cannot be too low.

Suppose that bequests were prohibited. If the no-bequest equilibrium interest rate is lower than the modified golden rule, the prohibition is irrelevant. Allowing for bequests would not change the situation of "*dynamic inefficiency*". If it is higher, then allowing for bequests would lead to a decrease in interest rates to the modified golden rule and to positive bequests in equilibrium. It is not the finiteness of lives that causes inefficient equilibria, but the absence of future generations' utility from the utility of the present generations. When parents incorporate their children's utility in their own preferences to an extent that is sufficient to induce bequests, the equilibrium becomes efficient (steady state is at the modified golden rule).

Two-Sided Altruism:

Now we allow gifts in both directions, from parents to children and from children to parents. Does this ensure that equilibria are Pareto optimal? We will see that it doesn't.

Designate by W_t the direct utility of generation t:

$W_t = u(c_{1t}) + \frac{1}{(1+\theta)} \cdot u(c_{2t+1})$.

With two-sided altruism, the utility of a generation is both affected by the utility of the next and the previous generations:

$V_t = W_t + \frac{1}{(1+R)} \cdot V_{t+1} + \frac{1}{(1+\varphi)} \cdot V_{t-1}$.

Kimball (1987) analyzed the possibility of expressing total utility as:

$V_t = \sum_{i=-\infty}^{\infty} (\gamma_i \cdot W_{t+i})$.

We impose three requirements: all γ_i positive; γ_i converging to zero as i tends to infinity; and with γ_i as a geometric series for all positive i. The meaning of the last requirement is simply that the family behavior is time-consistent.

To calculate the effect of an increase in W_t in V_t, we must take account of an "*hall of mirrors*" effect. People care for their children, and care for their parents who also care for their grandchildren, and so on.

Thus, besides the direct effect (one-to-one), an increase in W_t also increases V_{t+1} and V_{t-1}, and, consequently, increases V_t (the two effects together have a factor of $\frac{2}{(1+R)(1+\varphi)}$). This new increase has a similar effect, further increasing V_t (now with a factor of $\frac{4}{(1+R)^2(1+\varphi)^2}$). The triangle of Pascal can throw light on the global effect.

Expressing total utility as a weighted sum of direct utilities:

$$\sum_{i=-\infty}^{\infty} (\gamma_i \cdot W_{t+i}) = W_t + \frac{1}{1+R} \cdot \sum_{i=-\infty}^{\infty} (\gamma_i \cdot W_{t+i+1}) + \frac{1}{1+\varphi} \cdot \sum_{i=-\infty}^{\infty} (\gamma_i \cdot W_{t+i-1}) \Rightarrow$$
$$\Rightarrow \sum_{i=-\infty}^{\infty} (\gamma_i \cdot W_{t+i}) = W_t + \frac{1}{1+R} \cdot \sum_{i=-\infty}^{\infty} (\gamma_{i-1} \cdot W_{t+i}) + \frac{1}{1+\varphi} \cdot \sum_{i=-\infty}^{\infty} (\gamma_{i+1} \cdot W_{t+i}) \Rightarrow$$
$$\Rightarrow \frac{\partial V_t}{\partial W_{t+i}} = \gamma_i = \frac{\gamma_{i-1}}{1+R} + \frac{\gamma_{i+1}}{1+\varphi} \Rightarrow$$

$$\Rightarrow \begin{cases} \gamma_{-i} = \dfrac{\gamma_{-i-1}}{1+R} + \dfrac{\gamma_{-i+1}}{1+\varphi} \\ ... \\ \gamma_{-1} = \dfrac{\gamma_{-2}}{1+R} + \dfrac{\gamma_0}{1+\varphi} \\ \gamma_0 = 1 + \dfrac{\gamma_{-1}}{1+R} + \dfrac{\gamma_1}{1+\varphi} \\ \gamma_1 = \dfrac{\gamma_0}{1+R} + \dfrac{\gamma_2}{1+\varphi} \\ ... \\ \gamma_i = 1 + \dfrac{\gamma_{i-1}}{1+R} + \dfrac{\gamma_{i+1}}{1+\varphi} \end{cases}$$

Since the γ_i are a geometric series, and we allow for different ratios for negative and positive gaps ($\lambda > 1$ and $\mu < 1$, respectively):

$$\begin{cases} 1 = \frac{1}{(1+R)\cdot\lambda} + \frac{\lambda}{1+\varphi} \\ ... \\ 1 = \frac{1}{(1+R)\cdot\lambda} + \frac{\lambda}{1+\varphi} \\ 1 = \frac{1}{\gamma_0} + \frac{1}{(1+R)\cdot\lambda} + \frac{\mu}{1+\varphi} \\ 1 = \frac{1}{(1+R)\cdot\mu} + \frac{\mu}{1+\varphi} \\ ... \\ 1 = \frac{1}{(1+R)\cdot\mu} + \frac{\mu}{1+\varphi} \end{cases}$$

These conditions resume to three:

$$\begin{cases} 1 = \frac{1}{(1+R)\cdot\lambda} + \frac{\lambda}{1+\varphi} \\ 1 = \frac{1}{\gamma_0} + \frac{1}{(1+R)\cdot\lambda} + \frac{\mu}{1+\varphi} \\ 1 = \frac{1}{(1+R)\cdot\mu} + \frac{\mu}{1+\varphi} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{1}{1+\varphi} \cdot \lambda^2 - \lambda + \frac{1}{1+R} = 0 \\ \lambda = \frac{1}{1+R} + \frac{\lambda}{\gamma_0} + \frac{\mu\cdot\lambda}{1+\varphi} \\ \frac{1}{1+\varphi} \cdot \mu^2 - \mu + \frac{1}{1+R} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = \frac{1 \pm \sqrt{1 - \frac{4}{(1+\varphi)\cdot(1+R)}}}{\frac{2}{1+\varphi}} \\ \lambda = \frac{1}{1+R} + \frac{\lambda}{\gamma_0} + \frac{\mu\cdot\lambda}{1+\varphi} \\ \mu = \frac{1 \pm \sqrt{1 - \frac{4}{(1+\varphi)\cdot(1+R)}}}{\frac{2}{1+\varphi}} \end{cases}$$

To simplify these expressions, we define $A \equiv \sqrt{1 - \frac{4}{(1+\varphi)\cdot(1+R)}}$. Of course that the existence of finite positive weights γ_i demands a non-negative A. Resolution of the system gives us the γ_i.

The first restriction implies that altruism declines with distance, and the second implies that A is positive, and, consequently, the existence of finite positive weights:

$$\begin{cases} \frac{1}{1+\varphi} + \frac{1}{1+R} < 1 \\ \frac{1}{1+\varphi} \cdot \frac{1}{1+R} < \frac{1}{4} \end{cases}$$

With g_t representing gifts from the young to the old generation in period t, the budget constraint of generation t becomes:

$$\begin{cases} c_{1t} = w_t + b_t - g_t - s_t \\ c_{2t+1} = (1 + r_{t+1}) \cdot s_t - (1 + n) \cdot b_{t+1} + (1 + n) \cdot g_{t+1} \end{cases}$$

At time t, the objective of the individual (generation) is to maximize:

$$\frac{\mu_{-1}}{1+\theta} \cdot u(c_{2t}) + \mu_0 \cdot \left[u(c_{1t}) + \frac{1}{1+\theta} \cdot u(c_{2t+1})\right] + \frac{\mu_1}{u(c_{1t+1})} + ...$$

This implies the following first order conditions in s_t, g_t and b_{t+1}:

$$\begin{cases} u'(c_{1t}) = \frac{1+r_{t+1}}{1+\theta} \cdot u'(c_{2t+1}) \\ u'(c_{1t}) \geq \frac{\mu_{-1}}{\mu_0} \cdot \frac{1}{1+\theta} \cdot u'(c_{2t}) \quad \text{, with equality if } g_t > 0 \\ \frac{1+n}{1+\theta} u'(c_{2t+1}) \geq \frac{\mu_1}{\mu_0} \cdot u'(c_{1t+1}) \quad \text{, with equality if } b_{t+1} > 0 \end{cases}$$

Steady state then implies:

$$\begin{cases} \frac{u'_1}{u'_2} = \frac{1+r^*}{1+\theta} \\ \frac{u'_1}{u'_2} \geq \frac{\mu_{-1}}{\mu_0} \cdot \frac{1}{1+\theta} \\ \frac{u'_1}{u'_2} \leq \frac{\mu_0}{\mu_1} \cdot \frac{1+n}{1+\theta} \end{cases} \Longrightarrow$$

$$\frac{\mu_{-1}}{\mu_0} \leq 1 + r^* \leq (1 + n) \cdot \frac{\mu_0}{\mu_1} \Rightarrow$$

$$\Rightarrow \frac{1}{\lambda} \leq 1 + r^* \leq \frac{1+n}{\mu} \quad , \lambda > 1 \ , \mu < 1.$$

These results are an extension to the ones obtained with one-sided altruism. The second inequality tells us that the interest rate cannot be too high: if it were, bequests and further capital accumulation would take place to restore the equality. The first inequality tells us that it cannot be too low (gifts would take place and restore equality). If the interest rate of the original economy satisfies both strict inequalities, neither gifts nor bequests take place (if these possibilities are introduced). The interval in which this happens includes the golden rule interest rate, $r = n$. So, the inclusion of two-sided altruism does not ensure Pareto optimality in the overlapping generations model.

2 Social Security and Capital Accumulation

The introduction of a social security system alters, in general, the path of income received by individuals, having an effect on savings and, thus, on capital accumulation. In this section, we seek to characterize the impact on the economy of two types of social security systems.

We will study the impact of a "*fully funded*" social security system and of a "*pay as you go*" system. In the "*fully funded*" system, contributions are returned with interest to the same generation in the next period. The "*pay as you go*" system transfers the contributions of the young directly to the old. In simple terms, the first system consists of forced savings, while the second consists of forced transfers from the young to the old.

