

Introduction to Dynamic Programming Applied to Economics

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Chapter 1

Introduction

We will study the two workhorses of modern macro and financial economics, using dynamic programming methods:

- the intertemporal allocation problem for the representative agent in a finance economy;
- the Ramsey model

in four different environments:

- discrete time and continuous time;
- deterministic and stochastic

methodology

- we use analytical methods
- some heuristic proofs
- and derive explicit equations whenever possible.

1.1 A general overview

We will consider the following types of problems:

1.1.1 Discrete time deterministic models

In the space of the sequences $\{u_t, x_t\}_{t=0}^{\infty}$, such that $u_t \in \mathbb{R}^m$ where $t \mapsto u_t$ and $x_t \in \mathbb{R}$ where $t \mapsto x_t$, choose a sequence $\{u_t^*, x_t^*\}_{t=0}^{\infty}$ that maximizes the sum

$$\max_{\{u\}} \sum_{t=0}^{\infty} \beta^t f(u_t, x_t)$$

subject to the sequence of budget constraints

$$\begin{aligned} x_{t+1} &= g(x_t, u_t), \quad t = 0, \dots, \infty \\ x_0 &\text{ given} \end{aligned}$$

where $0 < \beta \equiv \frac{1}{1+\rho} < 1$, where $\rho > 0$.

By applying the **principle of dynamic programming** the first order necessary conditions for this problem are given by the **Hamilton-Jacobi-Bellman (HJB) equation**,

$$V(x_t) = \max_{u_t} \{f(u_t, x_t) + \beta V(g(u_t, x_t))\}$$

which is usually written as

$$V(x) = \max_u \{f(u, x) + \beta V(g(u, x))\} \quad (1.1)$$

If an optimal control u^* exists, it has the form $u^* = h(x)$, where $h(x)$ is called the **policy function**. If we substitute back in the HJB equation, we get a functional equation

$$V(x) = f(h(x), x) + \beta V[g(h(x), x)].$$

Then **solving the HJB equation** means finding the function $V(x)$ which solves the functional equation. If we are able to determine $V(x)$ (explicitly or numerically) then we can also determine $u_t^* = h(x_t)$. If we substitute in the difference equation, $x_{t+1} = g(x_t, h(x_t))$, starting at x_0 , the solution $\{u_t^*, x_t^*\}_{t=0}^{\infty}$ of the optimal control problem.

Only in very rare cases we can find $V(\cdot)$ explicitly.

1.1.2 Continuous time deterministic models

In the space of (piecewise-)continuous functions of time $(u(t), x(t))$ choose an optimal flow $\{(u^*(t), x^*(t)) : t \in \mathbb{R}_+\}$ such that $u^*(t)$ maximizes the functional

$$V[u] = \int_0^\infty f(u(t), x(t))e^{-\rho t} dt$$

where $\rho > 0$, subject to the instantaneous budget constraint and the initial state

$$\begin{aligned} \frac{dx}{dt} &\equiv \dot{x}(t) = g(x(t), u(t)), t \geq 0 \\ x(0) &= x_0 \text{ given} \end{aligned}$$

hold.

By applying the principle of the dynamic programming the first order conditions for this problem are given by the HJB equation

$$\rho V(x) = \max_u \left\{ f(u, x) + V'(x)g(u, x) \right\}.$$

Again, if an optimal control exists it is determined from the policy function $u^* = h(x)$ and the HJB equation is equivalent to the functional differential equation¹

$$\rho V(x) = f(h(x), x) + V'(x)g(h(x), x).$$

Again, if we can find $V(x)$ we can also find $h(x)$ and can determine the optimal flow $\{(u^*(t), x^*(t)) : t \in \mathbb{R}_+\}$ from solving the ordinary differential equation $\dot{x} = g(h(x), x)$ given $x(0) = x_0$.

1.1.3 Discrete time stochastic models

The variables are **random sequences** $\{u_t(\omega), x_t(\omega)\}_{t=0}^\infty$ which are adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^\infty$ over a probability space (Ω, \mathcal{F}, P) . The domain of the variables is $\omega \in \mathbb{N} \times (\Omega, \mathcal{F}, P, \mathbb{F})$, such that $(t, \omega) \mapsto u_t$ and $x_t \in \mathbb{R}$ where $(t, \omega) \mapsto x_t$. Then $u_t \in \mathbb{R}$ is a **random variable**.

An economic agent chooses a random sequence $\{u_t^*, x_t^*\}_{t=0}^\infty$ that maximizes the sum

$$\max_u E_0 \left[\sum_{t=0}^\infty \beta^t f(u_t, x_t) \right]$$

subject to the contingent sequence of budget constraints

$$\begin{aligned} x_{t+1} &= g(x_t, u_t, \omega_{t+1}), t = 0.. \infty, \\ x_0 &\text{ given} \end{aligned}$$

¹We use the convention $\dot{x} = dx/dt$, if $x = x(t)$, is a time-derivative and $V'(x) = dV/dx$, if $V = V(x)$ is the derivative for any other argument.

where $0 < \beta < 1$.

By applying the principle of the dynamic programming the first order conditions of this problem are given by the HJB equation

$$V(x_t) = \max_u \{f(u_t, x_t) + \beta E_t[V(g(u_t, x_t, \omega_{t+1}))]\}$$

where $E_t[V(g(u_t, x_t, \omega_{t+1}))] = E[V(g(u_t, x_t, \omega_{t+1}))|\mathcal{F}_t]$. If it exists, the optimal control can take the form $u_t^* = f(E_t[v(x_{t+1})])$.

1.1.4 Continuous time stochastic models

The most common problem used in economics and finance is the following: in the space of the flows $\{(u(\omega, t), x(\omega, t)) : \omega = \omega(t) \in (\Omega, \mathcal{F}, P, \mathcal{F}(t)), t \in \mathbb{R}_+\}$ choose a flow $u^*(t)$ that maximizes the functional

$$V[u] = E_0 \left[\int_0^\infty f(u(t), x(t)) e^{-\rho t} dt \right]$$

where $u(t) = u(\omega(t), t)$ and $x(t) = x(\omega(t), t)$ are It processes and $\rho > 0$, such that the instantaneous budget constraint is represented by a stochastic differential equation

$$\begin{aligned} dx &= g(x(t), u(t), t)dt + \sigma(x(t), u(t))dB(t), t \in \mathbb{R}_+ \\ x(0) &= x_0 \text{ given} \end{aligned}$$

where $\{dB(t) : t \in \mathbb{R}_+\}$ is a Wiener process.

By applying the stochastic version of the principle of DP the HJB equation is a second order functional equation

$$\rho V(x) = \max_u \left\{ f(u, x) + g(u, x)V'(x) + \frac{1}{2}(\sigma(u, x))^2 V''(x) \right\}.$$

1.2 References

First contribution: Bellman (1957)

Discrete time: Bertsekas (1976), Sargent (1987), Stokey and Lucas (1989), Ljungqvist and Sargent (2000), Bertsekas (2005a), Bertsekas (2005b)

Continuous time: Fleming and Rishel (1975), Kamien and Schwartz (1991), Bertsekas (2005a), Bertsekas (2005b)

Part I

Deterministic Dynamic
Programming

Chapter 2

Discrete Time

2.1 Optimal control and dynamic programming

General description of the optimal control problem:

- assume that time evolves in a discrete way, meaning that $t \in \{0, 1, 2, \dots\}$, that is $t \in \mathbb{N}_0$;
- the economy is described by two variables that evolve along time: a state variable x_t and a control variable, u_t ;
- we **know** the initial value of the state variable, x_0 , and the law of evolution of the state, which is a function of the control variable ($u(\cdot)$): $x_{t+1} = g_t(x_t, u_t)$;
- we assume that there are m control variables, and that they belong to the set $u_t \in \mathcal{U} \subset \mathbb{R}^m$ for any $t \in \mathbb{N}_0$,
- then, for any sequence of controls, u

$$u \equiv \{u_0, u_1, \dots : u_t \in \mathcal{U}\}$$

the economy can follow a large number of feasible paths,

$$x \equiv \{x_0, x_1, \dots\}, \text{ with } x_{t+1} = g_t(x_t, u_t), u_t \in \mathcal{U}$$

- however, if we have a criteria that allows for the evaluation of all feasible paths

$$U(x_0, x_1, \dots, u_0, u_1, \dots)$$

- and if there is at least an optimal control $u^* = \{u_0^*, u_1^*, \dots\}$ which maximizes U ,

- then there is at least an optimal path for the state variable

$$x^* \equiv \{x_0, x_1^*, \dots\}$$

The most common dynamic optimization problems in economics and finance have the following common assumptions

- timing: the state variable x_t is usually a stock and is measured at the beginning of period t and the control u_t is usually a flow and is measured in the end of period t ;
- horizon: can be finite or is infinite ($T = \infty$). The second case is more common;
- objective functional:
 - there is an intertemporal utility function is additively separable, stationary, and involves time-discounting (impatience):

$$\sum_{t=0}^T \beta^t f(u_t, x_t),$$

- $0 < \beta < 1$, models impatience as $\beta^0 = 1$ and $\lim_{t \rightarrow \infty} \beta^t = 0$;
- $f(\cdot)$ is well behaved: it is continuous, continuously differentiable and concave in (u, x) ;
- the economy is described by an autonomous difference equation

$$x_{t+1} = g(x_t, u_t)$$

where $g(\cdot)$ is autonomous, continuous, differentiable, concave. Then the DE verifies the conditions for existence and uniqueness of solutions;

- the non-Ponzi game condition holds:

$$\lim_{t \rightarrow \infty} \varphi^t x_t \geq 0$$

holds, for a discount factor $0 < \varphi < 1$;

- there may be some side conditions, v.g., $x_t \geq 0$, $u_t \geq 0$, which may produce corner solutions. We will deal only with the case in which the solutions are interior (or the domain of the variables is open).

These assumptions are formalized as **optimal control problems**:

Definition 1. The simplest optimal control problem (OCP): Find $\{u_t^*, x_t\}_{t=0}^T$: which solves

$$\max_{\{u_t\}_{t=0}^T} \sum_{t=0}^T \beta^t f(u_t, x_t)$$

such that $u_t \in \mathcal{U}$ and

$$x_{t+1} = g(x_t, u_t)$$

for x_0, x_T given and T free.

Definition 2. The free terminal state optimal control problem (OCP):

Find $\{u_t^*, x_t\}_{t=0}^{T-1}$: which solves

$$\max_{\{u_t\}_{t=0}^T} \sum_{t=0}^T \beta^t f(u_t, x_t)$$

such that $u_t \in \mathcal{U}$ and

$$x_{t+1} = g(x_t, u_t)$$

for x_0, T given and x_T free. If $T = \infty$ we have the **infinite horizon discounted optimal control problem**

Feasible candidate solutions: paths of $\{x_t, u_t\}$ that verify

$$x_{t+1} = g(x_t, u_t), x_0 \text{ given}$$

for any choice of $\{u_t \in \mathcal{U}\}_{t=0}^T$.

Methods for solving the OCP in the sense of obtaining necessary conditions or necessary and sufficient conditions:

- method of Lagrange (for the case T finite)
- Pontryagin's maximum principle
- dynamic programming principle.

Necessary and sufficient optimality conditions Intuitive meaning:

- necessary conditions: assuming that we know the optimal solution, $\{u_t^*, x_t^*\}$ which optimality conditions should the variables of the problem verify ? (This means that they hold for every extremum feasible solutions);
- sufficient conditions: if the functions defining the problem, $f(\cdot)$ and $g(\cdot)$, verify some conditions, then feasible paths verifying some optimality conditions are solutions of the problem.

In general, if the behavioral functions $f(\cdot)$ and $g(\cdot)$ are well behaved (continuous, continuously differentiable and concave) then necessary conditions are also sufficient.

2.1.1 Dynamic programming

The **Principle of dynamic programming** (Bellman (1957)):

an optimal trajectory has the following property:

for any given initial values of the state variable and for a given value of the state and control variables in the beginning of any period, the control variables should be chosen optimally for the remaining period, if we take the optimal values of the state variables which result from the previous optimal decisions.

We next follow an heuristic approach for deriving necessary conditions for problem 1, following the principle of DP.

Finite horizon Assume that we know a solution optimal control $\{u^*, x_t^*\}_{t=0}^T$. Which properties should the optimal solution have ?

Definition 3. Definition: value function for time τ

$$V_{T-\tau}(x_\tau) = \sum_{t=\tau}^T \beta^{t-\tau} f(u_t^*, x_t^*)$$

Proposition 1. *Given an optimal solution to the optimal control problem, solution optimal control $\{u^*, x_t^*\}_{t=0}^T$, then it verifies **Hamilton-Jacobi-Equation***

$$V_{T-t}(x_t) = \max_{u_t} \{f(x_t, u_t) + \beta V_{T-t-1}(x_{t+1})\} \quad (2.1)$$

Proof. If we know a solution for problem 1, then at time $\tau = 0$, we have

$$\begin{aligned} V_T(x_0) &= \sum_{t=0}^T \beta^t f(u_t^*, x_t^*) = \\ &= \max_{\{u_t\}_{t=0}^T} \sum_{t=0}^T \beta^t f(u_t, x_t) = \\ &= \max_{\{u_t\}_{t=0}^T} (f(x_0, u_0) + \beta f(x_1, u_1) + \beta^2 f(x_2, u_2) + \dots) = \\ &= \max_{\{u_t\}_{t=0}^T} \left(f(x_0, u_0) + \beta \sum_{t=1}^T \beta^{t-1} f(x_t, u_t) \right) = \\ &= \max_{u_0} \left(f(x_0, u_0) + \beta \max_{\{u_t\}_{t=1}^T} \sum_{t=1}^T \beta^{t-1} f(x_t, u_t) \right) \end{aligned}$$

by the principle of dynamic programming. Then

$$V_T(x_0) = \max_{u_0} \{f(x_0, u_0) + \beta V_{T-1}(x_1)\}$$

We can apply the same idea for the value function for any time $0 \leq t \leq T$ to get equation (2.1), which holds for feasible solutions, i.e., verifying $x_{t+1} = g(x_t, u_t)$ and given x_0 . \square

Intuition: we transform the maximization of a functional into a recursive two-period problem. We solve the control problem by solving the HJB equation. To do this we have to find the sequence $\{V_T, \dots, V_0\}$, through the recursion

$$V_{t+1}(x) = \max_u \{f(x, u) + \beta V_t(g(x, u))\} \quad (2.2)$$

Infinite horizon For the infinite horizon discounted optimal control problem, the limit function $V = \lim_{j \rightarrow \infty} V_j$ is independent of j so the Hamilton Jacobi Bellman equation becomes

$$V(x) = \max_u \{f(x, u) + \beta V[g(x, u)]\} = \max_u H(x, u)$$

Properties of the value function: it is usually difficult to get the properties of $V(\cdot)$. In general continuity is assured but not differentiability (this is a subject for advanced courses on DP, see Stokey and Lucas (1989)).

If some regularity conditions hold, we may determine the optimal control through the **optimality condition**

$$\frac{\partial H(x, u)}{\partial u} = 0$$

if $H(\cdot)$ is C^2 then we get the **policy function**

$$u^* = h(x)$$

which gives an optimal rule for changing the optimal control, given the state of the economy. If we can determine (or prove that there exists such a relationship) then we say that our problem is **recursive**.

In this case the HJB equation becomes a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta V[g(x, h(x))].$$

Solving the HJB: means finding the value function $V(x)$. Methods: analytical (in some cases exact) and mostly numerical (value function iteration).

2.2 Applications

2.2.1 The cake eating problem

This is, possibly, the simplest optimal control problem. Assume that there is a cake whose size at time t is denoted by W_t and a muncher who wants to eat in T periods. The initial size of the cake is $W_0 = \phi$ and $W_T = 0$. The eater has a psychological discount factor $0 < \beta < 1$ and a static logarithmic utility function. What is the optimal eating strategy ?

The problem is to find the optimal paths $C^* = \{C_t^*\}_{t=0}^{T-1}$ and $W^* = \{W_t^*\}_{t=0}^T$ that solve the problem

$$\max_C \sum_{t=0}^{T-1} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t, W_0 = \phi, W_T = 0. \quad (2.3)$$

In order to solve the cake eating problem by using dynamic programming we have to determine a particular version of the HJB. In this case, we get

$$V_{T-t}(W_t) = \max_{C_t} \{ \ln(C_t) + \beta V_{T-t-1}(W_{t+1}) \}, t = 0, 1, \dots, T-1,$$

To solve it, we should take into account the restriction $W_{t+1} = W_t - C_t$ and the initial and terminal conditions.

We get the optimal policy function for consumption from

$$\frac{\partial}{\partial C_t} (\ln(C_t) + \beta V_{T-t-1}(W_{t+1})) = 0$$

to get

$$C_t^* = C_t(W_{t+1}) = \left(\beta V'_{T-t-1}(W_{t+1}) \right)^{-1}$$

Then the HJB equation becomes

$$V_{T-t}(W_t) = \ln(C_t(W_{t+1})) + \beta V_{T-t-1}(W_{t+1}), t = 0, 1, \dots, T-1 \quad (2.4)$$

which is a partial difference equation.

In order to solve it we make the conjecture that the solution is of the kind

$$V_{T-t}(W_t) = A_{T-t} + \left(\frac{1 - \beta^{T-t}}{1 - \beta} \right) \ln(W_t), t = 0, 1, \dots, T-1$$

and apply the method of the undetermined coefficients. Then

$$\begin{aligned} C_t^* &= \left(\beta V'_{T-t-1}(W_{t+1}) \right)^{-1} = \\ &= \left(\beta \frac{1 - \beta^{T-t-1}}{1 - \beta} \right)^{-1} W_{t+1} = \\ &= \left(\frac{1 - \beta}{\beta - \beta^{T-t}} \right) W_{t+1}, t = 0, 1, \dots, T-1 \end{aligned}$$

which implies that, from $W_{t+1} = W_t - C_t$ that

$$W_t = \left(\frac{\beta^{t+1} - \beta^T}{\beta^t - \beta^T} \right) W_{t+1}.$$

If we substitute back into the equation (2.4) we get an equivalent HJB equation

$$\begin{aligned} A_{T-t} + \left(\frac{1 - \beta^{T-t}}{1 - \beta} \right) \left[\ln \left(\frac{\beta^{t+1} - \beta^T}{\beta^t - \beta^T} \right) + \ln W_{t+1} \right] = \\ = \ln \left(\frac{1 - \beta^{T-t}}{1 - \beta} \right) + \ln W_{t+1} + \beta A_{T-t-1} + \left(\frac{\beta - \beta^{T-t}}{1 - \beta} \right) \ln W_{t+1} \end{aligned}$$

The terms in $\ln W_{t+1}$ this indicates that our conjecture was right. In order to determine the independent term, we should solve the difference equation for the coefficient (which in this case is variable)

$$A_{T-t} = \beta A_{T-t-1} + f(T-t)$$

which is a non-autonomous equation, because

$$f(T-t) \equiv \frac{1}{1-\beta} \ln \left[\left(\frac{\beta - \beta^{T-t}}{1 - \beta^{T-t}} \right)^{1-\beta^{T-t}} \left(\frac{1 - \beta}{\beta - \beta^{T-t}} \right)^{1-\beta} \right].$$

We don't need to solve the equation, because, we already have the optimal policy for consumption

$$\begin{aligned} C_t^* &= \left(\beta \frac{1 - \beta^{T-t}}{1 - \beta} \right)^{-1} W_{t+1} = \\ &= \left(\beta \frac{1 - \beta^{T-t}}{1 - \beta} \right)^{-1} \left(\frac{\beta^{t+1} - \beta^T}{\beta^t - \beta^T} \right)^{-1} W_t = \\ &= \left(\frac{1 - \beta}{1 - \beta^{T-t}} \right) W_t. \end{aligned}$$

2.2.2 Representative agent problem

Assumptions:

- there are $T > 1$ periods;
- consumers are homogeneous and have an additive intertemporal utility functional;
- the instantaneous utility function is continuous, differentiable, increasing, concave and is homogenous of degree n ;

- consumers have a stream of endowments, $y \equiv \{y_t\}_{t=0}^T$, known with certainty;
- institutional setting: there are spot markets for the good and for a financial asset. The financial asset is an entitlement to receive the dividend D_t at the end of every period t . The spot prices are P_t and S_t for the good and for the financial asset, respectively.
- Market timing, we assume that the good market opens in the beginning and that the asset market opens at the end of every period.

The consumer's problem

- choose a sequence of consumption $\{c_t\}_{t=0}^T$ and of portfolios $\{\theta_t\}_{t=1}^T$, which is, in this simple case, the quantity of the asset bought at the beginning of time t , in order to find

$$\max_{\{c_t, \theta_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t)$$

- subject to the sequence of budget constraints:

$$\begin{aligned} A_0 + y_0 &= c_0 + \theta_1 S_0 \\ \theta_1(S_1 + D_1) + y_1 &= c_1 + \theta_2 S_1 \\ &\dots \\ \theta_t(S_t + D_t) + y_t &= c_t + \theta_{t+1} S_t \\ &\dots \\ \theta_T(S_T + D_T) + y_T &= c_T \end{aligned}$$

where A_t is the stock of financial wealth (in real terms) at the beginning of period t .

If we denote

$$A_{t+1} = \theta_{t+1} S_t$$

then the generic **period budget constraint** is

$$A_{t+1} = y_t - c_t + R_t A_t, \quad t = 0, \dots, T \quad (2.5)$$

where the asset return is

$$R_{t+1} = 1 + r_{t+1} = \frac{S_{t+1} + D_{t+1}}{S_t}.$$

Then the HJB equation is

$$V(A_t) = \max_{c_t} \{u(c_t) + \beta V(A_{t+1})\} \quad (2.6)$$

We can write the equation as

$$V(A) = \max_c \left\{ u(c) + \beta V(\tilde{A}) \right\} \quad (2.7)$$

where $\tilde{A} = y - c + RA$ then $V(\tilde{A}) = V(y - c + RA)$.

Deriving an intertemporal arbitrage condition

The optimality condition is:

$$u'(c^*) = \beta V'(\tilde{A})$$

if we could find the optimal policy function $c^* = h(A)$ and substitute it in the HJB equation we would get

$$V(A) = u(h(A)) + \beta V(\tilde{A}^*), \quad \tilde{A}^* = y - h(A) + RA.$$

Using the Benveniste and Scheinkman (1979) trick , we differentiate for A to get

$$\begin{aligned} V'(A) &= u'(c^*) \frac{\partial h}{\partial A} + \beta V'(\tilde{A}^*) \frac{\partial \tilde{A}^*}{\partial A} = \\ &= u'(c^*) \frac{\partial h}{\partial A} + \beta V'(\tilde{A}^*) \left(R - \frac{\partial h}{\partial A} \right) = \\ &= \beta R V'(\tilde{A}^*) = \\ &= R u'(c^*) \end{aligned}$$

from the optimality condition. If we shift both members of the last equation we get

$$V'(\tilde{A}^*) = R u'(\tilde{c}^*),$$

and, then

$$R u'(\tilde{c}^*) = \beta^{-1} u'(c^*).$$

Then, the optimal consumption path (we delete the * from now on) verifies the arbitrage condition

$$u'(c) = \beta R u'(\tilde{c}).$$

In the literature the relationship is called the **consumer's intertemporal arbitrage condition**

$$u'(c_t) = \beta u'(c_{t+1}) R_t \quad (2.8)$$

Observe that

$$\beta R_t = \frac{1 + r_t}{1 + \rho}$$

is the ratio between the market return and the psychological factor.

If the utility function is homogenous of degree η , it has the properties

$$\begin{aligned} u(c) &= c^\eta u(1) \\ u'(c) &= c^{\eta-1} u'(1) \end{aligned}$$

the arbitrage condition is a linear difference equation

$$c_{t+1} = \lambda c_t, \quad \lambda \equiv (\beta R_t)^{\frac{1}{1-\eta}}$$

Determining the optimal value function

In some cases, we can get an explicit solution for the HJB equation (2.7). We have to determine jointly the optimal policy function $h(A)$ and the optimal value function $V(A)$.

We will use the same non-constructive method to derive both functions: first, we make a conjecture on the form of $V(\cdot)$ and then apply the method of the undetermined coefficients.

Assumptions Let us assume that the utility function is logarithmic: $u(c) = \ln(c)$ and assume for simplicity that $y = 0$.

In this case the optimality condition becomes

$$c^* = [\beta V'(RA - c^*)]^{-1}$$

Conjecture: let us assume that the value function is of the form

$$V(x) = B_0 + B_1 \ln(A) \tag{2.9}$$

where B_0 and B_1 are undetermined coefficients.

From this point on we apply the **method of the undetermined coefficients**: if the conjecture is right then we will get an equation without the independent variable x , and which would allow us to determine the coefficients, B_0 and B_1 , as functions of the parameters of the HJB equation.

Then

$$V' = \frac{B_1}{A}.$$

Applying this to the optimality condition, we get

$$c^* = h(A) = \frac{RA}{1 + \beta B_1}$$

then

$$\tilde{A}^* = RA - c^* = \left(\frac{\beta B_1}{1 + \beta B_1} \right) RA$$

which is a linear function of A .

Substituting in the HJB equation, we get

$$\begin{aligned} B_0 + B_1 \ln(A) &= \ln\left(\frac{RA}{1 + \beta B_1}\right) + \beta \left[B_0 + B_1 \ln\left(\frac{\beta B_1 RA}{1 + \beta B_1}\right) \right] = \\ &= \ln\left(\frac{R}{1 + \beta B_1}\right) + \ln(A) + \beta \left[B_0 + B_1 \ln\left(\frac{\beta B_1 R}{1 + \beta B_1}\right) + \ln(A) \right] \end{aligned} \quad (2.10)$$

The term in $\ln(A)$ can be eliminated if $B_1 = 1 + \beta B_1$, that is if

$$B_1 = \frac{1}{1 - \beta}$$

and equation (2.10) reduces to

$$B_0(1 - \beta) = \ln(R(1 - \beta)) + \frac{\beta}{1 - \beta} \ln(R\beta)$$

which we can solve for B_0 to get

$$B_0 = (1 - \beta)^{-2} \ln(R\Theta), \text{ where } \Theta \equiv (1 - \beta)^{1-\beta} \beta^\beta$$

Finally, as our conjecture proved to be right, we can substitute B_0 and B_1 in equation (2.9) the optimal value function and the optimal policy function are

$$V(A) = (1 - \beta)^{-1} \ln\left(\left((R\Theta)^{(1-\beta)^{-1}} A\right)\right)$$

and

$$c^* = (1 - \beta)RA$$

then optimal consumption is linear in financial wealth.

We can also determine the optimal asset accumulation, by noting that $c_t = c^*$ and substituting in the period budget constraint

$$A_{t+1} = \beta R_t A_t$$

If we assume that $R_t = R$ then the solution for that DE is

$$A_t = (\beta R)^t A_0, \quad t = 0, \dots, \infty$$

and, therefore the optimal path for consumption is

$$c_t = (1 - \beta)\beta^t R^{t+1} A_0$$

Observe that the transversality condition holds,

$$\lim_{t \rightarrow \infty} R^{-t} A_t = \lim_{t \rightarrow \infty} A_0 \beta^t = 0$$

because $0 < \beta < 1$.

Exercises

1. solve the HJB equation for the case in which $y > 0$
2. solve the HJB equation for the case in which $y > 0$ and the utility function is CRRA: $u(c) = c^{1-\theta}/(1-\theta)$, for $\theta > 0$;
3. try to solve the HJB equation for the case in which $y = 0$ and the utility function is CARA: $u(c) = B - e^{-\beta c}/\beta$, for $\beta > 0$
4. try to solve the HJB equation for the case in which $y > 0$ and the utility function is CARA: $u(c) = B - e^{-\beta c}/\beta$, for $\beta > 0$.

2.2.3 The Ramsey problem

Find a sequence $\{c_t, k_t\}_{t=0}^{\infty}$ which solves the following optimal control problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = f(k_t) - c_t + (1 - \delta)k_t$$

where $0 \leq \delta \leq 1$ given k_0 . Both the utility function and the production function are neoclassical: continuous, differentiable, smooth and verify the Inada conditions. These conditions would ensure that the necessary conditions for optimality are also sufficient.

The HJB function is

$$V(k) = \max_c \left\{ u(c) + \beta V(k') \right\}$$

where $k' = f(k) - c + (1 - \delta)k$.

The optimality condition is

$$u'(c) = \beta V'(k'(k))$$

If it allows us to find a policy function $c = h(k)$ then the HJB becomes

$$V(k) = u[h(k)] + \beta V[f(k) - h(k) + (1 - \delta)k] \quad (2.11)$$

This equation has no explicit solution for generic utility and production function. Even for explicit utility and production functions the HJB has not an explicit solution. Next we present a benchmark case where we can find an explicit solution for the HJB equation.

Benchmark case: Let $u(c) = \ln(c)$ and $f(k) = Ak^\alpha$ for $0 < \alpha < 1$, and $\delta = 1$.