Our starting point is the equilibrium conditions of the decentralized economy:

$$\begin{cases} u'(w_t - s_t) = \frac{1+r_{t+1}}{1+\theta} \cdot u'((1 + r_{t+1}) \cdot s_t) \\ s_t = (1 + n) \cdot k_{t+1} \\ w_t = f(k_t) - k_t \cdot f'(k_t) \\ r_t = f'(k_t) \end{cases}$$

And we introduce some new notation. Let d_t be the contribution of the young generation and b_t be the benefit received by the old generation in period t.

A Fully Funded System:

In the "*fully funded*" system, the government raises contributions d_t, invests them as capital, and pays $b_t = (1 + r_t) \cdot d_{t-1}$ to the old. This modifies the equilibrium conditions:

$$\begin{cases} u'(w_t - (s_t + d_t)) = \frac{1+r_{t+1}}{1+\theta} \cdot u'((1 + r_{t+1}) \cdot (s_t + d_t)) \\ s_t + d_t = (1 + n) \cdot k_{t+1} \end{cases}$$

If the savings in the original economy are larger than the requested contributions, the path of capital accumulation remains unaltered. The only effect of the "*fully funded*" social security system is to force savings. This kind of system only has an effect on capital accumulation when d_t is higher than the s_t of the original economy. In this case, it increases savings and capital accumulation.

A Pay-As-You-Go System:

A "*pay-as-you-go*" social security system is not funded. Income is directly transfered from the young to the old in the same period. This kind of system is similar to forced gifts from the young to the old - which appeared in the analysis of two-sided altruism. With the introduction of a "*pay-as-you-go*" social security system, the equilibrium conditions become:

$$\begin{cases} u'(w_t - s_t - d_t) = \frac{1+r_{t+1}}{1+\theta} \cdot u'((1+r_{t+1}) \cdot s_t + (1+n) \cdot d_{t+1}) \\ s_t = (1+n) \cdot k_{t+1} \end{cases}$$

This is a system of pure transfers. Capital accumulation is determined only by private savings s_t. Naturally, the forced transfers diminish voluntary savings. Differentiating the previous relation and assuming $d_{t+1} = d_t$:

$$u_1'' \cdot \left(-\frac{\partial s_t}{\partial d_t} - 1 \right) = \frac{1+r_{t+1}}{1+\theta} \cdot u_2'' \cdot \left((1+r_{t+1}) \cdot \frac{\partial s_t}{\partial d_t} + 1 + n \right) \Rightarrow$$

$$\Rightarrow -\frac{\partial s_t}{\partial d_t} \cdot \left(u_1'' + \frac{(1+r_{t+1})^2}{1+\theta} \cdot u_2'' \right) = \frac{(1+r_{t+1}) \cdot (1+n)}{1+\theta} \cdot u_2'' \Rightarrow$$

$$\Rightarrow \frac{\partial s_t}{\partial d_t} = -\frac{u_1'' + \frac{(1+r_{t+1}) \cdot (1+n)}{1+\theta} \cdot u_2''}{u_1'' + \frac{(1+r_{t+1})^2}{1+\theta} \cdot u_2''} < 0.$$

With $n > r$, the module is greater than 1, that is, the decrease in private savings more than offsets the increase in forced transfers. With $n < r$, the result is the opposite. But there are also secondary effects. The decrease in savings, and thus in capital, decreases wages and increases interest rates. The decrease in wages further decreases savings, while the increase in the interest rates has an ambiguous effect. What, then, is the general equilibrium effect of an increase in the transfers imposed by the social security system on the capital stock?

Consider the dynamic equation:

$$(1+n) \cdot k_{t+1} = s[w_t(k_t), r_{t+1}(k_{t+1}), d_t].$$

Holding k_t constant and differentiating:

$$(1+n) \cdot \tfrac{dk_{t+1}}{dd_t} = s_r \cdot \tfrac{dr_{t+1}}{dk_{t+1}} \cdot \tfrac{dk_{t+1}}{dd_t} + \tfrac{\partial s_t}{\partial d_t}.$$

$$\frac{dk_{t+1}}{dd_t} = \frac{\partial s_t / \partial d_t}{1+n-s_r \cdot f''} < 0.$$

The numerator is negative, while the assumption of stability renders the sign of the denominator positive. Therefore, an increase in social security lowers the *savings locus*, slowing capital accumulation and reducing the steady state capital stock.

If, without social security, $r < n$, the introduction of a *pay-as-you-go* system can reduce or eliminate dynamic inefficiency. If $r > n$, social security benefits the first old generation at the expense of the following generations.

The Bequest Motive and The Effect of Social Security:

In the "*pay-as-you-go*" system, social security contributions are negative bequests, transfers from the young to the old. If (before the introduction of social security) the market has positive bequests, the old generation increases these bequests exactly by the amount it receives from social security system. The net transfers between generations remain, therefore, unaffected. The forced transfers may limit the consumption of the young generation in the first period. If original savings are higher than the forced transfers, the social security doesn't limit consumption(*). In these conditions, the "*pay-as-you-go*" system has no impact on the economy. In this case, the action of the government is completely offset by private sector responses.

3 A Model of Perpetual Youth

The previous model, in which life is split in two periods, and in which only two generations co-exist, is a somewhat gross simplification. It would be more realistic to consider many periods of life and more generations co-existing in each period. The problem is that overlapping generations models with many periods tend to be analytically intractable.

In this section, a continuous-time model of overlapping generations is introduced. We start by describing the demographics of the model and the structure of markets. Then we derive individual and aggregate consumption and saving. Finally, we examine the dynamic adjustment toward the steady state.

3.1 The Structure of the Model

Population:

In any unit of time, an individual faces a probability of dying equal to p, that is constant throughout life. This assumption is crucial for the tractability of the model, and is also the origin of the designation: "*model of perpetual youth*".

The probability of surviving during a period t is $e^{-p\cdot\Delta t}$:

$$\lim_{n\to\infty}\left(1-\frac{p\cdot\Delta t}{n}\right)^n = e^{-p\cdot\Delta t}$$

So, the probability of dying during a time interval is, of course, $1-e^{-p\cdot\Delta t}$. The probability of dying in moment t_d is the derivative: $p\cdot e^{-p\cdot t_d}$. Which is equal to the product of two probabilities: of dying in t_d, equal to p; and of surviving until t_d, equal to $e^{-p\cdot t_d}$.

Life expectancy is, then:

$$E(\chi) = \int_0^{\infty} t \cdot p \cdot e^{-pt} \cdot dt$$

With integration by parts, using $u = -t$ and $dv = -e^{-pt}$:

$$\begin{aligned} E(\chi) &= \left[-t \cdot e^{-pt}\right]_0^{\infty} - \int_0^{\infty} -e^{-pt} \cdot dt = \\ &= (0-0) - \left[p^{-1} \cdot e^{-pt}\right]_0^{\infty} \\ &= -(0 - p^{-1}) = p^{-1} \end{aligned}$$

Whatever the age of the individual, life expectancy is always equal to p^{-1} years (with $p = 0$, we are back in the Ramsey model). At each instant of time, a new cohort is born, being large enough so that each individual is negligible. In these conditions, p may be seen as the rate of decrease of the size of the cohort. Although each person is uncertain about his or her death, the size of a cohort evolves deterministically. A usual normalization consists in considering that the population is constant and equal to 1. In this case, consistency demands that we assume the size of each cohort to be equal to p, as we confirm below ($e^{-p(t-s)}$ is the fraction of the generation born in s that is still alive in $t > s$):

$$P = \int_{-\infty}^{t} p \cdot e^{-p(t-s)} \cdot ds = 1$$

The Availability of Insurance:

In the absence of insurance, given the uncertainty about the time of death, individuals would leave unintended bequests (negative if they died in debt). Here we consider an insurance industry with zero profit and free entry, thus paying a premium p per unit of time: individuals receive (pay) a rate p to pay (receive) one good contingent on death. In the absence of a bequest motive, and with negative bequests prohibited, individuals contract to transmit all of their wealth,

v_t to the insurance company contingent on their deaths. In exchange, the insurance company will pay them $p \cdot v_t$ per unit time. The insurance company has no profits. In each unit time, receives $p \cdot v_t$ from those who die, and pays premiums of $p \cdot v_t$ to those that remain alive. Notice that the insurance company faces no uncertainty, because the population is very large.

3.2 Individual and Aggregate Consumption

Individual Consumption:

We denote by $c(s,t)$, $y(s,t)$, $v(s,t)$ and $h(s,t)$ the consumption, labor income, tangible wealth and human wealth, respectively, at time t of an individual born at time s. Dealing with a generic generation, we omit the s from the notation.

Individuals face a maximization problem under uncertainty. At time t they maximize:

$$E(U) = E\left[\int_t^{\infty} u(c(z)) \cdot e^{-\theta(z-t)} \cdot dz\right]$$

The probability of being alive at z is $e^{-p(z-t)}$. And for simplicity we assume $u(c) = log(c)$, restricting our analysis to an elasticity of substitution between periods equal to 1. The objective function becomes:

$$E(U) = \int_t^{\infty} log(c(z)) \cdot e^{-(\theta+p)(z-t)} \cdot dz.$$

The constant probability of death simply increases the individual's rate of time preference.

An individual with tangible wealth $v(z)$ receives $r(z) \cdot v(z)$ of interest and $p \cdot v(z)$ from the insurance company. The dynamic budget constraint is:

$$\frac{dv(z)}{dz} = [r(z) + p] \cdot v(z) + y(z) - c(z).$$

To prevent individuals from going infinitely into debt by protecting themselves with life insurance, we need a no-Ponzi-game (NPG) condition. An individual cannot accumulate debt at a rate higher than the effective rate of interest:

$$\lim_{z \to \infty} e^{-\int_t^z [r(\mu)+p] \cdot d\mu} \cdot v(z) = 0.$$

It is convenient to define the discount factor, which (among other uses) allows us to calculate human wealth as the discounted value of wages:

$$R(t, z) \equiv e^{-\int_t^z [r(\mu)+p] \cdot d\mu}.$$

We integrating the dynamic budget condition by parts using:

$$\begin{cases} u = R \text{ and } du = -(r+p)R \cdot dz \\ w = v \text{ and } dw = \frac{dv}{dz} \cdot dz \\ h(t) = \int_t^\infty R(t, z) \cdot y(z) dz \end{cases}$$

$$\begin{aligned}
\int_t^\infty R(t,z)\frac{dv(z)}{dz}dz = \int_t^\infty R(t,z)[r(z)+p]v(z) + R(t,z)y(z) - R(t,z)c(z) \cdot dz \ \Rightarrow \\
\Rightarrow \int_t^\infty R(t,z)c(z)dz = \\
= \int_t^\infty R(t,z)[r(z)+p]v(z) \cdot dz - \int_t^\infty R(t,z)\frac{dv(z)}{dz} \cdot dz + h(t) = \\
= -\int_t^\infty w \cdot du - \int_t^\infty u \cdot dw + h(t) = -\left[R(t,z) \cdot v(z)\right]_t^\infty + h(t) = \\
= -\lim_{z\to\infty}\left[e^{-\int_t^z [r(\mu)+p]\cdot d\mu} \cdot v(z)\right] + e^{-\int_t^t [r(\mu)+p]\cdot d\mu} \cdot v(t) + h(t) \ \Rightarrow \\
\Rightarrow \int_t^\infty R(t,z)c(z)dz = -0 + e^0 \cdot v(t) + h(t) = v(t) + h(t).
\end{aligned}$$

This is the budget constraint of the maximization problem. Compared to the problem of infinitely lived consumers (model of Ramsey), here the future utility is more discounted ($\theta + p$ instead of θ), and the effective rate of interest is greater ($r + p$ instead of r).

Defining the Hamiltonian like in the model of Ramsey:

$H = log(c(z)) + q(z) \cdot \{[r(z) + p] \cdot v(z) + y(z) - c(z)\}.$

Necessary conditions for optimization, besides the transversality condition, are:

$$\begin{cases} H_c = 0 \\ \dot{q}(z) = -H_v + q(z) \cdot (\theta + p) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} c(z)^{-1} + q(z) \cdot (-1) = 0 \Rightarrow c(z)^{-1} = q(z) \\ \dot{q}(z) = q(z) \cdot [-r(z) - p + \theta + p] \Rightarrow \dot{q}(z) = q(z) \cdot [-r(z) + \theta] \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{q}(z) = -\frac{\dot{c}(z)}{c(z)^2} \\ \frac{\dot{c}(z)}{c(z)^2} = \frac{1}{c(z)} \cdot [r(z) - \theta] \Rightarrow \dot{c}(z) = [r(z) - \theta] \cdot c(z) \end{cases}$$

Individual consumption evolves at a rate that is equal to the difference between the interest rate and the individual discount rate, growing with age if the rate of interest is greater than that of discount.

Integrating and replacing the budget constraint we obtain:

$$\frac{\dot{c}(\mu)}{c(\mu)} = r(\mu) - \theta \Rightarrow \int_t^z \frac{\dot{c}(\mu)}{c(\mu)} \cdot d\mu = \int_t^z [r(\mu) - \theta] \cdot d\mu \Rightarrow$$

$$\Rightarrow log(c(z)) - log(c(t)) = \int_t^z [r(\mu) - \theta] \cdot d\mu \Rightarrow \frac{c(t)}{c(z)} = e^{-\int_t^z [r(\mu)-\theta] \cdot d\mu} \Rightarrow$$

$$\Rightarrow c(t) = e^{-\int_t^z [r(\mu)+p-\theta-p] \cdot d\mu} \cdot c(z) = R(t,z) \cdot e^{\int_t^z (\theta+p) \cdot d\mu} \cdot c(z) \Rightarrow$$

$$\Rightarrow c(t) \cdot e^{-(z-t)\cdot(\theta+p)} = R(t,z) \cdot c(z) \Rightarrow \int_t^\infty c(t) \cdot e^{(z-t)\cdot(\theta+p)} \cdot dz = v(t) + h(t) \Rightarrow$$

$$\Rightarrow c(t) \cdot \int_0^\infty e^{-(z)\cdot(\theta+p)} \cdot dz = v(t) + h(t) \Rightarrow$$

$$\Rightarrow c(t) \cdot \frac{-1}{\theta + p} \cdot (e^{-\infty} - e^0) = v(t) + h(t) \Rightarrow.$$

$$\Rightarrow c(t) = (\theta + p) \cdot [v(t) + h(t)].$$

Propensity to consume out of wealth is given by $(\theta + p)$, which is independent of the interest rate.

Aggregate Consumption:

Now we want to find the aggregate variables, $C(t)$, $Y(t)$, $V(t)$ and $H(t)$. Recall that in t, the size of generation born in s is $p \cdot e^{-p(t-s)}$. Thus, aggregate consumption is:

$$C(t) = \int_{-\infty}^{t} c(s,t) \cdot p \cdot e^{-p(t-s)} \cdot ds.$$

Integrating the individual path of consumption (we recall that the propensity to consume, $\theta + p$ is independent of age):

$$c(t) = (\theta + p) \cdot [v(t) + h(t)] \Rightarrow C(t) = (\theta + p) \cdot \int_{-\infty}^{t} [v(s,t) + h(s,t)] ds \Rightarrow$$
$$\Rightarrow C(t) = (\theta + p) \cdot [H(t) + V(t)].$$

We start with the study of the dynamic behavior of human wealth, $H(t)$. Recall that the population is constant and equal to 1, therefore, the aggregate and average values are equal. Assuming that labor income varies at a constant, non-increasing rate:

$$y(s,t) = a \cdot Y(t) \cdot e^{-\alpha(t-s)} \quad , \alpha \geq 0.$$

To find a, we use the definition of $Y(t)$:

$$Y(t) = \int_{-\infty}^{t} y(s,t) \cdot p \cdot e^{-p(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow Y(t) = a \cdot \int_{-\infty}^{t} Y(t) \cdot e^{-\alpha(t-s)} \cdot p \cdot e^{-p(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow \frac{1}{a \cdot p} = \int_{-\infty}^{t} e^{-(\alpha+p)(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow \frac{1}{a \cdot p} = \left[\frac{1}{\alpha + p} \cdot e^{(\alpha+p)(s)} \right]_{-\infty}^{0} \Rightarrow$$
$$\Rightarrow \frac{1}{a \cdot p} = \frac{1}{\alpha + p} \cdot (1 - 0) \Rightarrow a = \frac{\alpha + p}{p}.$$

Using this in the definition of $h(t,s)$:

$$h(s,t) = \int_t^{\infty} a \cdot Y(z) \cdot e^{-\alpha(z-s)} \cdot R(t,z) \cdot dz \Rightarrow$$
$$\Rightarrow h(s,t) = a \cdot e^{-\alpha(t-s)} \cdot \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot R(t,z) \cdot dz$$

So human wealth $H(t)$ is given by:

$$H(t) = \int_{-\infty}^{t} h(s,t) \cdot p \cdot e^{-p(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow H(t) = a \cdot p \cdot \int_{-\infty}^{t} \left\{ e^{-(\alpha+p)(t-s)} \cdot \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot R(t,z) \cdot dz \right\} \cdot ds$$

The interior integral is a constant in terms of the exterior one, so:

$$H(t) = (\alpha+p) \cdot \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot R(t,z) \cdot dz \cdot \int_{-\infty}^{t} e^{-(\alpha+p)(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow H(t) = \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot R(t,z) \cdot dz \cdot \int_{-\infty}^{t} (\alpha+p) \cdot e^{-(\alpha+p)(t-s)} \cdot ds \Rightarrow$$
$$\Rightarrow H(t) = \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot R(t,z) \cdot dz \cdot \left[e^{(\alpha+p)(s)} \right]_{-\infty}^{0} \Rightarrow$$
$$\Rightarrow H(t) = \int_t^{\infty} Y(z) \cdot e^{-\alpha(z-t)} \cdot e^{-\int_t^z [p+r(\mu)]d\mu} \cdot dz \cdot (1-0) \Rightarrow$$
$$\Rightarrow H(t) = \int_t^{\infty} Y(z) \cdot e^{-\int_t^z [\alpha+p+r(\mu)]d\mu} \cdot dz$$

We arrived at an intuitive result. Aggregate human wealth equals the present value of future aggregate wages, discounted at the rate $(\alpha+p+r)$.

Differentiating with respect to time and imposing a limit on the growth of $H(t)$:

$$H(t) = \int_{a(t)}^{b(t)} F(s,t) ds \Rightarrow$$
$$\Rightarrow \dot{H}(t) = \int_{a(t)}^{b(t)} \frac{dF(s,t)}{dt} ds - \frac{da(t)}{dt} \cdot F(a(t),t) + \frac{db(t)}{dt} \cdot F(b(t),t)$$
$$\frac{dH(t)}{dt} = \int_t^{\infty} Y(z) \cdot [\alpha+p+r(t)] \cdot e^{-\int_t^z [\alpha+p+r(\mu)]d\mu} \cdot dz - Y(t) \Rightarrow$$
$$\Rightarrow \frac{dH(t)}{dt} = [\alpha+p+r(t)] \cdot \int_t^{\infty} Y(z) \cdot e^{-\int_t^z [\alpha+p+r(\mu)]d\mu} \cdot dz - Y(t) \Rightarrow$$
$$\Rightarrow \frac{dH(t)}{dt} = [\alpha+p+r(t)] \cdot H(t) - Y(t)$$

A second condition guarantees that $H(t)$ is bounded:

$$\begin{cases} \dfrac{dH(t)}{dt} = [r(t) + p + \alpha] \cdot H(t) - Y(t) \\ \lim\limits_{z \to \infty} H(z) \cdot e^{-\int_t^z [\alpha + p + r(\mu)] d\mu} = 0 \end{cases}$$

Finally, we analyze tangible wealth:

$$V(t) = \int_{-\infty}^{t} v(s,t) \cdot p \cdot e^{-p \cdot (t-s)} \cdot ds$$

Differentiating with respect to time:

$$\frac{dV(t)}{dt} = \int_{-\infty}^{t} \frac{dv(s,t)}{dt} \cdot p \cdot e^{-p \cdot (t-s)} \cdot ds - p \cdot V(t) + p \cdot v(t,t)$$

The last term is the initial tangible wealth of the cohort born in t, equal to zero. The variation of $v(s,t)$ with time has already been studied:

$$\frac{dV(t)}{dt} = \int_{-\infty}^{t} \{[r(t) + p] \cdot v(s,t) + y(s,t) - c(s,t)\} \cdot p \cdot e^{-p \cdot (t-s)} \cdot ds - p \cdot V(t) \Rightarrow$$
$$\Rightarrow \frac{dV(t)}{dt} = [r(t) + p] \cdot V(t) + Y(t) - C(t) - p \cdot V(t) = r(t) \cdot V(t) + Y(t) - C(t)$$

Individual tangible wealth accumulates at the rate $r + p$ if the individual remains alive. Aggregate wealth accumulates only at the rate r, because of the transfer, through insurance companies, of $p \cdot V(t)$ from those who die to those who remain alive. This difference between the social and private returns on wealth is crucial to some results that will be derived.