Conjecture:

$$V(k) = B_0 + B_1 \ln(k)$$

In this case the optimality condition is

$$h(k) = \frac{Ak^\alpha}{1 + \beta B_1}$$

If we substitute in equation (2.11) we get

$$B_0 + B_1 \ln(k) = \ln \left(\frac{Ak^\alpha}{1 + \beta B_1} \right) + \beta \left[B_0 + B_1 \ln \left(\frac{Ak^\alpha \beta B_1}{1 + \beta B_1} \right) \right] \quad (2.12)$$

Again we can eliminate the term in $\ln(k)$ by making

$$B_1 = \frac{\alpha}{1 - \alpha\beta}$$

Thus, (2.12) changes to

$$B_0(1 - \beta) = \ln(A(1 - \alpha\beta)) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta A)$$

Finally the optimal value function and the optimal policy function are

$$V(A) = (1 - \alpha\beta)^{-1} \ln \left((A\Theta)^{(1-\beta)^{-1}} A \right), \quad \Theta \equiv (1 - \alpha\beta)^{1-\alpha\beta} (\alpha\beta)^{\alpha\beta}$$

and

$$c^* = (1 - \alpha\beta) A k^\alpha$$

Then the optimal capital accumulation is governed by the equation

$$k_{t+1} = \alpha\beta A k_t^\alpha$$

This equation generates a forward path starting from the known initial capital stock k_0

$$\{k_0, \alpha\beta A k_0^\alpha, (\alpha\beta A)^{\alpha+1} k_0^{\alpha^2}, \dots, (\alpha\beta A)^{\alpha^{t-1}+1} k_0^{\alpha^t}, \dots\}$$

which converges to a stationary solution: $\bar{k} = (\alpha\beta A)^{1/(1-\alpha)}$.

Chapter 3

Continuous Time

3.1 The dynamic programming principle and the HJB equation

3.1.1 Simplest problem optimal control problem

In the space of the functions $(u(t), x(t))$ for $t_0 \leq t \leq t_1$ find functions $(u^*(t), x^*(t))$ which solve the problem:

$$\max_{u(t)} \int_{t_0}^{t_1} f(t, x(t), u(t)) dt$$

subject to

$$\dot{x} \equiv \frac{dx(t)}{dt} = g(t, x(t), u(t))$$

given $x(t_0) = x_0$. We assume that t_1 is know and that $x(t_1)$ is free.

The value function is, for the initial instant

$$\mathcal{V}(t_0, x_0) = \int_{t_0}^{t_1} f(t, x^*, u^*) dt$$

and for the terminal time $\mathcal{V}(t_1, x(t_1)) = 0$.

Lemma 1. First order necessary conditions for optimality from the Dynamic Programming principle

Let $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$. Then the value function which is associated to the optimal path $\{(x^*(t), u^*(t)) : t_0 \leq t \leq t_1\}$ verifies the fundamental partial differential equation or the **Hamilton-Jacobi-Bellman equation**

$$-\mathcal{V}_t(t, x) = \max_u [f(t, x, u) + \mathcal{V}_x(t, x)g(t, x, u)].$$

Proof. Consider the value function

$$\begin{aligned} \mathcal{V}(t_0, x_0) &= \max_u \left(\int_{t_0}^{t_1} f(t, x, u) dt \right) \\ &= \max_u \left(\int_{t_0}^{t_0+\Delta t} f(\cdot) dt + \int_{t_0+\Delta t}^{t_1} f(\cdot) dt \right) = \quad (\Delta t > 0, \text{small}) \\ &= \max_u \left[\int_{t_0}^{t_0+\Delta t} f(\cdot) dt + \max_u \left(\int_{t_0+\Delta t}^{t_1} f(\cdot) dt \right) \right] = \\ &\quad (t_0 \leq t \leq t_0 + \Delta t \quad t_0 + \Delta t \leq t \leq t_1) \\ &\quad \text{(from dynamic prog principle)} \\ &= \max_u \left[\int_{t_0}^{t_0+\Delta t} f(\cdot) dt + \mathcal{V}(t_0 + \Delta t, x_0 + \Delta x) \right] = \\ &\quad (t_0 \leq t \leq t_0 + \Delta t) \\ &\quad \text{(approximating } x(t_0 + \Delta t) \approx x_0 + \Delta x) \\ &= \max_u [f(t_0, x_0, u)\Delta t + \mathcal{V}(t_0, x_0) + \mathcal{V}_t(t_0, x_0)\Delta t + \mathcal{V}_x(t_0, x_0)\Delta x + \text{h.o.t}] \\ &\quad (t_0 \leq t \leq t_0 + \Delta t) \end{aligned}$$

if $u \approx \text{constant}$ and $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$. Passing $\mathcal{V}(t_0, x_0)$ to the second member, dividing by Δt and taking the limit $\lim_{\Delta t \rightarrow 0}$ we get, for every $t \in [t_0, t_1]$,

$$0 = \max_u [f(t, x, u) + \mathcal{V}_t(t, x) + \mathcal{V}_x(t, x)\dot{x}].$$

□

The policy function is now $u^* = h(t, x)$. If we substitute in the HJB equation then we get a first order partial differential equation

$$-\mathcal{V}_t(t, x) = f(t, x, h(t, x)) + \mathcal{V}_x(t, x)g(t, x, h(t, x)).$$

Though the differentiability of \mathcal{V} is assured for the functions f and g which are common in the economics literature, we can get explicit solutions, for $V(\cdot)$ and for $h(\cdot)$, only in very rare cases. Proving that \mathcal{V} is differentiable, even in the case in which we cannot determine it explicitly is hard and requires proficiency in Functional Analysis.

Relationship with the Pontryagin's principle:

- (1) If we apply the transformation $\lambda(t) = \mathcal{V}_x(t, x(t))$ we get the following relationship with the Hamiltonian function which is used by the Pontryagin's principle: $-\mathcal{V}_t(t, x) = \mathcal{H}^*(t, x, \lambda)$;
- (2) If \mathcal{V} is sufficiently differentiable, we can use the principle of DP to get necessary conditions for optimality similar to the Pontryagin principle. The maximum condition is

$$f_u + \mathcal{V}_x g_u = f_u + \lambda g_u = 0$$

and the canonical equations are: as $\dot{\lambda} = \frac{\partial \mathcal{V}_x}{\partial t} = \mathcal{V}_{xt} + \mathcal{V}_{xx}g$ and differentiating the HJB as regards x , implies $-\mathcal{V}_{tx} = f_x + \mathcal{V}_{xx}g + \mathcal{V}_x g_x$, therefore the canonical equation results

$$-\dot{\lambda} = f_x + \lambda g_x.$$

- (3) Differently from the Pontryagin's principle which defines a dynamic system of the form $\{\mathbb{T}, \mathbb{R}^2, \varphi_t = (q(t), x(t))\}$, the principle of dynamic programming defines a dynamic system as $\{(\mathbb{T}, \mathbb{R}), \mathbb{R}, v_{t,x} = \mathcal{V}(t, x)\}$. That is, it defines a recursive mechanism in all or in a subset of the state space.

3.1.2 Infinite horizon discounted problem

Lemma 2. First order necessary conditions for optimality from the Dynamic Programming principle

Let $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$. Then the value function associated to the optimal path $\{(x^*(t), u^*(t)) : t_0 \leq t < +\infty\}$ verifies the fundamental non-linear ODE called the **Hamilton-Jacobi-Bellman equation**

$$\rho V(x) = \max_u [f(x, u) + V'(x)g(x, u)].$$

Proof. Now, we have

$$\begin{aligned} \mathcal{V}(t_0, x_0) &= \max_u \left(\int_{t_0}^{+\infty} f(x, u) e^{-\rho t} dt \right) = \\ &= e^{-\rho t_0} \max_u \left(\int_{t_0}^{+\infty} f(x, u) e^{-\rho(t-t_0)} dt \right) = \\ &= e^{-\rho t_0} V(x_0) \end{aligned} \tag{3.1}$$

where $V(\cdot)$ is independent from t_0 and only depends on x_0 . We can do

$$V(x_0) = \max_u \left(\int_0^{+\infty} f(x, u) e^{-\rho t} dt \right).$$

If we let, for every (t, x) $\mathcal{V}(t, x) = e^{-\rho t} V(x)$ and if we substitute the derivatives in the HJB equation for the simplest problem, we get the new HJB. \square

Observations:

- if we determine the policy function $u^* = h(x)$ and substitute in the HJB equation, we see that the new HJB equation is a non-linear ODE

$$\rho V(x) = f(x, h(x)) + V'(x)g(x, h(x)).$$

- Differently from the solution from the Pontryagin's principle, the HJB defines a recursion over x . Intuitively it generates a rule which says : if we observe the state x the optimal policy is $h(x)$ in such a way that the initial value problem should be equal to the present value of the variation of the state.
- It is still very rare to find explicit solutions for $V(x)$. There is a literature on how to compute it numerically, which is related to the numerical solution of ODE's and not with approximating value functions as in the discrete time case.

3.1.3 Bibliography

See Kamien and Schwartz (1991).

3.2 Applications

3.2.1 The cake eating problem

The problem is to find the optimal flows of cake munching $C^* = \{C^*(t) : t \in [0, T]\}$ and of the size of the cake $W^* = \{W^*(t) : t \in [0, T]\}$ such that

$$\max_C \int_0^T \ln(C(t))e^{-\rho t} dt, \text{ subject to } \dot{W} = -C, t \in (0, T), W(0) = \phi W(T) = 0 \quad (3.2)$$

where $\phi > 0$ is given. The problem can be equivalently written as a calculus of variations problem,

$$\max_W \int_0^T \ln(-\dot{W}(t))e^{-\rho t} dt, \text{ subject to } W(0) = \phi W(T) = 0$$

Consider again problem (3.2). Now, we want to solve it by using the principle of the dynamic programming. In order to do it, we have to determine the value function $V = V(t, W)$ which solves the HJB equation

$$-\frac{\partial V}{\partial t} = \max_C \left\{ e^{-\rho t} \ln(C) - C \frac{\partial V}{\partial W} \right\}$$

The optimal policy for consumption is

$$C^*(t) = e^{-\rho t} \left(\frac{\partial V}{\partial W} \right)^{-1}$$

If we substitute back into the HJB equation we get the partial differential equation

$$-e^{\rho t} \frac{\partial V}{\partial t} = \ln \left[e^{-\rho t} \left(\frac{\partial V}{\partial W} \right)^{-1} \right] - 1$$

To solve it, let us use the method of undetermined coefficients by conjecturing that the solution is of the type

$$V(t, W) = e^{-\rho t} (a + b \ln W)$$

where a and b are constants to be determined, if our conjecture is right. With this function, the HJB equation comes

$$\rho(a + b \ln W) = \ln(W) - \ln b - 1$$

if we set $b = 1/\rho$ we eliminate the term in $\ln W$ and get

$$a = -(1 - \ln(\rho))/\rho.$$

Therefore, solution for the HJB equation is

$$V(t, W) = \frac{-1 + \ln(\rho) + \ln W}{\rho} e^{-\rho t}$$

and the optimal policy for consumption is

$$C^*(t) = \rho W(t).$$

3.2.2 The representative agent problem

Assumptions:

- $\mathbb{T} = \mathbb{R}_+$, i.e., decisions and transactions take place continuously in time, and variables are represented by flows or trajectories $x \equiv \{x(t), t \in \mathbb{R}_+\}$ where $x(t)$ is a mapping $t \mapsto x(t)$;
- deterministic environment: the agents have perfect information over the flow of endowments $y \equiv \{y(t), t \in \mathbb{R}_+\}$ and the relevant prices;
- agents are homogeneous: they have the same endowments and preferences;
- preferences over flows of consumption are evaluated by the intertemporal utility functional

$$V[c] = \int_0^\infty u(c(t))e^{-\rho t} dt$$

which displays impatience (the discount factor $e^{-\rho t} \in (0, 1)$), stationarity ($u(\cdot)$ is not directly dependent on time) and time independence and the instantaneous utility function ($u(\cdot)$) is continuous, differentiable, increasing and concave;

- observe that mathematically the intertemporal utility function is in fact a functional, or a generalized function, i.e., a mapping whose argument is a function (not a number as in the case of functions). Therefore, solving the consumption problem means finding an optimal function. In particular, it consists in finding an **optimal trajectory** for consumption;
- institutional setting: there are spot real and financial markets that are continuously open. The price $P(t)$ clear the real market instantaneously. There is an asset market in which a single asset is traded which has the price $S(t)$ and pays a dividend $V(t)$.

Derivation of the budget constraint:

The consumer chooses the number of assets $\theta(t)$. If we consider a small increment in time h and assume that the flow variables are constant in the interval then

$$S(t+h)\theta(t+h) = \theta(t)S(t) + \theta(t)D(t)h + P(t)(y(t) - c(t))h.$$

Define $A(t) = S(t)\theta(t)$ in nominal terms.

The budget constraint is equivalently

$$A(t+h) - A(t) = i(t)A(t)h + P(t)(y(t) - c(t))h$$

where $i(t) = \frac{D(t)}{S(t)}$ is the nominal rate of return. If we divide by h and take the limit when $h \rightarrow 0$ then

$$\lim_{h \rightarrow 0} \frac{A(t+h) - A(t)}{h} \equiv \frac{dA(t)}{dt} = i(t)A(t) + P(t)(y(t) - c(t)).$$

If we define real wealth and the real interest rate as $a(t) \equiv \frac{A(t)}{P(t)}$ and $r(t) = i(t) + \frac{\dot{P}}{P(t)}$, then we get the **instantaneous budget constraint**

$$\dot{a}(t) = r(t)a(t) + y(t) - c(t) \tag{3.3}$$

where we assume that $a(0) = a_0$ given.

Define the **human wealth**, in real terms, as

$$h(t) = \int_t^\infty e^{-\int_t^s r(\tau)d\tau} y(s)ds$$

as from the Leibniz's rule

$$\dot{h}(t) \equiv \frac{dh(t)}{dt} = r(t) \int_t^\infty e^{-\int_t^s r(\tau)d\tau} y(s)ds - y(t) = r(t)h(t) - y(t)$$

then **total wealth** at time t is

$$w(t) = a(t) + h(t)$$

and we may represent the budget constraint as a function of $w(\cdot)$

$$\begin{aligned} \dot{w} &= \dot{a}(t) + \dot{h}(t) = \\ &= r(t)w(t) - c(t) \end{aligned}$$

The instantaneous budget constraint should not be confused with the **intertemporal budget constraint**. Assume the solvability condition holds at time $t = 0$

$$\lim_{t \rightarrow \infty} e^{-\int_0^t r(\tau)d\tau} a(t) = 0.$$

Then it is equivalent to the following intertemporal budget constraint,

$$w(0) = \int_0^{\infty} e^{-\int_0^t r(\tau) d\tau} c(t) dt,$$

the present value of the flow of consumption should be equal to the initial total wealth.