Aggregate Behavior:

The aggregate equations of this economy are:

$$\begin{cases} C = (p + \theta) \cdot (H + V) \ , \\ \dfrac{dV}{dt} = r \cdot V + Y - C \ , \\ \dfrac{dH}{dt} = (r + p + \alpha) \cdot H - Y \ , \\ \lim\limits_{z \to \infty} H(z) \cdot e^{-\int_t^z [\alpha + p + r(\mu)] d\mu} = 0 \ . \end{cases}$$

An alternative characterization of aggregate consumption will be useful:

$$\frac{dC}{dt} = (p+\theta)\cdot[(r+p+\alpha)\cdot H - Y + r\cdot V + Y - C] \Rightarrow$$
$$\Rightarrow \frac{dC}{dt} = (p+\theta)\cdot[(r+p+\alpha)\cdot(H+V) - (p+\alpha)\cdot V - C] \Rightarrow$$
$$\Rightarrow \frac{dC}{dt} = (r+p+\alpha)\cdot C - (p+\theta)\cdot(p+\alpha)\cdot V - (p+\theta)\cdot C \Rightarrow$$
$$\Rightarrow \frac{dC}{dt} = (r+\alpha-\theta)\cdot C - (p+\theta)\cdot(p+\alpha)\cdot V.$$

3.3 Dynamics and Steady State with Constant Relative Labor Income

The production function of the economy is assumed concave with constant returns to scale and depreciations. The value of capital is equal to the total tangible wealth, $K = V$. Accordingly, the interest rate equals the marginal product of capital.

$F(K) \equiv F(K,1) - \delta \cdot K.$

We begin by considering constant labor income, that is, $\alpha = 0$. We will analyze the effect of $\alpha > 0$ afterward.

$$\begin{cases} \dfrac{dC}{dt} = [F'(K) - \theta]\cdot C - p\cdot(p+\theta)\cdot K \ , \\ \dfrac{dV}{dt} = F(K) - C \ . \end{cases}$$

The following system determines the steady state.

$$\begin{cases} [F'(K^*) - \theta]\cdot C^* = p\cdot(p+\theta)\cdot K^* \ , \\ F(K^*) = C^* \ . \end{cases}$$

The phase diagram (in figure 3.5) shows a saddle path and a unique equilibria (except the origin).

The first condition implies that $F'(K) > \theta$. From the concavity of $F(\cdot)$, we have that the average slope of the production function from 0 to K^* is greater than the slope at K^*:

$$F'(K^*) < \frac{F(K^*)}{K^*} = \frac{C^*}{K^*} = \frac{p \cdot (p+\theta)}{F'(K^*) - \theta} \Rightarrow$$

$$\Rightarrow [F'(K^*) - \theta] \cdot F'(K^*) < p \cdot (p+\theta) \Rightarrow.$$

$$\Rightarrow F'(K^*) - \theta < p.$$

We have, then, $\theta < F'(K^*) < \theta + p$.

With finite horizons and $\alpha = 0$, the interest rate is greater than the individual discount rate. Since $n = 0$, the equilibrium is dynamically efficient. Notice that the shorter the individual's horizon, the higher the interest rate and the lower the capital stock.

3.4 Dynamics and Steady State with Declining Relative Labor Income

With $\alpha < 0$, the equilibrium may become dynamically inefficient. This happens because the individuals desire to accumulate more capital to ensure a high quality of life in later years. The dynamics are given by:

$$\begin{cases} \dot{C} = [F'(K) + \alpha - \theta] \cdot C - (p+\alpha) \cdot (p+\theta) \cdot K \ , \\ \dot{V} = F(K) - C \ . \end{cases}$$

The path of $\frac{dC}{dt} = 0$ now approaches $F' = \theta - \alpha$ instead of $F' = \theta$. So, if $\alpha > \theta$, we may have dynamic inefficiency (see figure 3.6).

4 Fiscal Policy: Debt and Deficit Finance

Does it matter whether the government finances its spending by lump-sum taxes or borrowing? In the Ramsey model it doesn't: consumption and savings are independent of the timing of taxes. Here we show that deficit finance does have real effects in the life-cycle model.

In the life-cycle model, taxes levied at different times are levied on different sets of people. The burden of taxes is on the present generations, while the deficit is also paid by the taxes of the future generations. Deficit finance thus increases the income of present generations and decreases that of the future, so economic behavior is affected by the timing of taxes.

4.1 The Government Budget Constraint

Let $B(t)$ be the government debt. The deficits of the government are bounded by the dynamic budget constraint and the no-Ponzi game (NPG) condition:

$$\begin{cases} \dot{B}(t) = r(t) \cdot B(t) + G(t) - T(t) \ ; \\ \lim\limits_{z \to \infty} B(z) \cdot e^{-\int_t^z r(\mu) \cdot d\mu} = 0 \ . \end{cases}$$

Notice that the government pays an interest rate of $r(t)$ for its debt, while the individual's return on assets is $r(t) + p$. This is crucial for the results that will be obtained.

Integrating the first condition by parts (and using the NPG condition):

$$\dot{B}(z) \cdot e^{-\int_t^z r(\mu) \cdot d\mu} = [r(z) \cdot B(z) + G(z) - T(z)] \cdot e^{-\int_t^z r(\mu) \cdot d\mu} \Rightarrow$$

$$\Rightarrow \int_t^\infty \dot{B}(z) \cdot e^{-\int_t^z r(\mu) \cdot d\mu} \cdot dz = \int_t^\infty [r(z) \cdot B(z) + G(z) - T(z)] \cdot e^{-\int_t^z r(\mu) \cdot d\mu} \cdot dz \Rightarrow$$

$$\Rightarrow \left[e^{-\int_t^z r(\mu)\cdot d\mu}\cdot B(z)\right]_t^{\infty} + \int_t^{\infty} B(z)\cdot r(z)\cdot e^{-\int_t^z r(\mu)\cdot d\mu} = ... \Rightarrow$$

$$\Rightarrow \left[e^{-\int_t^z r(\mu)\cdot d\mu}\cdot B(z)\right]_t^{\infty} = \int_t^{\infty} [G(z)-T(z)]\cdot e^{-\int_t^z r(\mu)\cdot d\mu}\cdot dz \Rightarrow$$

$$\Rightarrow B(t) = \int_t^{\infty} [T(z)-G(z)]\cdot e^{-\int_t^z r(\mu)\cdot d\mu}\cdot dz \ .$$

This is the intertemporal budget constraint of the government. The current value of debt must be equal to the present discounted value of primary surpluses. Debt is not necessarily repaid, but its rate of growth is bounded by the interest rate, so that the present value of the (infinitely) distant future debt is zero.

4.2 Fiscal Policy and Consumption: Partial Equilibrium

The government affects the total demand for goods through direct purchases of goods and by setting taxes (present and anticipated) that affect private consumption.

With the presence of the government, the equations that characterize the economy change. Tangible wealth of the individuals (V) must include government debt (B) besides other assets (K). And to calculate human wealth, we must consider income after taxes. For simplicity, assume $\alpha = 0$.

$$\begin{cases} C(t) = (p+\theta)\cdot[H(t)+V(t)] \ ; \\ V(t) = B(t) + K(t) \ ; \\ H(t) = \int_t^{\infty} [Y(z)-T(z)]\cdot e^{-\int_t^z [r(\mu)+p]\cdot d\mu}\cdot dz \ ; \\ \dot{V}(t) = r(t)\cdot V(t) + Y(t) - T(t) - C(t) \ . \end{cases}$$

Consider an intertemporal reallocation of taxes, from t to $t+s$. Given the intertemporal budget restriction, we must have:

$$dT(t+s) = -e^{\int_t^{t+s} r(\mu)\cdot d\mu} \cdot dT(t) \ .$$

That is, the increase in taxes in $t+s$ equals the decrease in t compounded at the rate of interest. Having no effect on $B(t)$, this reallocation affects aggregate demand in t through its effect on the human wealth of the present generations. Differentiating the relation that defines human wealth:

$$dH(t) = -dT(t) \cdot e^{-\int_t^t [r(\mu)+p]\cdot d\mu} - dT(t+s) \cdot e^{-\int_t^{t+s} [r(\mu)+p]\cdot d\mu} \Rightarrow$$

$$\Rightarrow dH(t) = -dT(t) + e^{\int_t^{t+s} r(\mu)\cdot d\mu} \cdot dT(t) \cdot e^{-\int_t^{t+s} [r(\mu)+p]\cdot d\mu} \Rightarrow$$

$$\Rightarrow dH(t) = -dT(t) \cdot (1 - e^{\int_t^{t+s} p\cdot d\mu}) = -dT(t) \cdot (1 - e^{-ps})$$

The reallocation of taxes from t to $t+s$ (notice that $dT(t) < 0$) has a positive effect on human wealth, raising private consumption. This positive effect arises from the difference between the discount rates in the budget constraint of the government (r), and the private rate of return on assets ($r+p$). This, in turn, reflects the fact that taxes are shifted to the future and the probability of the individuals not being around when its time to pay them. With $p = 0$ (Ramsey model), the effect is null - if individuals have lives of infinite length, they surely end up paying the future taxes.

The Ricardian equivalence is the proposition that deficit finance is no different from taxation because the individuals take into account the future taxes they will have to pay. In this model, with positive p, this proposition fails. However, if present generations care about their descendants, leaving them bequests, Ricardian Equivalence holds again. In case of a transference from taxes to deficit finance, the individuals make a corresponding increase in their savings, leaving an increased bequest that allows their children to pay the future tax increase.

Other reason for the failure of Ricardian equivalence is the relaxation of liquidity constraints that is associated with a reallocation of taxes from the present to the future. In reality, some people are childless, while other have many children and may care more or less about them. This complexity turns the issue into an essentially empirical one, of determining how important are the departures from Ricardian equivalence.

4.3 Fiscal Policy and Interest Rates: Steady State

With fiscal policy, the dynamic equations of the economy become:

$$\begin{cases} \dot{C} = (r-\theta)\cdot C - p\cdot(p+\theta)\cdot(B+K)\ ; \\ \dot{K} = F(K) - C - G\ ; \\ \dot{B} = r\cdot B + G - T\ ; \\ r = F'(K)\ . \end{cases}$$

Given the technical difficulty of solving this system, we first discuss the properties of the steady state and then consider the output as exogenous to examine the effects of fiscal policy on interest rates.

In steady state, we have:

$$\begin{cases} [F'(K^*)-\theta]\cdot C^* = p\cdot(p+\theta)\cdot(B^*+K^*)\ ; \\ F(K^*) = C^* + G^*\ ; \\ F'(K^*)\cdot B^* = T^* - G^*\ . \end{cases}$$

Assuming that individuals hold positive tangible assets B^*+K^* (government debt is not too negative), the first condition determines that $F'(K^*)-\theta > 0$. This assumption, equivalent to $r^* > \theta$, excludes the possibility of dynamic inefficiency.

The steady state level of government debt affects on the steady state capital stock (consider that taxes are variable while G is exogenously given):

$$F''(K^*) \cdot \frac{dK^*}{dB^*} \cdot [F(K^*) - G] + [F'(K^*) - \theta] \cdot F'(K^*) \cdot \frac{dK^*}{dB^*} =$$

$$= p \cdot (p + \theta) \cdot (1 + \frac{dK^*}{dB^*}) \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dB^*} \cdot \{F'' \cdot [F - G] + F' \cdot [F' - \theta] - p \cdot (p + \theta)\} = p \cdot (p + \theta) \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dB^*} = \frac{p \cdot (p + \theta)}{F'' \cdot [F - G] + F' \cdot [F' - \theta] - p \cdot (p + \theta)} \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dB^*} = \frac{p \cdot (p + \theta)}{F'' \cdot C^* + r^* \cdot (r^* - \theta) - p \cdot (p + \theta)}$$

With constant wealth, we have:

$$\dot{K} + \dot{B} = F(K) - C - G + r \cdot B + G - T = 0 \Rightarrow$$

$$\Rightarrow Y - C + r \cdot B - T = 0 \Rightarrow$$

$$\Rightarrow C^* = r^* \cdot (B + K) + Y - T \Rightarrow$$

$$\Rightarrow (r^* - \theta) \cdot C^* = (r^* - \theta) \cdot [r^* \cdot (B + K) + Y - T] = p \cdot (p + \theta) \cdot (B + K) \Rightarrow$$

$$\Rightarrow (r^* - \theta) \cdot \left(r^* + \frac{Y - T}{B + K}\right) = p \cdot (p + \theta) \Rightarrow$$

$$\Rightarrow r^* \cdot (r^* - \theta) - p \cdot (p + \theta) = (r^* - \theta) \cdot \frac{Y - T}{B + K}$$

Using this in the previous result we obtain:

$$\frac{dK^*}{dB^*} = \frac{p \cdot (p + \theta)}{F'' \cdot C^* + (r^* - \theta) \cdot \frac{Y - T}{B^* + K^*}}$$

With $Y > T$, a greater level of debt decreases the steady state stock of capital.

4.4 Fiscal Policy and Interest Rates: Dynamics

Now we assume constant output and no capital. The interest rate is not linked to capital accumulation. Actually it is such that aggregate demand equals exogenous supply. The economy is described by:

$$\begin{cases} \dot{C} = (r-\theta)\cdot C - p\cdot(p+\theta)\cdot B \ ; \\ \\ Y = C + G \ ; \\ \\ \dot{B} = r\cdot B + G - T \ . \end{cases}$$

With constant exogenous Y, if we consider constant G, then $\dot{C} = 0$. Recall that in equilibrium the interest rate is such that makes consumption constant. Solving:

$$0 = (r-\theta)\cdot C - p\cdot(p+\theta)\cdot B \Rightarrow$$

$$\Rightarrow r-\theta = p\cdot(p+\theta)\cdot\frac{B}{C} \Rightarrow$$

$$\Rightarrow r = \theta + p\cdot(p+\theta)\cdot\frac{B}{Y-G}.$$

In this simple exchange economy, the interest rate increases with debt and government spending, and decreases with output.

Consider the sequence of deficits implied by:

$$\dot{B} = r\cdot B + G - T(B,x) \ , \quad T_B > r + B\cdot\frac{dr}{dB} \ , \quad T_x > 0.$$

Taxes are assumed to be an increasing function of the debt and a parameter x. We consider a decrease in x, that is, a decrease in taxes. This increases the deficit and, thus, the debt. Taxes then rise and a balanced budget is again achieved. Notice that the rise in taxes is sufficient to offset the debt increase:

$$\frac{d\dot{B}}{dB} = r + B \cdot \frac{dr}{dB} - T_B < 0 \Rightarrow \text{Stability.}$$

Now we introduce a long-term interest rate - the yield on perpetuities paying a constant coupon flow of unity. Let R be their yield, and thus $1/R$ be their price:

$$P = 1 + 1 \cdot (1-R) + 1 \cdot (1-R)^2 + ... = \frac{1}{1-(1-R)} = \frac{1}{R}.$$

The instantaneous rate of return is:

$$\frac{1 + d(1/R)/dt}{1/R} = R + R \cdot (-1) \cdot \frac{dR/dt}{R^2} = R - \frac{\dot{R}}{R}.$$

The return on perpetuities is the sum of the yield and the expected rate of capital gain. By arbitrage, the returns on long term and short term bonds are equal:

$$R - \frac{\dot{R}}{R} = r \Rightarrow \frac{\dot{R}}{R} = R - r.$$

We arrive at a dynamical system in R and B:

$$\begin{cases} R - \dfrac{\dot{R}}{R} = \theta + p \cdot (p+\theta) \cdot \dfrac{B}{Y-G} \ ; \\ \dot{B} = \left(R - \dfrac{\dot{R}}{R}\right) \cdot B + G - T(B,x) \ . \end{cases}$$

The condition on T_B ensures that as B increases, the increase in taxes more than compensates for the increase in the interest burden. So B converges to B^*. The condition that makes R constant is a straight line in the graph of $\{B, R\}$:

$$\dot{R} = 0 \Rightarrow R = \frac{p \cdot (p+\theta)}{Y-G} \cdot B + \theta \ .$$

Figure 3.7 shows the phase diagram of the system. A decrease in x (taxes) shifts the vertical line of $\dot{B} = 0$ to the right, increasing the steady state interest rates and decreasing the steady state capital stock (figure 3.8 presents this dynamic effect).

5 Aggregate Saving and the Interest Rate

Whether an increase in the rate of return on saving would increase saving, and consequently investment and the capital stock, is the subject of much controversy. Some argue that the elasticity of aggregate saving with respect to the interest is zero, while others defend that its value is very high, perhaps infinite. In this section we examine the issue of the elasticity of saving with respect to its rate of return.

A change in the rate of return on savings leads, in general, to a change in the rate of saving, and to a change in the stock of wealth. For long-run analysis, the interesting question is about the stock of wealth. In a stationary economy, net saving is zero. So, when comparing steady-states we find no difference in the rates of saving. But the levels of wealth and, consequently, of capital and welfare may differ.

Figure 3.9 plots the tangible wealth of an individual through life. By assumption, she starts and ends its life with no wealth so that her lifelong net saving is zero. In steady state, the figure may be seen as a cross section of the wealth of the different generations, with area A representing aggregate wealth. Notice that this area may be affected by the interest rate, but aggregate saving remains necessarily equal to zero in equilibrium. With finite horizons, a necessary condition for aggregate saving to be positive is either population or productivity growth. In the first case average wealth is constant, but aggregate wealth grows at the rate of population growth. In the second case, the area A may grow according to some scale effect.

5.1 The Two-Period Model

We start by analyzing the elasticity of saving with respect to the interest rate in the context of the two-period-life, no bequest, overlapping generations model developed in section 1. In this model, supply of capital is given by the savings of the young:

$$s_t = s(w_t, r_{t+1}).$$

We saw that the sign of s_r depends on the relative importance of the wealth and substitution effects. These two effects are represented in figure 3.10. An increase in the interest rate shifts the budget restriction from AB to AB', and the equilibrium from E to E'. The total effect is decomposed into a substitution effect (along the indifference curve - E to E") and an income effect (from E" to E'). In this case, the income effect dominates, so saving decreases with the interest rate.

With a CRRA utility function, we have:

$$max\ U_t = \frac{c_{1t}^{\sigma}-1}{1-\sigma} + \frac{1}{(1+\theta)} \cdot \frac{c_{2t+1}^{\sigma}-1}{1-\sigma}$$

$$s.t. \begin{cases} c_{1t} + s_t = w_t \\ c_{2t+1} = (1+r_{t+1}) \cdot s_t \end{cases}$$

Considering s_t as the only decision variable:

$$max\ U_t = \frac{(w_t - s_t)^{\sigma}-1}{1-\sigma} + \frac{1}{(1+\theta)} \cdot \frac{[(1+r_{t+1})\cdot s_t)]^{\sigma}-1}{1-\sigma}$$

$$\Rightarrow \frac{dU_t}{ds_t} = 0 \Rightarrow \frac{1}{1-\sigma} \cdot \sigma \cdot (w_t - s_t)^{\sigma-1} \cdot (-1) + \frac{1}{(1+\theta)\cdot(1-\sigma)} \cdot \sigma \cdot [(1+r_{t+1}) \cdot s_t)]^{\sigma-1} \cdot (1+r_{t+1}) = 0.$$

Simplifying the first order condition, we obtain:

$$(w_t - s_t)^{\sigma-1} = \frac{1}{1+\theta} \cdot [(1+r_{t+1}) \cdot s_t)]^{\sigma-1} \cdot (1+r_{t+1}) \Rightarrow$$

$$\Rightarrow \left(\frac{w_t - s_t}{s_t}\right)^{\sigma-1} = \frac{(1+r_{t+1})^{\sigma}}{1+\theta}.$$

Saving increases, remains constant, or decreases depending on whether the elasticity of substitution $1/\sigma$ is greater, equal or less than unity. In the logarithmic utility case, the effect is null.