To prove this, solve the instantaneous budget constraint (3.3) to get

$$a(t) = a(0)e^{\int_0^t r(\tau) d\tau} + \int_0^t e^{\int_0^s r(\tau) d\tau} y(s) - c(s) ds$$

multiply by $e^{-\int_0^t r(\tau) d\tau}$, pass to the limit $t \rightarrow \infty$, apply the solvability condition and use the definition of human wealth.

Therefore the **intertemporal optimization problem for the representative agent** is to find $(c^*(t), w^*(t))$ for $t \in \mathbb{R}_+$ which maximizes

$$V[c] = \int_0^{+\infty} u(c(t))e^{-\rho t} dt$$

subject to the instantaneous budget constraint

$$\dot{w}(t) = r(t)w(t) - c(t)$$

given $w(0) = w_0$.

Solving the consumer problem using DP.

The HJB equation is

$$\rho V(w) = \max_c \left\{ u(c) + V'(w)(rw - c) \right\}$$

where $w = w(t)$, $c = c(t)$, $r = r(t)$.

We assume that the utility function is homogeneous of degree η . Therefore it has the properties:

$$\begin{aligned} u(c) &= c^\eta u(1) \\ u'(c) &= c^{\eta-1} u'(1) \end{aligned}$$

The optimality condition is:

$$u'(c^*) = V'(w)$$

then

$$c^* = \left(\frac{V'(w)}{u'(1)} \right)^{\frac{1}{\eta-1}}$$

substituting in the HJB equation we get the ODE, defined on $V(w)$,

$$\rho V(w) = rwV'(w) + (u(1) - u'(1)) \left(\frac{V'(w)}{u'(1)} \right)^{\frac{\eta}{\eta-1}}$$

In order to solve it, we **guess** that its solution is of the form:

$$V(w) = Bw^\eta$$

if we substitute in the HJB equation, we get

$$\rho Bw^\eta = \eta r Bw^\eta + (u(1) - u'(1)) \left(\frac{\eta Bw}{u'(1)} \right)^\eta.$$

Then we can eliminate the term in w^η and solve for B to get

$$B = \left[\left(\frac{u(1) - u'(1)}{\rho - \eta r} \right) \left(\frac{\eta}{u'(1)} \right)^\eta \right]^{\frac{1}{1-\eta}}.$$

Then, as B is a function of $r = r(t)$, we determine explicitly the value function

$$V(w(t)) = Bw(t)^\eta = \left[\left(\frac{u(1) - u'(1)}{\rho - \eta r(t)} \right) \left(\frac{\eta}{u'(1)} \right)^\eta \right]^{\frac{1}{1-\eta}} w(t)^\eta$$

as a function of total wealth.

Observation: this is one known case in which we can solve explicitly the HJB equation as it is a linear function on the state variable, w and the objective function $u(c)$ is homogeneous.

The optimal policy function can also be determined explicitly

$$c^*(t) = \left(\frac{\eta B(t)}{u'(1)} \right)^{\frac{1}{\eta-1}} w(t) \equiv \pi(t)w(t) \quad (3.4)$$

as it sets the control as a function of the state variable, and not as depending on the path of the co-state and state variables as in the Pontryagin's case, sometimes this solution is called **robust feedback control**.

Substituting in the budget constraint, we get the optimal wealth accumulation

$$\dot{w}^*(t) = w_0 e^{\int_0^t r(s) - \pi(s) ds} \quad (3.5)$$

which is a solution of

$$\dot{w}^* = r(t)w^*(t) - c^*(t) = (r(t) - \pi(t))w^*(t)$$

Conclusion: the optimal paths for consumption and wealth accumulation ($c^*(t)$, $w^*(t)$) are given by equations (3.4) and (3.5) for any $t \in \mathbb{R}_+$.

3.2.3 The Ramsey model

This is a problem for a centralized planner which chooses the optimal consumption flow $c(t)$ in order to maximize the intertemporal utility functional

$$\max V[c] = \int_0^{\infty} u(c)e^{-\rho t} dt$$

where $\rho > 0$ subject to

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) \\ k(0) &= k_0 \text{ given} \end{aligned}$$

The HJB equation is

$$\rho V(k) = \max_c \{u(c) + V'(k)(f(k) - c)\}$$

Benchmark assumptions: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$ and $f(k) = Ak^\alpha$ where $0 < \alpha < 1$.

The HJB equation is

$$\rho V(k) = \max_c \left\{ \frac{c^{1-\sigma}}{1-\sigma} + V'(k)(Ak^\alpha - c) \right\} \quad (3.6)$$

the optimality condition is

$$c^* = \left(V'(k) \right)^{-\frac{1}{\sigma}}$$

after substituting in equation (3.6) we get

$$\rho V(k) = V'(k) \left(\frac{\sigma}{1-\sigma} V'(k)^{-\frac{1}{\sigma}} + Ak^\alpha \right) \quad (3.7)$$

In some particular cases we can get explicit solutions, but in general we don't.

Particular case: $\alpha = \sigma$ Equation (3.7) becomes

$$\rho V(k) = V'(k) \left(\frac{\sigma}{1-\sigma} V'(k)^{-\frac{1}{\sigma}} + Ak^\sigma \right) \quad (3.8)$$

Let us conjecture that the solution is

$$V(k) = B_0 + B_1 k^{1-\sigma}$$

where B_0 and B_1 are undetermined coefficients. Then $V'(k) = B_1(1-\sigma)k^{-\sigma}$.

If we substitute in equation (3.8) we get

$$\rho(B_0 + B_1 k^{1-\sigma}) = B_1 \left[\sigma ((1-\sigma)B_1)^{-\frac{1}{\sigma}} k^{1-\sigma} + (1-\sigma)A \right] \quad (3.9)$$

Equation (3.9) is true only if

$$B_0 = \frac{A(1-\sigma)}{\rho} B_1 \quad (3.10)$$

$$B_1 = \left(\frac{1}{1-\sigma} \right) \left(\frac{\sigma}{\rho} \right)^\sigma. \quad (3.11)$$

Then the following function is indeed a solution of the HJB equation in this particular case

$$V(k) = \left(\frac{\sigma}{\rho} \right)^\sigma \left(\frac{A}{\rho} + \frac{1}{1-\sigma} k^{1-\sigma} \right)$$

The optimal policy function is

$$c = h(k) = \frac{\rho}{\sigma} k$$

We can determine the optimal flow of consumption and capital $(c^*(t), k^*(t))$ by substituting $c(t) = \frac{\rho}{\sigma} k(t)$ in the admissibility conditions to get the ODE

$$\dot{k}^* = Ak^*(t)^\alpha - \frac{\rho}{\sigma} k^*(t)$$

for given $k(0) = k_0$. This a Bernoulli ODE which has an explicit solution as

$$k^*(t) = \left[\frac{A\rho}{\sigma} + \left(k_0^{\frac{1}{1-\sigma}} - \frac{A\rho}{\sigma} \right) e^{-\frac{(1-\sigma)\rho t}{\sigma}} \right]^{\frac{1}{1-\sigma}}, \quad t = 0, \dots, \infty$$

and

$$c^*(t) = \frac{\rho}{\sigma} k^*(t), \quad t = 0, \dots, \infty.$$

Part II

**Stochastic Dynamic
Programming**

Chapter 4

Discrete Time

4.1 Introduction to stochastic processes

Assume that we have T periods: $\mathbb{T} = \{0, 1, \dots, T\}$, and consider an underlying probability space (Ω, \mathcal{F}, P) .

We introduce the family of random variables $X^t \equiv \{X_\tau : \tau = 0, 1, \dots, T\}$ where $X_t = X(w_t)$ where $w_t \in \mathcal{F}_t$. The information available at period $t \in \mathbb{T}$ will be represented by the σ -algebra $\mathcal{F}_t \subset \mathcal{F}$.

4.1.1 Information structure

Consider a state space with a finite number of states of nature $\Omega = \{\omega_1, \dots, \omega_N\}$.

Let the information, at time T be represented by the random sequence of events $w^t = \{w_\tau, \tau = 0, 1, \dots, t\}$ where $w_t \in \{A_t\}$ and $\{A_t\}$ is a subset of Ω :

- at $t = 0$ any state of nature $\omega \in \Omega$ is possible, $w_0 = \Omega$;
- at $t = T$ the true state of nature is known, $\{A_T\} \in \{\{\omega_1\}, \dots, \{\omega_N\}\}$, that is where $\omega_T \in \Omega$. This means that there is a correspondence between the number of information sequences and the number of states of nature N ;
- at $0 < t < T$ some states of nature can be discarded, which means that the amount of information is increasing, that is, the true state of nature will belong to a subset with a smaller number of elements, as time passes by. Formally, $A_0 \supseteq A_1 \supseteq \dots \supseteq A_T$, that is

$$A_{t+1} \subseteq A_t, \quad \forall t \in \mathbb{T}.$$

If we consider the set of all the states of nature in each period, we may set $w_t \in \mathcal{P}_t$ where $\{\mathcal{P}_t\}_{t=0}^T$ is a sequence of partitions over Ω . That is, we may establish a (one to one) correspondence between the sequence $\{w_t\}_{t=0}^T$ of information and the sequences of partitions, $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_T$, such that

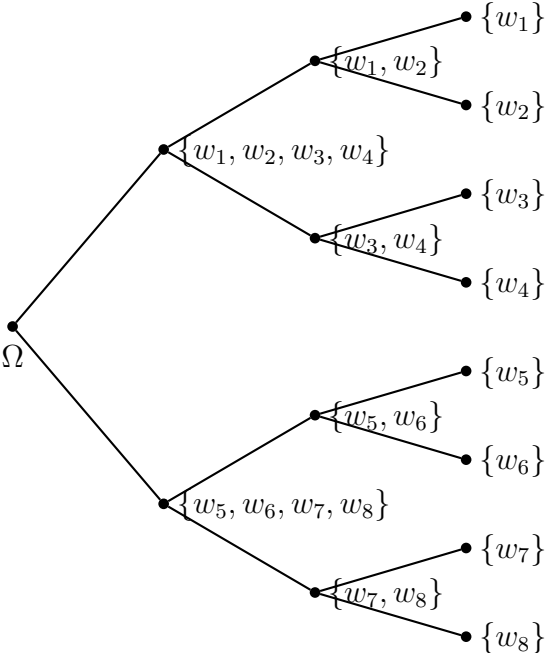


Figure 4.1: Information tree

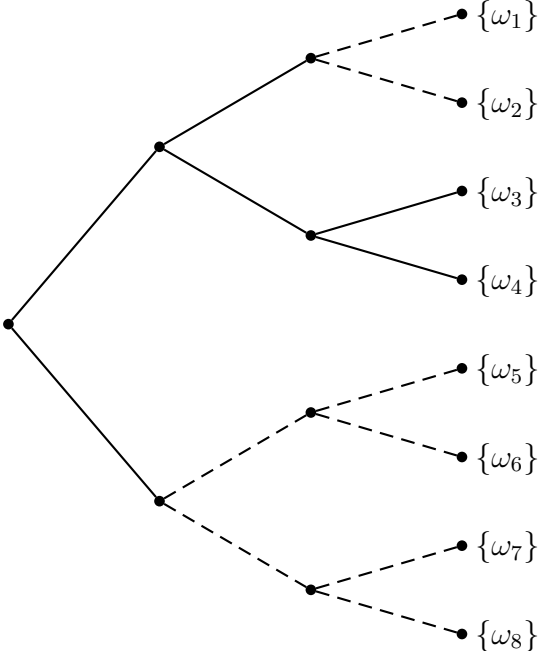


Figure 4.2: Information realization through time

- $\mathcal{P}_0 = \Omega$;
- the elements of \mathcal{P}_t are mutually disjoint and are equivalent to the union of elements of \mathcal{P}_{t+1} ;
- $\mathcal{P}_T = \{\{\omega_1\}, \{\omega_2\}, \dots, \{\omega_N\}\}$.

Example (vd Pliska (1997)) Let $N = 8$, $T = 3$, a admissible sequence of partitions is:

$$\begin{aligned} \mathcal{P}_0 &= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\} \\ \mathcal{P}_1 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \\ \mathcal{P}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\} \\ \mathcal{P}_3 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\} \end{aligned}$$

We may understand those partitions as sequences of two states of nature, a good state u and a bad state d . Then: $\{\omega_1\} = uuu$, $\{\omega_2\} = uud$, $\{\omega_3\} = udu$, $\{\omega_4\} = udd$, $\{\omega_5\} = duu$, $\{\omega_6\} = dud$, $\{\omega_7\} = ddu$ and $\{\omega_8\} = ddd$, $\{\omega_1, \omega_2\} = uu$, $\{\omega_3, \omega_4\} = ud$, $\{\omega_5, \omega_6\} = du$, $\{\omega_7, \omega_8\} = dd$, $\{\omega_1, \omega_2, \omega_3, \omega_4\} = u$ and $\{\omega_5, \omega_6, \omega_7, \omega_8\} = d$ and $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\} = \{u, d\}$. We can write all the potential sequences of information as a sequence os sets of events

$$\begin{aligned} \mathcal{P}_0 &= w_0 \\ \mathcal{P}_1 &= \{w_{1,1}, w_{1,2}\} \\ \mathcal{P}_2 &= \{w_{2,1}, w_{2,2}, w_{2,3}, w_{2,4}\} \\ \mathcal{P}_3 &= \{w_{3,1}, w_{3,2}, w_{3,3}, w_{3,4}, w_{3,5}, w_{3,6}, w_{3,7}, w_{3,8}\} \end{aligned}$$

□

A realization corresponds to the occurrence of a particular sequence of history $w^t = \{w_0, w_1, \dots, w_t\}$, where $w_t = (w_{t,1}, \dots, w_{t,N_t})$ and N_t is the number of elements of \mathcal{P}_t .

Given a partition, we may obtain several different histories, which correspond to building as as many subsets of Ω as possible, by means of set operations (complements, unions and intersections) and build a σ -algebra. There is, therefore, a correspondence between sequences of partitions over Ω and sequences σ -algebras $\mathcal{F}_t \subset \mathcal{F}$.

The information available at time $t \in \mathbb{T}$ will be represented by \mathcal{F}_t and by a **filtration**, for the sequence of periods $t = 0, 1, \dots, T$.

Definition 4. A **filtration** is a sequence of σ -algebras $\{\mathcal{F}_t\}$

$$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}.$$

A filtration is **non-anticipating** if

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$,
- $\mathcal{F}_s \subseteq \mathcal{F}_t$, if $s \leq t$,
- $\mathcal{F}_T = \mathcal{F}$.

Intuition:

- (1) Initially we have no information (besides knowing that an event is observable or not);
- (2) the information increases with time;
- (3) at the terminal moment we not only observe the true state of nature, but also know the past history.