5.2 The Model of Perpetual Youth

Now we turn to the model of perpetual growth studied in section 3 to examine the effect of the interest rate on saving. We assume logarithmic utility and constant labor income ($\alpha = 0$).

For simplicity, let the interest rate and the labor income be exogenous. This can be viewed as a partial equilibrium analysis or as a model of a small open economy.

Aggregate consumption is a linear function of wealth. Human wealth is the present discounted value of labor income (which equals $\frac{w}{r+p}$ with constant r and w). Wealth accumulation is equal to aggregate saving.

The equations of motion are:

$$\begin{cases} C = (p+\theta)\cdot\left(K + \dfrac{w}{r+p}\right) \ ; \\ \\ \dot{K} = S = r\cdot K - C + w \ . \end{cases}$$

In this setting, what are the dynamic effects of a permanent increase in interest rates? First we consider $\theta < r < \theta + p$. Figure 3.11 shows the stable dynamic system. The assumption $r < \theta + p$ implies that the consumption line is more sloped than the line of wealth accumulation. The effects of an increase in r are described in figure 3.12. The consumption line shifts down, due to the wealth effect caused by the increase in the discount rate. And the line of constant wealth, which has a slope equal to r, rotates upward.

The two effects are in opposite directions, but, nevertheless, the new steady state level of wealth is unambiguously higher. Solving for K^*, we see that it is positive with our condition $\theta < r < \theta + p$:

$$r \cdot K^* + w = (p+\theta) \cdot \left(K^* + \frac{w}{r+p}\right) \Rightarrow$$

$$\Rightarrow (r-p-\theta) \cdot K^* = \left(\frac{p+\theta}{r+p} - 1\right) \cdot w \Rightarrow$$

$$\Rightarrow (r-p-\theta) \cdot K^* = \frac{\theta - r}{r+p} \cdot w \Rightarrow$$

$$\Rightarrow K^* = \frac{r-\theta}{(r+p)(p+\theta-r)} \cdot w > 0.$$

Differentiating:

$$\frac{dK^*}{dr} = \frac{(r+p)(p+\theta-r) + [(r+p)-(p+\theta-r)](r-\theta)}{(r+p)^2(p+\theta-r)^2} \cdot w \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dr} = \frac{(r+p)(p+\theta-r) + (2r-\theta)(r-\theta)}{(r+p)^2(p+\theta-r)^2} \cdot w \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dr} = \frac{rp + p^2 + r\theta + p\theta - r^2 - pr + 2r^2 - 2r\theta - \theta r + \theta^2}{(r+p)^2(p+\theta-r)^2} \cdot w \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dr} = \frac{p^2 + p\theta + r^2 - 2r\theta + \theta^2}{(r+p)^2(p+\theta-r)^2} \cdot w \Rightarrow$$

$$\Rightarrow \frac{dK^*}{dr} = \frac{(r-\theta)^2 + p(p+\theta)}{(r+p)^2(p+\theta-r)^2} \cdot w > 0.$$

An increase in the interest rate increases income because of higher interest payments. Meanwhile, consumption decreases because of the decrease in human wealth. Both effects lead to increased saving. With the accumulation of wealth, consumption increases faster than income. In the new steady state, wealth is higher and saving is again zero.

Remember that in the two-period model, human wealth was unaffected by the interest rate, because all income was received in the first period. Now an increase in interest rates negatively affects human wealth.

With low elasticity of substitution, short run effects are quite different from those prevailing in the long run. The dynamic effects of the interest rate on savings may be slow: consumption may increase initially, but the higher interest rate implies a higher rate of wealth accumulation. This effect eventually dominates the first, leading to a positive long-run response of aggregate saving and wealth. Notice that there is some similarity between the short run effects and the predictions of the 2-period model.

In this model, the effect of the interest rate in steady state wealth can be quite substantial. With $p = \theta = 4\%$, an increase in r from 5% to 6% increases steady state wealth by a factor of 2.7.

Figure 3.13 shows what happens when $r > \theta + p$ (in section 3 we showed that in a closed economy $r < \theta + p$). There is an unstable equilibrium with negative wealth. Starting with zero wealth, accumulation is unending.

5.3 The Infinite Horizon Model

Now we examine the case of infinite horizons, that is, the case in which individuals care about their heirs enough to leave bequests. From the first order condition of the Ramsey problem, we see that $\dot{C}$ has the same sign as $r - \theta$. With $r > \theta$, individuals accumulate endlessly; with $r = \theta$, individuals do not accumulate at all; and with $r < \theta$, individuals consume their wealth. Thus, the elasticity of steady state wealth with respect to the interest rate is infinite.

The previous reasoning is valid for an exogenous interest rate. In a complete model, capital accumulation would have the effect of decreasing the interest rate, that would converge again to θ.

Consider the steady state condition of the Ramsey model, which is a modified golden rule:

$$f'(k^*) = \theta + n.$$

If a subsidy is given to capital, it becomes:

$$(1+\epsilon) \cdot f'(k^*) = \theta + n.$$

The effect of the subsidy on capital accumulation can be derived:

$$f'(k^*) + (1+\epsilon) \cdot f''(k^*) \cdot \frac{dk^*}{d\epsilon} = 0 \Rightarrow \frac{dk^*}{d\epsilon} = -\frac{f'(k^*)}{(1+\epsilon) \cdot f''(k^*)}.$$

For a constant returns to scale production function that allows us to write output as $f(k)$, elasticity of substitution is:

$$\sigma = -\frac{f'(k^*) \cdot w}{f''(k^*) \cdot f(k^*) \cdot k}.$$

Substituting, we obtain:

$$\frac{dk^*/d\epsilon}{k^*} = -\frac{f'(k^*)}{(1+\epsilon) \cdot f''(k^*) \cdot k^*} = -\frac{f'(k^*) \cdot w}{f''(k^*) \cdot k^*} \cdot f(k^*) \cdot \frac{f(k^*)}{(1+\epsilon) \cdot w} \Rightarrow$$

$$\Rightarrow \frac{dk^*/d\epsilon}{k^*} = \frac{\sigma \cdot f(k^*)}{(1+\epsilon) \cdot w}.$$

A subsidy to capital stimulates saving, which then reduces the interest rate. It is the production function that determines the steady state effects. The effectiveness of subsidies in raising capital accumulation is greater for greater elasticities of substitution and for smaller shares of wages in output.

If the production function is Cobb-Douglas with a labor's share of 75% and ϵ is increased from 0 to 25%, the steaty state capital stock increases by 33%.

The use of these three models suggests a positive elasticity of saving with respect to the interest rate. But empirical research has not discovered these saving and wealth elasticities. Our experiments considered permanent increases in the interest rate, while in reality, interest movements are mostly temporary (and seen as temporary). So the wealth effect is pretty small. Movements do not last, so we observe only short-run responses, which are highly dependent on the elasticity of substitution of consumption.

6 Exercises

Problem 6.1 *In the simplest two-period life-cycle model, assume that the utility function is nonseparable, and derive explicitly the expressions for s_w and s_r. Explain under what circumstances*

(a) $0 < s_w < 1$.

(b) $s_r > 0$.

Solution 6.1 *This is a problem of comparative statics. We seek to estimate the impact on equilibrium savings of small changes in wages and in the interest rate.*

The problem of the economic agents is the following:

$$\max_{c_{1t}, c_{2t+1}} u(c_{1t}, c_{2t+1}) \quad , \textit{ subject to} \begin{cases} c_{1t} + s_t = w_t \\ \\ c_{2t+1} = (1 + r_{t+1}) \cdot s_t. \end{cases}$$

Since s_t determines consumption in both periods, the problem can be formulated in a more convenient way:

$$\max_{s_t} u(w_t - s_t, (1 + r_{t+1}) \cdot s_t) \quad , \textit{ with} \quad 0 \leq s_t \leq w_t.$$

Assume for now that an optimal consumption exists and is positive in both periods. This makes the first order condition on s_t necessary and sufficient:

$$u_1(w_t - s_t, (1 + r_{t+1}) \cdot s_t) \cdot (-1) + u_2(w_t - s_t, (1 + r_{t+1}) \cdot s_t) \cdot (1 + r_{t+1}) = 0 \Rightarrow$$

$$\Rightarrow u_1(w_t - s_t, (1 + r_{t+1}) \cdot s_t) = u_2(w_t - s_t, (1 + r_{t+1}) \cdot s_t) \cdot (1 + r_{t+1}).$$

To derive s_w, differentiate this expression with respect to w:

$$u_{11} \cdot (1 - s_w) + u_{12} \cdot (1 + r_{t+1}) \cdot s_w =$$

$$= (1 + r_{t+1}) \cdot [u_{21} \cdot (1 - s_w) + u_{22} \cdot (1 + r_{t+1}) \cdot s_w].$$

With some manipulation, s_w can be made explicit:

$$s_w \cdot [-u_{11} + u_{12} \cdot (1 + r_{t+1})] + u_{11} =$$

$$= s_w \cdot (1 + r_{t+1}) \cdot [-u_{21} + u_{22} \cdot (1 + r_{t+1})] + (1 + r_{t+1}) \cdot u_{21} \Rightarrow$$

$$\Rightarrow s_w \cdot \left[-u_{11} + (u_{12} + u_{21}) \cdot (1 + r_{t+1}) - u_{22} \cdot (1 + r_{t+1})^2\right] =$$

$$= -u_{11} + (1 + r_{t+1}) \cdot u_{21} \Rightarrow$$

$$\Rightarrow s_w = \frac{-u_{11} + (1 + r_{t+1}) \cdot u_{21}}{-u_{11} + (u_{12} + u_{21}) \cdot (1 + r_{t+1}) - u_{22} \cdot (1 + r_{t+1})^2}.$$