Example Taking the last example, we have:

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\}, \\ \mathcal{F}_2 &= \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \\ &\quad \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_5, \omega_6\}, \\ &\quad \{\omega_1, \omega_2, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_3, \omega_4, \omega_7, \omega_8\}, \\ &\quad \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_7, \omega_8\}, \\ &\quad \{\omega_1, \omega_2, \omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}\} \end{aligned}$$

□

Then \mathcal{F}_t is the set of all the histories up until time t ,

$$\mathcal{F}_t = \{\{w_\tau\}_{\tau=0}^t : w_\tau \in \mathcal{P}_\tau, 0 \leq \tau \leq t\}$$

Probabilities can be determined from two perspectives:

- for events y , occuring in a moment in time, $w_y = y$ $P(w_t = y)$ or probabilities associated to sequences of events $w^t = y^t$ means $\{w_0 = y_0, w_1 = y_1, \dots, w_t = y_t\}$, $P(w^t = y^t)$;
- with information taken at time $t = 0$, i.e. Ω ; or associated to a particular history $w^t \in \mathcal{F}_t$.

Unconditional probabilities, $\pi_0^t(y)$ denotes the probability that the event y occurs at time t , assuming the information at time $t = 0$. If we consider the information available at time t the probability of $w_t = y$, at time t , is

$$\pi_0^t(y) = P(w_t = y)$$

where $w_t \in \mathcal{P}_t \subset \mathcal{F}_t$. As a probability, we have

$$0 \leq \pi_0^t(\cdot) \leq 1.$$

From the properties of $\mathcal{P}_t \subset \mathcal{F}_t$, as the finer partition of \mathcal{F}_t , we readily see that

$$\pi_0^t(\mathcal{P}_t) = P(\cup_{s=1}^{N_t} w_{t,s}) = \sum_{s=1}^{N_t} P(w_t = w_{t,s}) = \sum_{s=1}^{N_t} \pi_0^t(w_{t,s}) = 1$$

We denote by $\pi_0^t(w^t)$ the probability that history $w^t \in \mathcal{F}_t$ occurs. In this case, we have $\pi_0^T(\mathcal{F}_T) = 1$,

Conditional probabilities, $\pi_s^t(y)$ denotes the probability that the event y occurs at time t conditional on the information available at time $s < t$

$$\pi_s^t(y) = P(w_t = y | \mathcal{F}_s), \quad s < t$$

where $w_t \in \mathcal{P}_t \subset \mathcal{F}_t$ such that

$$0 \leq \pi_s^t(\cdot) \leq 1.$$

In particular, the conditional probability that we will have event y_t at time t , given a sequence sequence of events y_{t-1}, \dots, y_0 from time $t = 0$ until time $t - 1$ is denoted as

$$\pi_{t-1}^t(y) = P(w_t = y_t | w_{t-1} = y_{t-1}, \dots, w_0 = y_0),$$

In order to understand the meaning of the conditional probability, consider a particular case in which it is conditional on the information for a particular previous moment $s < t$. Let $\pi_s^t(y|z)$ be the probability that event y occurs at time t given that event z has occurred at time $s < t$,

$$\pi_s^t(y|z) = P(w_t = y | w_s = z)$$

Let us denote \mathcal{P}_t^y and \mathcal{P}_s^z the partitions of \mathcal{P}_t and \mathcal{P}_s which contain, respectively, y and z . Then, clearly $\pi_s^t(y|z) = 0$ if $\mathcal{P}_t^y \cap \mathcal{P}_s^z = \emptyset$.

Let us assume that there are at t , $n_s^t(z)$ subsets of \mathcal{P}_t such that $\cup_{j=1}^{n_s^t(z)} w_{t,j} = z$. Then

$$\pi_s^t(z|z) = P\left(\cup_{j=1}^{n_s^t(z)} w_{t,j} \mid w_s = z\right) = \sum_{j=1}^{n_s^t(z)} P(w_t = w_{t,j} | w_s = z) = \sum_{j=1}^{n_s^t(z)} \pi_s^t(w_{t,j}) = 1$$

Alternatively, if we use the indicator function $1_s^t(z)$

$$1_s^t(z) = \begin{cases} 1, & \text{if } \mathcal{P}_t^y \subset \mathcal{P}_s^z \\ 0, & \text{if } \mathcal{P}_t^y \not\subset \mathcal{P}_s^z \end{cases}$$

we can write

$$\pi_s^t(z|z) = \sum_{j=1}^{N_t} P(w_t = w_{t,j} | w_s = z) 1_s^t(z) = \sum_{j=1}^{N_t} \pi_s^t(w_{t,j}) 1_s^t(z) = 1$$

This means that $\pi_s^t(y|z)$ is a probability measure starting from a particular node $w_s = z$.

There is a **relationship between unconditional and conditional probabilities**. Consider the unconditional probabilities, $\pi_0^t(w_t) = P(w_t = y)$ and $\pi_0^{t-1}(w_{t-1}) = P(w_{t-1} = z)$. We assume that the following relationship between unconditional and conditional probabilities hold ¹

$$\pi_0^t(y) = \pi_{t-1}^t(y|z) \pi_0^{t-1}(z).$$

For sequences of events we have the following relationship between conditional and unconditional probabilities, if we take information up to time $s = t - 1$

$$P(w_t = y_t | w_{t-1} = y_{t-1}, \dots, w_0) = \frac{P(w_t = y_t, w_{t-1} = y_{t-1}, \dots, w_0)}{P(w_{t-1} = y_{t-1}, \dots, w_0)} \quad (4.1)$$

or

$$\pi_{t-1}^t(y_t | y^{t-1}) = \pi_0^t(y^t) / \pi_0^{t-1}(y^{t-1})$$

Therefore, if we consider a sequence of events starting from $t = 0$, $\{w_\tau\}_{\tau=0}^t$, we have associated a sequence of conditional probabilities, or transition probabilities

$$\{\pi_0^1, \pi_1^2, \dots, \pi_{t-1}^t\}$$

where

$$\pi_0^t = \prod_{s=1}^t \pi_{s-1}^s$$

¹This is a consequence of the Bayes' rule.

4.1.2 Stochastic processes

Definition 5. A **stochastic process** is a function $X : \mathbb{T} \times \Omega \rightarrow \mathbb{R}$. For every $\omega \in \Omega$ the mapping $t \mapsto X_t(\omega)$ defines a trajectory and for every $t \in \mathbb{T}$ the mapping $\omega \mapsto X(t, \omega)$ defines a random variable.

A common representation is the sequence $X = \{X_0, X_1, \dots, X_T\} = \{X_t : t = 0, 1, \dots, T\}$, which is a sequence of random variables.

Definition 6. The stochastic process $X^t = \{X_\tau : \tau = 0, 1, \dots, t\}$ is an **adapted process** as regards the filtration $\mathbb{F} = \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ if the random variable X_t is measurable as regards \mathcal{F}_t , for all $t \in \mathbb{T}$. That is

$$X_t = X(w_t) \quad w_t \in \mathcal{F}_t$$

that is $X_t = (X_1, \dots, X_{N_t}) = (X(w_1), \dots, X(w_{N_t}))$

Example: In the previous example, the stochastic process

$$X(\omega) = \begin{cases} 6, & \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 7, & \omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}$$

is adapted to the filtration \mathbb{F} and is measurable as regards \mathcal{F}_1 , but the process

$$Y(\omega) = \begin{cases} 6, & \omega \in \{\omega_1, \omega_3, \omega_5, \omega_7\} \\ 7, & \omega \in \{\omega_2, \omega_4, \omega_6, \omega_8\} \end{cases}$$

is not.

Example: Again, in the previous example, a stochastic process adapted to the filtration, has the following possible realizations

$$\begin{aligned} X_0 = X(w_0) &= x_0 = 1, \quad w_0 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \\ X_1 = X(w_1) &= \begin{cases} x_{1,1} = 1.5, & w_1 = w_{1,1} = \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ x_{1,2} = 0.5, & w_1 = w_{1,2} = \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases} \\ X_2 = X(w_2) &= \begin{cases} x_{2,1} = 2, & w_2 = w_{2,1} = \{\omega_1, \omega_2\} \\ x_{2,2} = 1.2, & w_2 = w_{2,2} = \{\omega_3, \omega_4\} \\ x_{2,3} = 0.9, & w_2 = w_{2,3} = \{\omega_5, \omega_6\} \\ x_{2,4} = 0.3, & w_2 = w_{2,4} = \{\omega_7, \omega_8\} \end{cases} \\ X_3 = X(w_3) &= \begin{cases} x_{3,1} = 4, & w_3 = w_{3,1} = \{\omega_1\} \\ x_{3,2} = 3, & w_3 = w_{3,2} = \{\omega_2\} \\ x_{3,3} = 2.5, & w_3 = w_{3,3} = \{\omega_3\} \\ x_{3,4} = 2, & w_3 = w_{3,4} = \{\omega_4\} \\ x_{3,5} = 1.5, & w_3 = w_{3,5} = \{\omega_5\} \\ x_{3,6} = 1, & w_3 = w_{3,6} = \{\omega_6\} \\ x_{3,7} = 0.5, & w_3 = w_{3,7} = \{\omega_7\} \\ x_{3,8} = 0.125, & w_3 = w_{3,8} = \{\omega_8\} \end{cases} \end{aligned}$$

Therefore, depending on the particular history, $\{w_t\}_{t=0}^3$ we have a particular realization $\{X_t\}_{t=0}^3$, that is a sequence, for example $\{1, 0.50.9, 1\}$, if the process is adapted. \square

Observations:

- (1) Seen as a function, we may write $X : \Omega \rightarrow \mathbb{T} \times \mathbb{R}^{N_t}$ where N_t is the number of elements of the partition \mathcal{P}_t .
- (2) The stochastic process takes a constant value for any $\omega \in A_t$ such that $A_t \in \mathcal{P}_t$, i.e., for every element of the partition \mathcal{P}_t .
- (3) For each t , we have $X_t \in \mathbb{R}^{N_t}$, is a random variable that can take as many values as the elements of \mathcal{P}_t .
- (4) At a given period t the past and present values of X_t are known. This is the meaning of "measurability as regards \mathcal{F}_t ". In addition, in any given moment we know that the true state belongs to a particular subset of the partition \mathcal{P}_t (as the past history had led us to it). As the partitions are increasingly "thinner", the observation of past values of X may allow us to infer what the subsequent states of nature will be observed (and the true state will be one of them). Therefore, analogously to a random variable, which also induces a measure over Ω , a stochastic process also generates a filtration (which is based on the past history of the process).

Definition 7. *The stochastic process $X = \{X_t : t = 0, 1, \dots, T\}$ is a **predictable process** as regards the filtration $\mathbb{F} = \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ if the random variable X_t is measurable as regards \mathcal{F}_{t-1} , for every $t \in \mathbb{T}$. That is, for time $0 \leq t \leq T$*

$$X_t = X(w_{t-1}) = (X_{t1}, \dots, X_{t, N_{t-1}})$$

Observation: As $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$ then the predictable processes are also adapted as regards the filtration \mathbb{F} .

As the underlying space (Ω, \mathcal{F}, P) is a probability space, there is an associated probability of following a particular sequence $\{x_0, \dots, x_t\}$ ²

$$P(X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0), t \in [0, T].$$

We can write, the conditional probability for $t + 1$

$$P(X_{t+1} = x_{t+1} | X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = \frac{P(X_{t+1} = x_{t+1}, X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0)}{P(X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0)} \quad (4.2)$$

We will characterize it next through its conditional and unconditional moments.

²Off course the same definitions apply for any subsequence t_0, \dots, t_n .

4.1.3 Conditional probabilities

Definition 8. The **unconditional mathematical expectation** of the random variable Y as regards \mathbb{F} is written as $E_0(Y) = E(Y | \mathcal{F}_0)$, for any $t \in \mathbb{T}$.

Definition 9. The **conditional mathematical expectation** of the random variable Y as regards \mathbb{F} is written as $E_t(Y) = E(Y | \mathcal{F}_t)$, for any $t \in \mathbb{T}$.

The conditional expectation is a random variable which is measurable as regards $\mathcal{G} \subseteq \mathcal{F}_t$, for any $t \in \mathbb{T}$ and has the following **properties**:

- $E(X | \mathcal{G}) \geq 0$ if $X \geq 0$;
- $E(aX + bY | \mathcal{G}) = aE(X | \mathcal{G}) + bE(Y | \mathcal{G})$ for a and b constant;
- $E(X | \mathcal{F}_0) = E(X) = \sum_{s=1}^S P(\omega_s)X(\omega_s)$;
- $E(1 | \mathcal{G}) = 1$;
- **law of iterated expectations:** given $\mathcal{F}_{t-s} \subseteq \mathcal{F}_t$, then

$$E(X | \mathcal{F}_{t-s}) = E(E(X | \mathcal{F}_t) | \mathcal{F}_{t-s}),$$

or, equivalently,

$$E_{t-s}(E_t(X)) = E_{t-s}(X);$$

- if Y measurable as regards \mathcal{F}_t then $E(Y | \mathcal{F}_t) = Y$;
- if Y is independent as regards \mathcal{F}_t then $E(Y | \mathcal{F}_t) = E(Y)$;
- if Y is measurable as regards \mathcal{F}_t then $E(YX | \mathcal{F}_t) = YE(X | \mathcal{F}_t)$.

For a stochastic process, the **unconditional mathematical expectation** of X_t is

$$E_0(X_t) = E(X_t | \mathcal{F}_0) = \sum_{s=1}^{N_t} P(X_t = x_{t,s})x_{t,s} = \sum_{s=1}^{N_t} \pi_0^t(x_{t,s})x_{t,s}$$

where $x_{t,s} = X(w_t = w_{t,s})$, and the **unconditional variance** of X_t is

$$V_0(X_t) = V(X_t | \mathcal{F}_0) = E_0[(X_t - E_0(X_t))^2] = \sum_{s=1}^{N_t} \pi_0^t(x_{t,s})(x_{t,s} - E_0(X_t))^2.$$

The **conditional mathematical expectation** of X_t as regards \mathcal{F}_s with $s \leq t$ is denoted by

$$E_s(X_t) = E(X_t | \mathcal{F}_s).$$

Using our previous notation, we immediately see that $E_s(X_t)$ is a random variable, measurable as regards \mathcal{F}_s , that is

$$E_s(X_t) = (E_{s,1}(X_t), \dots, E_{s,N_s}(X_t)).$$

Again, we can consider the case in which the expectation is taken relative to a given history $X^s = Y^s$ that is to a particular path $\{x_s = y_s, x_{s-1} = y_{s-1}, \dots, x_0 = y_0\}$,

$$E_{s,i}(X_t) = \sum_{j=0}^{N_t} P(X_t = x_{t,j} | Y^s) X_{t,j} = \sum_{j=0}^{N_t} \pi_s^t(X_{t,j} | Y^s) X_{t,j}, \quad i = 1, \dots, N_{t-1}$$

or relative to a given value of the process at time s , X_s

$$E_{s,i}(X_t) = \sum_{j=0}^{N_t} P(X_t = x_{t,j} | X_s = x_{s,i}) X_{t,j} = \sum_{j=0}^{N_t} \pi_s^t(X_{t,j} | X_{s,i}) X_{t,j}, \quad i = 1, \dots, N_{t-1}$$

4.1.4 Some important processes

Stationary process: a stochastic process $\{X_t, t \in \mathbb{T}\}$ is stationary if the joint probability is invariant to time shifts

$$P(X_{t+h} = x_{t+h}, X_{t+h-1} = x_{t+h-1} \dots X_{s+h} = x_{s+h}) = P(X_t = x_t, X_{t-1} = x_{t-1} \dots X_s = x_s)$$

For example, a sequence of independent and identically distributed random variables generates a stationary process.