Imposing the second order condition:

$$u_{11} - u_{12} \cdot (1 + r_{t+1}) + (1 + r_{t+1}) \cdot [u_{21} \cdot (-1) + u_{22} \cdot (1 + r_{t+1})] < 0 \Rightarrow$$

$$\Rightarrow u_{11} - (u_{12} + u_{21}) \cdot (1 + r_{t+1}) + u_{22} \cdot (1 + r_{t+1})^2 < 0.$$

With decreasing marginal utility, the numerator in the expression of s_w is positive. Assuming that the second order condition is satisfied, the denominator is also positive. In these conditions, $0 < s_w < 1$ is satisfied because, obviously:

$$u_{12} - u_{22} \cdot (1 + r_{t+1}) > 0.$$

We found that the effect of wages on savings is always positive, but the effect of the interest rate on savings is more complex. On one hand, there is an income effect that increases consumption in both periods. But on the other hand, second period consumption becomes cheaper relatively to consumption in the first period, so there is a substitution effect that decreases consumption in the first period and *increases it in the second period. The income effect diminishes saving while the substitution effect increases saving.*

To derive s_r, differentiate the optimality condition with respect to r:

$$u_{11} \cdot (-s_r) + u_{12} \cdot [s_t + (1 + r_{t+1}) \cdot s_r] =$$

$$= u_2 + (1 + r_{t+1}) \cdot \{u_{21} \cdot (-s_r) + u_{22} \cdot [s_t + (1 + r_{t+1}) \cdot s_r]\}.$$

Manipulating, we make s_r explicit:

$$s_r \cdot [-u_{11} + u_{12} \cdot (1 + r_{t+1})] + u_{12} \cdot s_t =$$

$$= s_r \cdot \left[-(1 + r_{t+1}) \cdot u_{21} + u_{22} \cdot (1 + r_{t+1})^2\right] + u_2 + u_{22} \cdot (1 + r_{t+1}) \cdot s_t \Rightarrow$$

$$\Rightarrow s_r = \frac{u_2 - u_{12} \cdot s_t + u_{22} \cdot (1 + r_{t+1}) \cdot s_t}{-u_{11} + (u_{12} + u_{21}) \cdot (1 + r_{t+1}) - u_{22} \cdot (1 + r_{t+1})^2}.$$

From the second order condition, the denominator is positive. Therefore, $s_r > 0$ is true if:

$$u_2 > [u_{12} - u_{22} \cdot (1 + r_{t+1})] \cdot s_t.$$

That is, if the substitution effect (second period consumption becomes cheaper relatively to first period consumption) more than offsets the income effect of the increase in the interest rate.

Problem 6.2 *Consider an overlapping generation economy in which each individual lives for two periods.*

Population is constant. The individuals' endowments in each period are exogenous. The first-period endowment of an individual born at time t is equal to e_t, *and the second period endowment of the same individual to* $e_t \cdot (1+g)$, *where* g *can be negative. Each individual saves by investing in a constant returns technology, where each unit invested yields* $1+r$ *units of output in the following period.*

An individual born at time t *maximizes:*

$$U(c_{1t}, c_{2t+1}) = log(c_{1t}) + \frac{1}{1+d} \cdot log(c_{2t+1}), \quad d > 0.$$

Finally, the first period endowments grow at rate m:

$$e_t = (1+m) \cdot e_{t-1}.$$

(a) How does an increase in the growth rate of income expected by one individual, g, *affect his saving rate?*

(b) How does an increase in m *affect aggregate saving?*

(c) Assuming $g = m$ *(or* $g = m - x$, *for a given* x*), how does an increase in* m *affect aggregate saving?*

(d) In light of these results, assess the theoretical validity of the claim that high growth is responsible for the high Japanese saving rate.

(e) "The reason why the saving rate has gone down in the United States in the 1980s despite the supply side incentives is that growth prospects are much more favorable than in the 1970s." Comment.

Solution 6.2 *Start by reformulating the problem of the individual, considering that the decision is only on* s_t:

$$\max_{s_t} \; log(e_t - s_t) + \frac{1}{1+d} \cdot log(e_t \cdot (1+g) + (1+r_{t+1}) \cdot s_t) \quad , s.t. \quad 0 < s_t < e_t.$$

From the first order condition we obtain:

$$-\frac{1}{e_t - s_t} + \frac{1}{1+d} \cdot \frac{1+r_{t+1}}{e_t \cdot (1+g) + (1+r_{t+1}) \cdot s_t} = 0 \Rightarrow$$

$$\Rightarrow (1 + r_{t+1}) \cdot (e_t - s_t) = (1 + d) \cdot [e_t \cdot (1 + g) + (1 + r_{t+1}) \cdot s_t].$$

(a) Differentiating the optimality condition with respect to g, we can derive s_g:

$$(1 + r_{t+1}) \cdot (-s_g) = (1 + d) \cdot [e_t + (1 + r_{t+1}) \cdot s_g] \Rightarrow$$

$$\Rightarrow (1 + r_{t+1}) \cdot (-2 - d) \cdot s_g = (1 + d) \cdot e_t \Rightarrow$$

$$\Rightarrow s_g = -\frac{(1 + d) \cdot e_t}{(1 + r_{t+1}) \cdot (2 + d)} < 0.$$

The impact on the saving rate of an increase in second period income is unambiguously negative. The increase in total income leads the agent to raise consumption (decrease saving) in the first period.

(b) In the 2-period economy, aggregate saving is simply the saving of the generation born in t. It is obvious that if e_t is given, variations of m have no effect on the behavior of the individuals born in t. What we suppose is that e_{t-1} is given, so an increase in m leads to an increase in e_t and consequently affects saving. To find this last effect we express the optimality condition in terms of e_{t-1} and then differentiate it with respect to m:

$$(1 + r_{t+1}) \cdot [(1 + m) \cdot e_{t-1} - s_t] =$$

$$= (1 + d) \cdot [(1 + m) \cdot e_{t-1} \cdot (1 + g) + (1 + r_{t+1}) \cdot s_t].$$

Differentiating:

$$\Rightarrow (1 + r_{t+1}) \cdot (e_{t-1} - s_m) = (1 + d) \cdot [e_{t-1} \cdot (1 + g) + (1 + r_{t+1}) \cdot s_m] \Rightarrow$$

$$\Rightarrow -(2 + d) \cdot (1 + r_{t+1}) \cdot s_m = [(1 + d) \cdot (1 + g) - (1 + r_{t+1})] \cdot e_{t-1} \Rightarrow$$

$$\Rightarrow s_m = \frac{(1 + r_{t+1}) - (1 + d) \cdot (1 + g)}{(2 + d) \cdot (1 + r_{t+1})} \cdot e_{t-1}.$$

With given e_t, the impact is obviously null. If the raise in m induces an increase in e_t, the impact on the saving rate is ambiguous. It is positive for high interest rates, low discounts on second period utility and low income growth.

(c) Considering e_t as given, we have the same problem as in a). Again, what is interesting is to consider e_{t-1} given. We want to examine the effect on saving of an increase in first period income together with an even greater increase in second period income. The optimality condition comes slightly modified:

$$(1+m) \cdot e_{t-1} - s_t = \frac{1+d}{1+r_{t+1}} \cdot [(1+m) \cdot e_{t-1} \cdot (1+m-x) + (1+r_{t+1}) \cdot s_t].$$

Differentiating:

$$e_{t-1} - s_m = \frac{1+d}{1+r_{t+1}} \cdot [e_{t-1} \cdot (1+m-x+1+m) + (1+r_{t+1}) \cdot s_m] \Rightarrow$$

$$\Rightarrow s_m = \frac{(1+r_{t+1}) - (1+d) \cdot (2+2m-x)}{(2+d) \cdot (1+r_{t+1})} \cdot e_{t-1}.$$

The effect is negative unless the interest rate is very high.

(d) The results that we have obtained contradict this hypothesis. In the 2-period overlapping generations model, with everything else constant, high (income) growth leads to a decrease in savings.

(e) The 2-period overlapping generations model supports the hypothesis that more favorable growth prospects induce lower savings. We showed how an expected increase of future incomes diminishes the marginal utility of savings, and, therefore, diminishes savings.

Problem 6.3 *a) Suppose that an individual receives wages w_1 and w_2 in the two periods of life and has a constant relative risk aversion utility function. Examine the effects of a change in the interest rate on saving, and contrast the results with those for $w_2 = 0$.*

b) Suppose that $w_1 = w_2$. Can the steady state in this model be dynamically inefficient? Why?