Example : A random walk, X , is a process such that $X_0 = 0$ and $X_{t+1} - X_t$, for $t = 0, 1, \dots, T-1$ are i.i.d. X is both a stationary process and a martingale.

Processes with independent variation: a process $\{X_t, t \in \mathbb{T}\}$ has independent variations if the random variation between any two moments $X_{t_j} - X_{t_{j-1}}$ is independent from any other sequence of time instants.

Wiener process

It is an example of a stationary stochastic process with independent variations (and continuous sample paths) $W^t = \{W_t, t \in [0, T]\}$ such that

$$W_0 = 0, \quad E_0(W_t) = 0, \quad V_0(W_t - W_s) = t - s,$$

for any pair $t, s \in [0, T]$.

Markov processes and Markov chains

Markov processes have the **Markov property**:

$$P(X_{t+h} = x_{t+h} | X^t = x^t) = P(X_{t+h} = x_{t+h} | X_t = x_t)$$

where $X^t = x^t$ denotes $\{X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0\}$ and $P(X_{t+h} = x_{t+h} | X^t = x^t)$ is called a **transition probability**.

That is, the conditional probability of any state in the future, conditional on the past history is only dependent on the present state of the process. In other words, the only relevant probabilities are transition probabilities. The sequences of transition probabilities are independent of the

If we assume that we have a finite number of states, that is X_t can only take a finite number of values

$$\mathcal{Y} = \{y_1, \dots, y_M\}$$

then we have a (discrete-time) **Markov chain**. Then, the transition probability can be denoted as

$$\pi_i^j(n) = P(X_{t_{n+1}} = y_j | X_{t_n} = y_i)$$

the transition probability from state y_i to state y_j at time t_n . Obviously,

$$0 \leq \pi_i^j(n) \leq 1, \quad \sum_{j=1}^M \pi_i^j(n) = 1$$

for any n .

The transitional probability $\pi_i^j(n)$ is conditional. We have the unconditional probability

$$\pi_i(n) = P(X_{t_n} = y_i)$$

and the recurrence relationship between conditional and unconditional probabilities

$$\pi_j(n+1) = \sum_{i=1}^M \pi_i^j(n) \pi_i(n)$$

determining the unconditional probability that the Markov chain will be in state y_j at $n+1$.

We can determine the vector of probabilities for all states in \mathcal{Y} through the transition probability matrix

$$\mathbf{P}(n+1) = \pi(n) \mathbf{P}(n)$$

where

$$\mathbf{P}(n) = \begin{pmatrix} \pi_1 \\ \dots \\ \pi_M \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_1^1 & \dots & \pi_1^M \\ \vdots & \vdots & \vdots \\ \pi_1^M & \dots & \pi_M^M \end{pmatrix}$$

4.2 Stochastic Dynamic Programming

From the set of all feasible random sequences $\{x_t, u_t\}_{t=0}^T$ where $x_t = x_t(w^t)$ and $u_t = u_t(w^t)$ are \mathcal{F}_t -adapted processes, choose a contingent plan $\{x_t^*, u_t^*\}_{t=0}^T$ such that

$$\max_{\{u_t\}_{t=0}^T} E_0 \left[\sum_{t=0}^T \beta^t f(u_t, x_t) \right]$$

where

$$E_0 \left[\sum_{t=0}^T \beta^t f(u_t, x_t) \right] = E \left[\sum_{t=0}^T \beta^t f(u_t, x_t) \mid \mathcal{F}_0 \right]$$

subject to the random sequence of constraints

$$\begin{aligned} x_1 &= g(x_0, u_0) \\ &\dots \\ x_{t+1} &= g(x_t, u_t, w^{t+1}), \quad t = 1, \dots, T-2 \\ &\dots \\ x_T &= g(x_{T-1}, u_{T-1}, w^T) \end{aligned}$$

where x_0 is given and w_t is a \mathcal{F}_t -adapted process representing the uncertainty affecting the agent decision.

Intuition: at the beginning of a period t x_t and u_t are **known** but the value of x_{t+1} , at the end of period t is **conditional** on the value of w^{t+1} . The values of this random process may depend on an exogenous variable which is given by a stochastic process.

Let us call $\{u_t^*\}_{t=0}^T$ the **optimal control**. This is, in fact, a contingent plan, i.e., a planned sequence of decisions **conditional** on the sequence of states of nature (or events).

At time $t = 0$ the optimal value of the state variable x_0 is

$$\begin{aligned} V_0 &= V(x_0) = E_0 \left[\sum_{t=0}^T \beta^t f(u_t^*, x_t) \right] = \\ &= \max_{\{u_t\}_{t=0}^T} E_0 \left[\sum_{t=0}^T \beta^t f(u_t, x_t) \right] = \\ &= \max_{\{u_t\}_{t=0}^T} E_0 \left[f(u_0, x_0) + \beta \sum_{t=1}^T \beta^{t-1} f(u_t^*, x_t) \right] = \\ &= \max_{u_0} \left\{ f(u_0, x_0) + \beta E_0 \left[\max_{\{u_t\}_{t=1}^T} E_1 \left[\sum_{t=1}^T \beta^{t-1} f(u_t^*, x_t) \right] \right] \right\} \end{aligned}$$

by the principle of DP, by the fact the the $t = 0$ variables are measurable as regards \mathcal{F}_0 and because of the law of iterated expectations. Then

$$V_0 = \max_{u_0} [f_0 + \beta E_0(V_1)]$$

or

$$V(x_0) = \max_{u_0} \{f(u_0, x_0) + \beta E_0[V(x_1)]\}$$

where u_0 , x_0 and V_0 are \mathcal{F}_0 -adapted and $x_1 = g(u_0, x_0, w^1)$, in V_1 , is \mathcal{F}_1 -adapted.

The same idea can be extended to any $0 \leq t \leq T$. Then

$$V(x_t) = \max_{u_t} \{f(u_t, x_t) + \beta E_t[V(x_{t+1})]\}$$

and, under boundness conditions, for the case in which $T \rightarrow \infty$.

Observe that $\{V(x_t)\}_{t=0}^{\infty}$ is a \mathcal{F}_t -adapted stochastic process and the operator $E_t(\cdot)$ is a probability measure conditional on the information available at t (represented by \mathcal{F}_t).

If $\{x\}_{t=0}^T$ follows a k -state Markov process then the HJB equation can be written as

$$V(x_t) = \max_{u_t} \left\{ f(u_t, x_t) + \beta \sum_{s=1}^k \pi(s) V(x_{t+1}(s)) \right\}$$

4.3 Applications

4.3.1 The representative consumer

Assumptions :

- there are K short run financial assets which have a price S_t^j , $j = 1, \dots, K$ at time t that entitle to a contingent payoff D_{t+1}^j at time $t + 1$;
- the value of the portfolio at the end of period t is $\sum_{j=1}^K \theta_{t+1}^j S_t^j$, and its conditional payoff, at the beginning of period $t + 1$ is $\sum_{j=1}^K \theta_{t+1}^j (S_{t+1}^j + V_{t+1}^j)$, where θ_{t+1}^j , S_t^j and V_t^j are \mathcal{F}_t -measurable,
- the stream of endowments, is $\{y_t\}_{t=0}^T$ where y_t is \mathcal{F}_t -measurable;
- $A_0 \neq 0$.

Budget constraints The consumer faces a (random) sequence of budget constraints, which defines his feasible contingent plans:

- At time $t = 0$

$$c_0 + \sum_{j=1}^K \theta_1^j S_0^j \leq y_0 + A_0$$

where all the components are scalars.

- At time $t = 1$, we have

$$c_1 + \sum_{j=1}^K \theta_2^j S_1^j \leq y_1 + \sum_{j=1}^K \theta_1^j (S_1^j + V_1^j)$$

where c_1 , y_1 and θ_2 are \mathcal{F}_1 -measurable, that is

$$c_1(s) + \sum_{j=1}^K \theta_2^j(s) S_1^j(s) \leq y_1(s) + \sum_{j=1}^K \theta_1^j (S_1^j(s) + V_1^j(s)), \dots, s = 1, \dots, N_1$$

if we assume, for simplicity, that $\dim(\Omega) = N$.

Looking at the two budget constraints, we get the non-human wealth at time $t = 1$ (which is also \mathcal{F}_1 -measurable) as

$$A_1 := \sum_{j=1}^K \theta_1^j (S_1^j + V_1^j)$$

Then, the sequence of instantaneous budget constraints is

$$y_t + A_t \geq c_t + \sum_{j=1}^K \theta_{t+1}^j S_t^j, \quad t \in [0, \infty) \quad (4.3)$$

$$A_{t+1} = \sum_{j=1}^K \theta_{t+1}^j (S_{t+1}^j + V_{t+1}^j), \quad t \in [0, \infty) \quad (4.4)$$

The representative consumer chooses a strategy of consumption, represented by the adapted process $c := \{c_t, t \in \mathbb{T}\}$ and of financial transactions in K financial assets, represented by the forecastable process $\theta := \{\theta_t, t \in \mathbb{T}\}$, where $\theta_t = (\theta_t^1, \dots, \theta_t^K)$ in order to solve the following problem

$$\max_{\{c, \theta\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to equations (4.3)-(4.4), where $A(0) = A_0$ is given and

$$\lim_{k \rightarrow \infty} E_t [\beta^k S_{t+k}^j] = 0.$$

The last condition prevents consumers from playing Ponzi games. It rules out the existence of arbitrage opportunities.

Solution by using dynamic programming

The HJB equation is

$$V(A_t) = \max_{c_t} \{u(c_t) + \beta E_t [V(A_{t+1})]\}.$$

In our case, we may solve it by determining the optimal transactions strategy

$$V(A_t) = \max_{\theta_{t+1}^j, j=1, \dots, K} \left\{ u \left[y_t + A_t - \sum_{j=1}^K \theta_{t+1}^j S_t^j \right] + \beta E_t [V(A_{t+1}(\theta_{t+1}^1, \dots, \theta_{t+1}^K))] \right\} \quad (4.5)$$

The optimality condition, is

$$-u'(c_t) S_t^j + \beta E_t [V'(A_{t+1})(S_{t+1}^j + V_{t+1}^j)] = 0,$$

for every asset $j = 1, \dots, K$.

In order to simplify this expression, we may apply the Benveniste-Scheinkman formula (see (Ljungqvist and Sargent, 2000, p. 237)) by substituting the optimality conditions in equation (4.5) and by differentiating it in order to A_t . Then we get

$$V'(A_t) = u'(c_t).$$

However, we cannot go further without specifying the utility function, a particular probability process and the stochastic processes for the asset prices and returns.

Intertemporal arbitrage condition

The optimality conditions may be rewritten as the following intertemporal arbitrage conditions for the representative consumer

$$u'(c_t)S_t^j = \beta E_t \left[u'(c_{t+1})(S_{t+1}^j + V_{t+1}^j) \right], \quad j = 1, \dots, K, \quad t \in [0, \infty). \quad (4.6)$$

This model is imbedded in a financial market institutional framework that prevents the existence of arbitrage opportunities.

This imposes conditions on the asymptotic properties of prices, which imposes conditions on the solution of the consumer's problem.

Taking equation (4.6) and operating recursively, the consumer chooses an optimal trajectory of consumption such that (remember that the asset prices and the payoffs are given to the consumer)

$$S_t^j = E_t \left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{u'(c_{t+\tau})}{u'(c_t)} V_{t+\tau}^j \right], \quad j = 1, \dots, K, \quad t \in [0, \infty) \quad (4.7)$$

In order to prove this note that by repeatedly applying the law of iterated

expectations

$$\begin{aligned}
u'(c_t)S_t^j &= \beta E_t \left[u'(c_{t+1})(S_{t+1}^j + V_{t+1}^j) \right] = \\
&= \beta E_t \left[u'(c_{t+1})S_{t+1}^j \right] + \beta E_t \left[u'(c_{t+1})V_{t+1}^j \right] = \\
&= \beta E_t \left\{ \beta E_{t+1} \left[u'(c_{t+2})(S_{t+2}^j + V_{t+2}^j) \right] \right\} + \\
&\quad + \beta E_t \left[u'(c_{t+1})V_{t+1}^j \right] = \\
&= \beta E_t \left\{ \beta E_{t+1} \left[u'(c_{t+2})S_{t+2}^j \right] \right\} + \\
&\quad + E_t \left\{ \beta u'(c_{t+1})V_{t+1}^j + \beta^2 E_{t+1} \left[u'(c_{t+2})V_{t+2}^j \right] \right\} = \\
&= \beta^2 E_t \left[u'(c_{t+2})S_{t+2}^j \right] + E_t \left[\sum_{\tau=1}^2 \beta^\tau u'(c_{t+\tau})V_{t+\tau}^j \right] = \\
&\dots \\
&= \beta^k E_t \left[u'(c_{t+k})S_{t+k}^j \right] + E_t \left[\sum_{\tau=1}^k \beta^\tau u'(c_{t+\tau})V_{t+\tau}^j \right] = \\
&\dots \\
&= \lim_{k \rightarrow \infty} \beta^k E_t \left[u'(c_{t+k})S_{t+k}^j \right] + E_t \left[\sum_{\tau=1}^{\infty} \beta^\tau u'(c_{t+\tau})V_{t+\tau}^j \right]
\end{aligned}$$

The condition for ruling out speculative bubbles,

$$\lim_{k \rightarrow \infty} \beta^k E_t \left[u'(c_{t+k})S_{t+k}^j \right] = 0$$

allows us to get equation (4.7).

References: Ljungqvist and Sargent (2000)

Chapter 5

Continuous time

5.1 Introduction to continuous time stochastic processes

Assume that $\mathbb{T} = \mathbb{R}_+$ and that the probability space is (Ω, \mathcal{F}, P) where Ω is of infinite dimension.

Let $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{T}\}$ be a filtration over the probability space (Ω, \mathcal{F}, P) . $(\Omega, \mathcal{F}, \mathbb{F}, P)$ may be called a filtered probability space.

A stochastic process is a flow $X = \{X(t, \omega), t \in \mathbb{T}, \omega \in \mathcal{F}_t\}$.

5.1.1 Brownian motions

Definition 10. Brownian motion Assume the probability space $(\Omega, \mathcal{F}, P^x)$, a sequence of sets $F_t \in \mathbb{R}$ and the stochastic process $B = \{B(t), t \in \mathbb{T}\}$ such that a sequence of distributions over B are given by

$$\begin{aligned} & P^x(B(t_1) \in F_1, \dots, B(t_k) \in F_k) = \\ = & \int_{F_1 \times \dots \times F_k} p(t_1, x, x_1) \dots p(t_2 - t_1, x_1, x_2) \dots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 dx_2 \dots dx_k, \end{aligned}$$

where the conditional probabilities are

$$P(B(t_j) = x_j \mid B(t_i) = x_i) = p(t_j - t_i, x_i, x_j) = (2\pi(t_j - t_i))^{-\frac{1}{2}} e^{-\frac{|x_j - x_i|^2}{2(t_j - t_i)}}.$$

Then B is a Brownian motion (or Wiener process), starting from the initial state x , where $(P^x(B(0) = x) = 1)$.

Remark We consider one-dimensional Brownian motions: that is, those for which the trajectories have continuous versions, $B(\omega) : \mathbb{T} \rightarrow \mathbb{R}^n$ where $t \mapsto B_t(\omega)$, with $n = 1$.

Properties of B

1. B is a gaussian process:
that is, $Z = (B(t_1), \dots, B(t_k))$ has a normal distribution with mean $M = E^x[Z] = (x, \dots, x) \in \mathbb{R}^k$ and variance-covariance matrix

$$E^x[(Z_j - M_j)(Z_i - M_i)]_{i,j=1,\dots,k} = \begin{bmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \dots & \dots & \dots & \dots \\ t_1 & t_1 & \dots & t_k \end{bmatrix}$$

and, for any moment $t \geq 0$

$$\begin{aligned} E^x[B(t)] &= x, \\ E^x[(B(t) - x)^2] &= t, \\ E^x[(B(t) - x)(B(s) - x)] &= \min(t, s), \\ E^x[(B(t) - B(s))^2] &= t - s. \end{aligned}$$

2. B has independent variations:
given a sequence of moments $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$ and the sequence of variations of a Brownian motion, $B_{t_2} - B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}}$ we have

$$E^x[(B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] = 0, \quad t_i < t_j$$

3. B has continuous versions;
4. B is a stationary process:
that is $B(t+h) - B(t)$, with $h \geq 0$ has the same distribution for any $t \in \mathbb{T}$;
5. B is not differentiable (with probability 1) in the Riemannian sense.

Observation: it is very common to consider $B(0) = 0$, that is $x = 0$.

5.1.2 Processes and functions of B

As with random variables and with stochastic processes over a finite number of periods and states of nature:

- (1) if we can define a filtration, we can build a stochastic process or,
- (2) given a stochastic process, we may define a filtration.

Observation: for random variables (if we define a measure we can define a random variable, or a random variables may induce measures in a measurable space).

Definition 11. (*filtration*)

$\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{T}\}$ is a filtration if it verifies:

- (1) $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$
- (2) $\mathcal{F}_t \subset \mathcal{F}$ for any $t \in \mathbb{T}$

Definition 12. (*Filtration over a Brownian motion*)

Consider a sequence of subsets of \mathbb{R} , F_1, F_2, \dots, F_k where $F_j \subset \mathbb{R}$ and let B be a Brownian motion of dimension 1. \mathcal{F}_t is a σ -algebra generated by $B(s)$ such that $s \leq t$, if it is the finest partition which contains the subsets of the form

$$\{\omega : B_{t_1}(\omega) \in F_1, \dots, B_{t_k}(\omega) \in F_k\}$$

if $t_1, t_2, \dots, t_k \leq t$.

Intuition \mathcal{F}_t is the set of all the histories of B_j up to time t .

Definition 13. (\mathcal{F}_t -measurable function)

A function $h(\omega)$ is called a \mathcal{F}_t -measurable if and only if it can be expressed as the limit of the sum of function of the form

$$h(t, \omega) = g_1(B(t_1))g_2(B(t_2)) \dots g_k(B(t_k)), \quad t_1, t_2, \dots, t_k \leq t.$$

Intuition h is a function of present and past values of a Brownian motion.

Definition 14. (\mathcal{F}_t -adapted process)

If $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{T}\}$ is a filtration then the process $g = \{g(t, \omega), t \in T, \omega \in \Omega\}$ is called \mathcal{F}_t -adapted if for every $t \geq 0$ the function $\omega \mapsto g_t(\omega)$ is \mathcal{F}_t -measurable.

For what is presented next, there are two important types of functions and processes:

Definition 15. (*Class N functions*)

Let $f : \mathbb{T} \times \Omega \rightarrow \mathbb{R}$. If

1. $f(t, \omega)$ is \mathcal{F}_t -adapted;
2. $E \left[\int_s^T f(t, \omega)^2 dt \right] < \infty$,

then $f \in N(s, T)$, is called a class $N(s, T)$ function.

Definition 16. (*Martingale*)

The stochastic process $M = \{M(t), t \in \mathbb{T}\}$ defined over (Ω, \mathcal{F}, P) is a martingale as regards the filtration \mathbb{F} if

1. $M(t)$ is \mathcal{F}_t -measurable, for any $t \in \mathbb{T}$,
2. $E[|M(t)|] < \infty$, for any $t \in \mathbb{T}$,
3. the martingale property holds

$$E[M(s) | \mathcal{F}_t] = M(t),$$

for any $s \geq t$.

5.1.3 Itô's integral

Definition 17. (*Itô's integral*)

Let f be a function of class N and $B(t)$ a one-dimensional Brownian motion. Then the Itô's integral is denoted as

$$I(f, \omega) = \int_s^T f(t, \omega) dB_t(\omega)$$

If f is a class N function, it can be proved that the sequence of elementary functions of class N ϕ_n where $\int_s^T \phi(t, \omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega)[B_{t_{j+1}}(\omega) - B_{t_j}(\omega)]$, verifying $\lim_{n \rightarrow \infty} E[\int_s^T |f - \phi_n|^2 dt] = 0$, such that Itô's integral is defined as

$$I(f, \omega) = \int_s^T f(t, \omega) dB_t(\omega) = \lim_{n \rightarrow \infty} \int_s^T \phi_n(t, \omega) dB_t(\omega).$$

Intuition: as f is not differentiable (in the Riemannian sense), we may have several definitions of integral. Itô's integral approximates the function f by step functions the e_j **evaluated at the beginning** of the interval (t_{j+1}, t_j) . The Stratonovich integral

$$\int_s^T f(t, \omega) \circ dB_t(\omega)$$

evaluates in the intermediate point of the intervals.

Theorem 1. (*Properties of Itô's integral*)

Consider two class N function $f, g \in N(0, T)$, then

1. $\int_s^T f dB_t = \int_s^U f dB_t + \int_U^T f dB_t$ for almost every ω and for $0 \leq s < U < T$;
2. $\int_s^T (cf + g) dB_t = c \int_s^T f dB_t + \int_s^T g dB_t$ for almost all ω and for c constant;
3. $E\left(\int_s^T f dB_t\right) = 0$;
4. has continuous versions up to time t , that is there is a stochastic process $J = \{J(t), t \in \mathbb{T}\}$ such that $P\left(J(t) = \int_0^t f dB_t\right) = 1$ for any $0 \leq t \leq T$;
5. $M(t, \omega) = \int_0^t f(s, \omega) dB_s$ is a martingale as regards the filtration \mathcal{F}_t .

5.1.4 Stochastic integrals

Up to this point we presented a theory of integration, and implicitly of differentiation. The Itô's presents a very useful stochastic counterpart of the chain rule of differentiation.

Definition 18. (*Itô's process or stochastic integral*)

Let B_t be a one-dimensional Brownian motion over (Ω, \mathcal{F}, P) . Let ν be a class N function (i.e., such that $P\left(\int_0^t \nu(s, \omega)^2 ds < \infty, \forall t \geq 0\right) = 1$) and let μ be a function of class \mathcal{H}_t (i.e., such that $P\left(\int_0^t |\mu(s, \omega)| ds < \infty, \forall t \geq 0\right) = 1$).

Then $X = \{X(t), t \in \mathbb{T}\}$ where $X(t)$ has the domain (Ω, \mathcal{F}, P) , is a stochastic integral of dimension one if it is a stochastic process with the following equivalent representations:

1. integral representation

$$X(t) = X(0) + \int_0^t \mu(s, \omega) ds + \int_0^t \nu(s, \omega) dB_s$$

2. differential representation

$$dX(t) = u(t, \omega)dt + \nu(t, \omega)dB(t)$$

Lemma 3. (*Itô's lemma*)

Let $X(t)$ be a stochastic integral in its differential representation

$$dX(t) = \mu dt + \nu dB(t)$$

and let $g(t, x)$ be a continuous differentiable function as regards its two arguments. Then

$$Y = \{Y(t) = g(t, X(t)), t \in \mathbb{T}\}$$

is a stochastic process that verifies

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2.$$

We apply the rule: $dt^2 = dt dB(t) = 0$ e $dB(t)^2 = dt$ then

$$dY(t) = \left(\frac{\partial g}{\partial t}(t, X(t)) + \frac{\partial g}{\partial x}(t, X(t))\mu + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))\nu^2 \right) dt + \frac{\partial g}{\partial x}(t, X(t))\nu dB(t).$$

Example 1 Let $X(t) = B(t)$ where B is a Brownian motion. Which process follows $Y(t) = (1/2)B(t)^2$? If we write $Y(t) = g(t, x) = (1/2)x^2$ and apply Itô's lemma we get

$$\begin{aligned} dY(t) &= d\left(\frac{1}{2}B(t)^2\right) = \\ &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB(t))^2 = \\ &= 0 + B(t)dB(t) + \frac{1}{2}dB(t)^2 \\ &= B(t)dB(t) + \frac{dt}{2}, \end{aligned}$$

or in the integral representation

$$Y(t) = \int_0^t dY(t) = \frac{1}{2}B(t)^2 = \int_0^t B(s)dB(s) + \frac{t}{2}.$$

Example 2 Let $dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$ and let $Y(t) = \ln(X(t))$. Find the SDE for Y . Applying the Itô's lemma

$$\begin{aligned} dY(t) &= \frac{\partial Y}{\partial X}dX(t) + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX(t))^2 = \\ &= \frac{dX(t)}{X(t)} - \frac{1}{2X(t)^2}(dX(t))^2 = \\ &= \mu dt + \sigma dB(t) - \frac{\sigma^2}{2}dt \end{aligned}$$

Then

$$dY(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB(t)$$

or, in the integral representation

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t \mu - \frac{\sigma^2}{2}ds + \int_0^t \sigma dB(s) = \\ &= Y(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t) \end{aligned}$$

if $B(0) = 0$.

The process X is called a **geometric Brownian motion**, as its integral (which is the $\exp(Y)$) is

$$X(t) = X(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}.$$

Example 3: Let $Y(t) = e^{aB(t)}$. Find a stochastic integral for Y . From the Itô's lemma

$$\begin{aligned} dY(t) &= ae^{aB(t)}dB(t) + \frac{1}{2}a^2e^{aB(t)}(dB(t))^2 = \\ &= \frac{1}{2}a^2e^{aB(t)}dt + ae^{aB(t)}dB(t) \end{aligned}$$

the integral representation is

$$Y(t) = Y(0) + \frac{1}{2}a^2 \int_0^t e^{aB(s)}ds + a \int_0^t e^{aB(s)}dB(s)$$

5.1.5 Stochastic differential equations

Stochastic differential equation's theory is a very vast field. Here we will only present some results that we will be useful afterwards.

Definition 19. (SDE)

A stochastic differential equation can be defined as

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t))W(t)$$

where $b(t, x) \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $W(t)$ represents a one-dimensional "noise".

Definition 20. (SDE: Itô's interpretation)

$X(t)$ satisfies a stochastic differential equation is

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$$

or in the integral representation

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s).$$

How to solve, or study qualitatively, the solution of those equations ?

There are two solution concepts: weak and strong. We say that the process X is a **strong** solution if $X(t)$ is \mathcal{F}_t -adapted, and if $B(t)$ is given, it verifies the representation of the SDE.

An important special case is the **diffusion equation**, which is the SDE with constant coefficients and multiplicative noise

$$dX(t) = aX(t)dt + \sigma X(t)dB(t).$$

As we already saw, the (strong) **solution** is the stochastic process X such that

$$X(t) = xe^{(a-\frac{\sigma^2}{2})t+\sigma B(t)}$$

where x is a random variable, which can be determined from $x = X(0)$, where $X(0)$ is a given initial distribution, and $B(t) = \int_0^t dB(s)$, if $B(0) = B_0 = 0$.

Properties:

1. asymptotic behavior:
 - if $a - \frac{\sigma^2}{2} < 0$ then $\lim_{t \rightarrow \infty} X(t) = 0$ a.s.
 - if $a - \frac{\sigma^2}{2} > 0$ then $\lim_{t \rightarrow \infty} X(t) = \infty$ a.s.
 - if $a - \frac{\sigma^2}{2} = 0$ then $\lim_{t \rightarrow \infty} X(t)$ will be finite a.s.
2. can we say anything about $E[X(t)]$?

$$E(X(t)) = E(X(0))e^{at}$$

To prove this, take Example 3 and observe that the stochastic integral of $Y(t) = e^{aB(t)}$ is

$$Y(t) = Y(0) + \frac{1}{2}a^2 \int_0^t e^{aB(s)} ds + a \int_0^t e^{aB(s)} dB(s)$$

taking expected values

$$\begin{aligned} E[Y(t)] &= E[Y(0)] + E \left[\frac{1}{2}a^2 \int_0^t e^{aB(s)} ds \right] + aE \left[\int_0^t e^{aB(s)} dB(s) \right] = \\ &= E[Y(0)] + \frac{1}{2}a^2 \int_0^t E [e^{aB(s)}] ds + 0 \end{aligned}$$

because $e^{aB(s)}$ is a f -class function (from the properties of the Brownian motion). Differentiating

$$\frac{dE[Y(t)]}{dt} = \frac{1}{2}a^2 E[Y(t)]$$

as $E[Y(0)] = 1$. Then

$$E[Y(t)] = e^{\frac{1}{2}a^2 t}$$

References: Oksendal (2003)

5.1.6 Stochastic optimal control

Finite horizon

We consider the stochastic optimal control problem, that consists in determining the value function, $J(\cdot)$,

$$J(t_0, x_0) = \max_u E_{t_0} \left[\int_{t_0}^T f(t, x, u) dt \right]$$

subject to

$$dx(t) = g(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dB(t)$$

given the initial distribution for the state variable $x(t_0, \omega) = x_0(\omega)$. We call $u(\cdot)$ the control variable and assume that the objective, the drift and the volatility functions, $f(\cdot)$, $g(\cdot)$ and $\sigma(\cdot)$, are of class H (the second) and N (the other two).