Solution 6.3 *The problem of the individual is the following:*

$$\max_{c_1,c_2} \frac{c_1^{1-R}}{1-R} + \frac{1}{1+d} \cdot \frac{c_2^{1-R}}{1-R} \quad \text{, subject to} \quad \begin{cases} c_1 + s = w_1 \\ \\ c_2 = w_2 + (1+r) \cdot s \ . \end{cases}$$

For convenience, we formulate an equivalent problem with s as the only decision variable:

$$\max_{s_t} \frac{(w_1 - s)^{1-R}}{1-R} + \frac{1}{1+d} \cdot \frac{(w_2 + (1+r) \cdot s)^{1-R}}{1-R} \quad \text{, with} \quad 0 \leq s \leq w_1 \ .$$

The first order condition is:

$$-(w_1 - s)^{-R} + \frac{1}{1+d} \cdot (w_2 + (1+r) \cdot s)^{-R} \cdot (1+r) = 0 \Rightarrow$$

$$\Rightarrow (w_1 - s)^{-R} = \frac{1+r}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R}.$$

Observe that it is equivalent to:

$$\frac{c_1}{c_2} = \left(\frac{1+d}{1+r}\right)^{1/R}.$$

Differentiating the first expression with respect to the interest rate:

$$R \cdot (w_1 - s)^{-R-1} \cdot s_r = \frac{1+r}{1+d} \cdot (-R) \cdot [w_2 + s \cdot (1+r)]^{-R-1} \cdot (s_r \cdot (1+r) + s) +$$

$$+ \frac{1}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R} \Rightarrow$$

$$\Rightarrow s_r \cdot R \cdot \left\{ (w_1 - s)^{-R-1} + \frac{(1+r)^2}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R-1} \right\} =$$

$$= \frac{1+r}{1+d} \cdot (-R) \cdot [w_2 + s \cdot (1+r)]^{-R-1} \cdot s + \frac{1}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R} =$$

$$= \frac{1}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R-1} \cdot [-R \cdot s \cdot (1+r) + w_2 + s \cdot (1+r)] =$$

$$= \frac{1}{1+d} \cdot [w_2 + s \cdot (1+r)]^{-R-1} \cdot [s \cdot (1+r) \cdot (1-R) + w_2] \Rightarrow$$

$$\Rightarrow s_r = \frac{[w_2 + s \cdot (1+r)]^{-R-1} \cdot [s \cdot (1+r) \cdot (1-R) + w_2]}{(1+d) \cdot R \cdot (w_1 - s)^{-R-1} + R \cdot (1+r)^2 \cdot [w_2 + s \cdot (1+r)]^{-R-1}} \Rightarrow$$

$$\Rightarrow s_r = \frac{1}{R} \cdot \frac{s \cdot (1+r) \cdot (1-R) + w_2}{(1+d) \cdot (\frac{c_1}{c_2})^{-R-1} + (1+r)^2} = \frac{1}{R} \cdot \frac{s \cdot (1+r) \cdot (1-R) + w_2}{(1+d) \cdot (\frac{1+r}{1+d})^{\frac{1+R}{R}} + (1+r)^2} \Rightarrow$$

$$\Rightarrow s_r = \frac{1}{R} \cdot \frac{s \cdot (1+r) \cdot (1-R) + w_2}{(1+d)^{-\frac{1}{R}} \cdot (1+r)^{\frac{1+R}{R}} + (1+r)^2}.$$

The sign of s_r is the same as the sign of R. With risk aversion ($R > 0$), an increase in the interest rate has a positive effect on savings.

With $w_2 = 0$ we have:

$$\frac{s \cdot (1+r)}{w_1 - s} = \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}} \Rightarrow$$

$$\Rightarrow s \cdot (1+r) = \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}} \cdot w_1 - \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}} \cdot s \Rightarrow$$

$$\Rightarrow s \cdot \left[1 + r + \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}}\right] = \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}} \cdot w_1 \Rightarrow$$

$$\Rightarrow s = \frac{\left(\frac{1+r}{1+d}\right)^{\frac{1}{R}} \cdot w_1}{1 + r + \left(\frac{1+r}{1+d}\right)^{\frac{1}{R}}}.$$

Substituting in the expression of s_r:

$$\Rightarrow s_r = \frac{1}{R} \cdot \frac{w_1 \cdot (1-R)}{(1+d)^{-\frac{1}{R}} \cdot (1+r)^{\frac{1}{R}} + 1 + r}.$$

The denominator is smaller, and the numerator should be greater than with $w_2 > 0$, as long as w (when $w_2 = 0$) is greater than the $s \cdot (1+r)$ of the case where $w_2 > 0$. So, the effect remains positive but is amplified.

Notice that with equal income: $w \cdot (1+r) = w_1 \cdot (1+r) + w_2$, the shift from income in period one to income in period 2 diminishes savings so that consumption remains equal. There is, of course, no income effect and also no substitution effect, because relative prices remain unchanged.

Now we consider $w_1 = w_2$ and investigate whether the steady state can be dynamically inefficient in a model of constant population.

With constant population and without discount on second period income, dynamic inefficiency would correspond to negative interest rates. This is impossible, as we assume always positive marginal productivity of capital.

In steady state we have:

$$\frac{c_1}{c_2} = \left(\frac{1+d}{1+r}\right)^{1/R}.$$

Consumption is equalin both periods if savings are null. With positive savings, consumption in second period is greater.

$$c_2 \geq c_1 \Rightarrow \frac{1+d}{1+r} \leq 1 \Rightarrow d \leq r.$$

The interest rate is higher than the individual discount, so the steady state cannot be dynamically inefficient.

Problem 6.4 *Assume a Cobb-Douglas production function, with share of labor α, and the simplest two-period-life overlapping generations model. The population grows at rate n. Individuals supply inelastically one unit of labor in the first period of their lives and have logarithmic utility over consumption:*

$$U = log(c_{1t}) + \frac{1}{1+\theta} \cdot log(c_{2t+1}).$$

a) Solve for the steady state capital stock.

b) Show how the introduction of pay-as-you-go social security, in which the government collects the amount d from each young person and and gives $(1+n) \cdot d$ to each old person, affects the steady state capital stock.

Solution 6.4 *a) Start by solving the problem of the individual, considering that the decision is only on s_t:*

$$\max_{s_t} \; log(w_t - s_t) + \frac{1}{1+\theta} \cdot log(s_t \cdot (1 + r_{t+1})) \quad \text{, subject to} \quad 0 \leq s_t \leq w_t \; .$$

From the first order condition, we derive a saving function that is independent of the interest rate:

$$\frac{-1}{w_t - s_t} + \frac{1}{1+\theta} \cdot \frac{1 + r_{t+1}}{s_t \cdot (1 + r_{t+1})} \Rightarrow$$

$$w_t - s_t = (1+\theta) \cdot s_t \;\Rightarrow\; s_t = \frac{w_t}{2+\theta}.$$

This means that a variation of the interest rate induces an income effect and a substitution effect that exactly cancel each other. Therefore, saving and first period consumption are independent of the interest rate.

The capital stock at $t+1$ is equal to the savings made at t. In per capita terms, we have:

$$k_{t+1} = \frac{s_t}{1+n} = \frac{w_t}{(2+\theta)\cdot(1+n)}.$$

And profit maximization implies equality between wages and marginal productivity of labor:

$$k_{t+1} = \frac{\alpha \cdot k_t^{1-\alpha}}{(2+\theta)\cdot(1+n)}.$$

In steady state, $k_{t+1} = k_t$:

$$\frac{\alpha \cdot k^{-\alpha}}{(2+\theta)\cdot(1+n)} = 1 \Rightarrow$$

$$\Rightarrow k^{-\alpha} = \frac{(2+\theta)\cdot(1+n)}{\alpha} \Rightarrow$$

$$\Rightarrow k^* = \left(\frac{\alpha}{(2+\theta)\cdot(1+n)}\right)^{\frac{1}{\alpha}}.$$

b) With the introduction of a "pay-as-you-go" social security system, the problem of the individual becomes:

$$\max_{s_t}\ log(w_t - s_t - d) + \frac{1}{1+\theta}\cdot log(s_t\cdot(1+r_{t+1}) + (1+n)\cdot d),$$

subject to:

$$0 \le s_t \le w_t - d\ .$$

The corresponding first order condition is:

$$\frac{-1}{w_t - s_t - d} + \frac{1}{1+\theta}\cdot\frac{1+r_{t+1}}{s_t\cdot(1+r_{t+1}) + (1+n)\cdot d} \Rightarrow$$

$$\Rightarrow (1+r_{t+1})\cdot(w_t - s_t - d) = (1+\theta)\cdot[s_t\cdot(1+r_{t+1}) + (1+n)\cdot d] \Rightarrow$$

$$\Rightarrow (1 + r_{t+1}) \cdot (w_t - (2 + \theta) \cdot s_t - d) = (1 + \theta) \cdot (1 + n) \cdot d \Rightarrow$$

$$\Rightarrow (2 + \theta) \cdot (1 + r_{t+1}) \cdot s_t = w_t \cdot (1 + r_{t+1}) - d \cdot [1 + r_{t+1} + (1 + \theta) \cdot (1 + n)] \Rightarrow$$

$$\Rightarrow s_t = \frac{w_t}{2 + \theta} - \frac{1 + r_{t+1} + (1 + \theta) \cdot (1 + n)}{(1 + r_{t+1}) \cdot (2 + \theta)} \cdot d.$$

The introduction of this social security system diminishes savings. On the other hand, the contributions for the social security raise the capital stock. So, the impact on the capital stock depends on whether the parameter that multiplies d is smaller or greater than 1. For capital stock to increase it is necessary that:

$$\frac{1 + r^* + (1 + \theta) \cdot (1 + n)}{(1 + r^*) \cdot (2 + \theta)} < 1 \Rightarrow$$

$$\Rightarrow 1 + r^* + (1 + \theta) \cdot (1 + n) < (1 + r^*) \cdot (2 + \theta) \Rightarrow$$

$$\Rightarrow 1 + r^* + 1 + \theta + n + n \cdot \theta < 2 + 2 \cdot r^* + \theta + \theta \cdot r^* \Rightarrow$$

$$\Rightarrow r^* + n + n \cdot \theta < 2 \cdot r^* + \theta \cdot r^* \Rightarrow$$

$$\Rightarrow n \cdot (1 + \theta) < r^* \cdot (1 + \theta) \Rightarrow n < r^*.$$

We conclude that if the steady state interest rate of the economy with social security is greater than the growth rate of the population (dynamic efficiency), which may be seen as the interest rate on social security forced savings, then the effect of social security on the capital stock is positive.

References

1. **O.J. Blanchard, S. Fischer (1989)**, *Lectures in Macroeconomics*, Cambridge: MIT Press, 1989.

2. Allais, M. (1947), *Economie et interet*, Paris: Imprimerie Nationale, 1947.

3. Diamond, P. (1965), *National Debt in a Neoclassical Growth Model*, American Economic Review, 55 (Dec. 1965), pp 1126-50.

4. Kimball, M.S. (1987), *Making Sense of Two-Sided Altruism*, Journal of Monetary Economics, 20 (Sept. 1987), pp 301-26.

5. Samuelson, P. (1958), *An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money*, Journal of Political Economy, 66 (Dec. 1958), pp 467-82.

6. Summers, L.H. (1981), *Capital Taxation and Accumulation in a Life-Cycle Growth Model*, American Economic Review, 71 (Sept. 1981), pp 533-544.