By applying the Bellman's principle, the following nonlinear partial differential equation over the value function, called the **Hamilton-Jacobi-Bellman** equation, gives us the necessary conditions for optimality

$$-\frac{\partial J(t, x)}{\partial t} = \max_u \left(f(t, x, u) + g(t, x, u) \frac{\partial J(t, x)}{\partial x} + \frac{1}{2} \sigma(t, x, u)^2 \frac{\partial^2 J(t, x)}{\partial x^2} \right).$$

In order to prove it, heuristically, observe that a solution of the problem verifies

$$\begin{aligned} J(t_0, x_0) &= \max_u E_{t_0} \left(\int_{t_0}^T f(t, x, u) dt \right) = \\ &= \max_u E_{t_0} \left(\int_{t_0}^{t_0+\Delta t} f(t, x, u) dt + \int_{t_0+\Delta t}^T f(t, x, u) dt \right) \end{aligned}$$

by the principle of the dynamic programming and the law of iterated expectations we have

$$\begin{aligned} J(t_0, x_0) &= \max_{u, t_0 \leq t_0+\Delta t} E_{t_0} \left[\int_{t_0}^{t_0+\Delta t} f(t, x, u) dt + \max_{u, t_0 \leq t_0+\Delta t} E_{t_0+\Delta t} \left[\int_{t_0+\Delta t}^T f(t, x, u) dt \right] \right] \\ &= \max_{u, t_0 \leq t_0+\Delta t} E_{t_0} [f(t, x, u)\Delta t + J(t_0 + \Delta t, x_0 + \Delta x)] \end{aligned}$$

if we write $x(t_0 + \Delta t) = x_0 + \Delta x$. If J is continuously differentiable of the second order, the Itô's lemma may be applied to get, for any t

$$J(t + \Delta t, x + \Delta x) = J(t, x) + J_t(t, x)\Delta t + J_x(t, x)\Delta x + \frac{1}{2} J_{xx}(t, x)(\Delta x)^2 + h.o.t$$

where

$$\begin{aligned}\Delta x &= g\Delta t + \sigma\Delta B \\ (\Delta x)^2 &= g^2(\Delta t)^2 + 2g\sigma(\Delta t)(\Delta B) + \sigma^2(\Delta B)^2 = \sigma^2\Delta t.\end{aligned}$$

Then,

$$\begin{aligned}J &= \max_u E \left[f\Delta t + J + J_t\Delta t + J_x g\Delta t + J_x\sigma\Delta B + \frac{1}{2}\sigma^2 J_{xx}\Delta t \right] \\ &= \max_u \left[f\Delta t + J + J_t\Delta t + J_x g\Delta t + \frac{1}{2}\sigma^2 J_{xx}\Delta t \right]\end{aligned}$$

as $E_0(dB) = 0$. Taking the limit $\Delta \rightarrow 0$, we get the HJB equation.

Infinite horizon

The autonomous discounted infinite horizon problem is

$$V(x_0) = \max_u E_0 \left[\int_0^\infty f(x, u) e^{-\rho t} dt \right]$$

subject to

$$dx(t) = g(x(t), u(t))dt + \sigma(x(t), u(t))dB(t)$$

given the initial distribution of the state variable $x(0, \omega) = x_0(\omega)$, and assuming the same properties for functions $f(\cdot)$, $g(\cdot)$ and $\sigma(\cdot)$. Also $\rho > 0$.

Applying, again, the Bellman's principle, now the HJB equation is the non-linear ordinary differential equation of the form

$$\rho V(x) = \max_u \left(f(x, u) + g(t, x, u)V'(x) + \frac{1}{2}\sigma(x, u)^2 V''(x) \right).$$

References Kamien and Schwartz (1991, cap. 22).

5.2 Applications

5.2.1 The representative agent problem

Here we present essentially the Merton (1971) model, which is a micro model for the simultaneous determination of the strategies of consumption and portfolio investment. We next present a simplified version with one risky and one riskless asset.

Let the exogenous processes be given to the representative consumer

$$\begin{aligned} d\beta(t) &= r\beta(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dB(t) \end{aligned}$$

where β and S are respectively the prices of the risky and the riskless assets, r is the interest rate, μ and σ are the constant rates of return and volatility for the equity.

The stock of financial wealth is denoted by $A(t) = \theta_0(t)\beta(t) + \theta_1(t)S(t)$, for any $t \in \mathbb{T}$. Assume that $A(0) = \theta_0(0)\beta(0) + \theta_1(0)S(0)$ is known.

Assume that the agent also gets an endowment $\{y(t), t \in \mathbb{R}\}$ which adds to the incomes from financial investments and that the consumer uses the proceeds for consumption. Then the value of financial wealth at time t is

$$A(t) = A(0) + \int_0^t (r\theta_0(s)\beta(s) + \mu\theta_1(s)S(s) + y(s) - c(s)) ds + \int_0^t \sigma\mu\theta_1(s)S(s)dB(s).$$

If the weight of the equity in total wealth is denoted by $w = \frac{\theta_1 S}{A}$ then $1 - w = \frac{\theta_0 \beta}{A}$. Then, we get the differential representation of the instantaneous budget constraint comes

$$dA(t) = [r(1 - w(t))A(t) + \mu w(t)A(t) + y(t) - c(t)]dt + w(t)\sigma A(t)dB(t). \quad (5.1)$$

The problem for the consumer-investor is

$$\max_{c,w} E_0 \left[\int_0^\infty u(c(t))e^{-\rho t} dt \right] \quad (5.2)$$

subject to the instantaneous budget constraint (5.1), given $A(0)$ and assuming that the utility function is increasing and concave.

This is a stochastic optimal control problem with infinite horizon, and has two control variables. The Hamilton-Jacobi-Bellman equation is

$$\rho V(A) = \max_{c,w} \left\{ u(c) + V'(A)[(r(1 - w) + \mu w)A + y - c] + \frac{1}{2}w^2\sigma^2 A^2 V''(A) \right\}.$$

The first order necessary conditions allows us to get the optimal controls, i.e. the optimal policies for consumption and portfolio composition

$$u'(c^*) = V'(A), \quad (5.3)$$

$$w^* = \frac{(r - \mu)V'(A)}{\sigma^2 AV''(A)} \quad (5.4)$$

If $u''(\cdot) < 0$ then the optimal policy function for consumption may be written as $c^* = h(V'(A))$. Plugging into the HJB equation, we get the differential equation over $V(A)$

$$\rho V(A) = u\left(h(V'(A))\right) - h(V'(A))V'(A)(y + rA)V'(A) - \frac{(r - \mu)^2(V'(A))^2}{2\sigma^2 V''(A)}. \quad (5.5)$$

In some cases the equation may be solve explicitly. In particular, let the utility function be CRRA as

$$u(c) = \frac{c^{1-\eta} - 1}{1-\eta}, \quad \eta > 0$$

and conjecture that the solution for equation (5.5) is of the type

$$V(A) = x(y + rA)^{1-\eta}$$

for x an unknow constant. If it is indeed a solution, there should be a constant, dependent upon the parameters of the model, such that equation (5.5) holds.

First note that

$$\begin{aligned} V'(A) &= (1 - \eta)rx(y + rA)^{-\eta} \\ V''(A) &= -\eta(1 - \eta)r^2x(y + rA)^{-\eta-1} \end{aligned}$$

then: the optimal consumption policy is

$$c^* = (xr(1 - \eta))^{-\frac{1}{\eta}} (y + rA)$$

and the optimal portfolio composition is

$$w^* = \left(\frac{\mu - r}{\sigma^2}\right) \frac{y + rA}{\eta r A}$$

Interestingly it is a linear function of the ratio of total (human plus financial wealth $\frac{y}{r} + a$) over financial wealth.

After some algebra, we get

$$V(A) = \Theta \left(\frac{y + rA}{r}\right)^{1-\eta}$$

where ¹

$$\Theta \equiv \frac{1}{1-\eta} \left[\frac{\rho r(1-\eta)}{\eta} - \frac{(1-\eta)}{2\eta^2} \left(\frac{\mu-r}{\sigma} \right)^2 \right]^{-\eta}$$

Then the optimal consumption is

$$c^* = \left(\frac{\rho r(1-\eta)}{\eta} - \frac{(1-\eta)}{2\eta^2} \left(\frac{\mu-r}{\sigma} \right)^2 \right) \left(\frac{y+rA}{r} \right)$$

If we set the total wealth as $W = \frac{y}{r} + A$, we may write the value function and the policy functions for consumption and portfolio investment

$$\begin{aligned} V(W) &= \Theta W^{1-\eta} \\ c^*(W) &= (1-\eta)\Theta^{-\frac{1}{\eta}}W \\ w^*(W) &= \left(\frac{\mu-r}{\eta\sigma^2} \right) \frac{W}{A}. \end{aligned}$$

Remark The value function follows a stochastic process which is a monotonous function for wealth. The optimal strategy for consumption follows a stochastic process which is a linear function of the process for wealth and the fraction of the risky asset in the optimal portfolio is a direct function of the premium of the risky asset relative to the riskless asset and is a inverse function of the volatility.

We see that the consumer cannot eliminate risk, in general. If we write $c^* = \chi A$, where $\chi \equiv (1-\eta)\Theta^{-\frac{1}{\eta}}\frac{W}{A}$, then the optimal process for wealth is

$$dA(t) = [r^* + (\mu-r)w^* - \chi]A(t)dt + \sigma w^* A(t)dB(t)$$

where $r^* = r\frac{W}{A}$, which is a linear SDE. Then as $c^* = c(A)$, if we apply the Itô's lemma we get

$$dc = \chi dA = c(\mu_c dt + \sigma_c dB(t))$$

where

$$\begin{aligned} \mu_c &= \frac{r-\rho}{\eta} + \frac{1+\eta}{2} \left(\frac{\mu-r}{\sigma\eta} \right)^2 \\ \sigma_c &= \frac{\mu-r}{\sigma\eta}. \end{aligned}$$

The sde has the solution

$$c(t) = c(0) \exp \left\{ \left(\mu_c - \frac{\sigma_c^2}{2} \right) t + \sigma_c B(t) \right\}$$

¹Of course, $x = r^{-(1-\eta)}\Theta$.

, where $\frac{\mu-r}{\sigma}$ is the Sharpe index, and the unconditional expected value for consumption at time t

$$E_0[C(t)] = E_0[C(0)]e^{\mu ct}.$$

References Merton (1971), Merton (1990), Duffie (1996) Cvitanić and Zapatero (2004)

5.2.2 The stochastic Ramsey model

Let us assume that the economy is represented by the equations

$$\begin{aligned} dK(t) &= (F(K(t), L(t)) - C(t))dt \\ dL(t) &= \mu L(t)dt + \sigma L dB(t) \end{aligned}$$

where we assume that $F(K, L)$ is linearly homogeneous, given the (deterministic) initial stock of capital and labor $K(0) = K_0$ and $L(0) = L_0$.

The growth of the labor input (or its productivity) is stochastic.

Let us define the variables in intensity terms

$$k(t) \equiv \frac{K(t)}{L(t)}, \quad c(t) \equiv \frac{C(t)}{L(t)}$$

We can get the restriction of the economy as a single equation on k by using the Itô's lemma

$$\begin{aligned} dk &= \frac{\partial k}{\partial K} dK + \frac{\partial k}{\partial L} dL + \frac{1}{2} \frac{\partial^2 k}{\partial K^2} (dK)^2 + \frac{1}{2} \frac{\partial^2 k}{\partial K \partial L} dK dL + \frac{1}{2} \frac{\partial^2 k}{\partial L^2} (dL)^2 \\ &= \frac{F(K, L) - C}{L} dt - \frac{K(\mu L dt + \sigma L dB(t))}{L^2} + \sigma^2 \frac{K}{L} dt \\ &= (f(k) - c - (\mu - \sigma^2)k) dt - \sigma^2 k dB(t) \end{aligned}$$

if we set $f(k) = F\left(\frac{K}{L}, 1\right)$.

The HJB equation is

$$\rho V(k) = \max_c \left\{ u(c) + V'(k) (f(k) - c - (\mu - \sigma^2)k) + \frac{1}{2} (k\sigma)^2 V''(k) \right\}$$

the optimality condition is again

$$u'(c) = V'(k)$$

and, we get again a 2nd order ODE

$$\rho V(k) = u(h(k)) + V'(k) (f(k) - h(k) - (\mu - \sigma^2)k) + \frac{1}{2} (k\sigma)^2 V''(k).$$

Again, we assume the benchmark **particular case**: $u(c) = \frac{c^{1-\theta}}{1-\theta}$ and $f(k) = k^\alpha$. Then the optimal policy function becomes

$$c^* = V'(k)^{-\frac{1}{\theta}}$$

and the HJB becomes

$$\rho V(k) = \frac{\theta}{1-\theta} V'(k)^{\frac{\theta-1}{\theta}} + V'(k) (k^\alpha - (\mu - \sigma^2)k) + \frac{1}{2} (k\sigma)^2 V''(k)$$

We can get, again, a closed form solution if we assume further that $\theta = \alpha$. Again we conjecture that the solution is of the form

$$V(k) = B_0 + B_1 k^\alpha$$

Using the same methods as before we get

$$\begin{aligned} B_0 &= (1-\alpha) \frac{B_1}{\rho} \\ B_1 &= \frac{1}{1-\alpha} \left[\frac{(1-\alpha)\theta}{(1-\theta)(\rho - (1-\alpha)^2\sigma^2)} \right]^\alpha. \end{aligned}$$

Then

$$V(k) = B_1 \left(\frac{1-\alpha}{\rho} + k^{1-\alpha} \right)$$

and

$$c^* = c(k) = \left(\frac{(1-\theta)(\rho - (1-\alpha)^2\sigma^2)}{(1-\alpha)\theta} \right) k \equiv \varrho k$$

as we see an increase in volatility decreases consumption for every level of the capital stock.

Then the optimal dynamics of the per capita capital stock is the SDE

$$dk^*(t) = (f(k^*(t)) - (\mu + \varrho - \sigma^2)k^*(t)) dt - \sigma^2 k^*(t) dB(t).$$

In this case we can not solve it explicitly as in the deterministic case.

References: Brock and Mirman (1972), Merton (1975), Merton (1990)

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