Modeling Horizontal Shareholding with Ownership Dispersion*

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Abstract

The dominant formulation for modeling the objective function of managers of competing firms with horizontal shareholding has been critiqued for producing the result that, if non-horizontal shareholders are highly dispersed and the interests of horizontal shareholders in the different firms coincide, managers would mimic the interests of horizontal shareholders even if they own a share of the firm that does not induce full control. We show that this issue can be avoided with an alternative formulation that is derived from a probabilistic voting model that assumes shareholders with higher financial stakes will take greater interest in the managerial actions, which yields the result that managers maximize a control-weighted sum of the shareholders' relative returns.

JEL Classification: L13, L41

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1 Introduction

Horizontal shareholding is common ownership in competing firms. Such horizontal shareholding can induce a conflict in the firm-specific interests of shareholders, wherein horizontal shareholders in any given firm want that firm to pursue a less competitive strategy than the strategy desired by non-horizontal shareholders.[1] Hence, firm managers must weigh the conflicting interests of different shareholders according to their relative influence over firm decision-making.

Schmalz (2018) discusses the desirable properties for the weighting scheme (objective function) used by managers: (*i*) absent horizontal shareholding, managers would maximize their firm's own profit, which implies no weight would be given to rival firm profits; (*ii*) with horizontal shareholding, managers would internalize the impact of their firm's strategy on rival firm profits when their firm's controlling shareholders have financial rights in the rival; (*iii*) the weight that managers assign to rival firm profits would be continuous on the financial and control rights of the firm's controlling shareholders; (*iv*) managers would maximize industry profit when all controlling shareholders are fully diversified across rivals; and (*v*) the weight that managers assign to rival firm profits would reflect relatively more the interests of relatively large shareholders.[2] Gramlich and Grundl (2017), O'Brien and Waehrer (2017) and Crawford *et al.* (2018) discuss an additional property: (*vi*) the weight that managers assign to rival firm profits when the interests of horizontal shareholders in the different firms coincide would not mimic those interests when horizontal shareholders own a share of the firm that, even if non-horizontal shareholders are highly dispersed, does not induce full control.

The dominant formulation of the objective function of managers is due to O'Brien and Salop (2000, henceforth O&S), who incorporating features from both Rotemberg (1984) and Bresnahan and Salop (1986), assume that the interests of each shareholder can be captured by the return from her financial investments and, as such, *the manager would decide the strategy of the firm to maximize a control-weighted sum of the firm's shareholders returns*:

$$\max_{x_j} \sum\nolimits_{k\in\Theta} \gamma_{kj} R_k = \sum\nolimits_{k\in\Theta_j} \gamma_{kj} R_k, \tag{1}$$

[1] Although non-horizontal shareholders may favor a different firm-specific strategy, that does not mean they are harmed by horizontal shareholding because horizontal shareholding also reduces the competitiveness of rival firms, and non-horizontal shareholders benefit from a mutual reduction of competition at both the firm and its rivals.

[2] Property (*v*) implies that the weight that managers assign to rival firm profits would not decrease - and would typically increase - as horizontal shareholders become relatively larger. The former accounts, for example, for ownership structures in which horizontal shareholders already hold full control and, thus, becoming relatively larger would not impact the weight that managers assign to rival firm profits.

where Θ denotes the set of existing shareholders and Θ_j denotes the subset of shareholders that hold financial rights in firm j, x_j denotes the strategy of firm j, γ_{kj} denotes the control rights of shareholder k in firm j, and R_k denotes the returns of shareholder k. Because those returns are a function of the profits of the firms in which shareholders hold financial rights, this implies that the manager of any firm j would maximize a weighted sum of the profits of (potentially) all the firms in the industry:

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} R_k = \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \left(\sum\nolimits_{g\in\Im} \phi_{kg}\pi_g\right) \propto \max_{x_j} \pi_j + \sum\nolimits_{g\in\Im, g\neq j} \frac{\sum_{k\in\Theta_j} \gamma_{kj}\phi_{kg}}{\sum_{k\in\Theta_j} \gamma_{kj}\phi_{kj}} \pi_g, \tag{2}$$

which makes use of the fact that $R_k = \sum_{g\in\Im} \phi_{kg}\pi_g$ denotes the returns of shareholder k, where $\Im$ denotes the set of existing firms, ϕ_{kj} denotes the financial rights of shareholder k in firm j, and π_j denotes the operating profit of firm j.

The financial rights ϕ_{kj} have very clear empirical counterparts and can thus be measurable. However, the control rights γ_{kj} have less clear empirical counterparts. These rights are, naturally, a function of the distribution of voting rights in the firm, but the particular function is an open question. Azar (2016, 2017) addresses this question by showing that the dominant formulation can be microfounded through a probabilistic voting model in which shareholders vote to elect one of two potential managers, an incumbent and a challenger, with conceivably differing plans.[3] Further, if managers maximize their vote share, he shows that the control rights of shareholder k in firm j would be proportional to their voting rights (proportional control): $\gamma_{kj} = \upsilon_{kj}$, where υ_{kj} denotes the voting rights of shareholder k in firm j.[4] If, alternatively, managers maximize their odds of election, he shows that the control rights of shareholder k in firm j would equal the odds that their vote will be pivotal in the election, i.e., would equal their Banzhaf (1965) power index: $\gamma_{kj} = \lambda^p_{kj} / \sum_{h\in\Theta_j} \lambda^p_{hj}$, where λ^p_{kj} denotes the number of subsets of Θ_j that can award victory to a manager/candidate of firm j and in which shareholder k is pivotal.[5]

The weight w_{jg} that the manager of firm j assigns to rival firm g profits is, under this dominant formulation, thus, given by $w_{jg} = \sum_{k\in\Theta_j} \gamma_{kj}\phi_{kg} / \sum_{k\in\Theta_j} \gamma_{kj}\phi_{kj}$. It satisfies, as discussed in Appendix A, properties (i) to (iv) above. However, it may not necessarily satisfy property (v) as we show in our illustrative example below. Further, it fails property (vi). In order to see why, note that the weight's numerator accounts for the interests of the horizontal shareholders in firm g (since the financial rights of non-horizontal sharehold-

[3] Azar (2017) also considers a probabilistic voting model in which shareholders vote on whether to approve a manager-proposed change in the firm's strategic plan.

[4] The voting rights of a shareholder may not necessarily coincide with her financial rights.

[5] Brito *et al.* (2018a) generalize Azar (2016, 2017)'s framework to jointly capture common-ownership and cross-ownership by rival firms.

ers in the rival firm are null) whereas the denominator accounts for the interests of both horizontal and non-horizontal shareholders in firm j. As the ownership of non-horizontal shareholders becomes dispersed, the denominator tends to reflect solely the interests of the (the non-dispersed) horizontal shareholders (as the summation referent to the non-horizontal shareholders approximates zero). If the rights of the horizontal shareholders in firms j and g coincide, this implies that the weight w_{jg} will approximate one, which reflects the exact interests of horizontal shareholders in the two firms, even when the voting rights of the horizontal shareholders do not induce full control of the firm.

In this paper, we propose an alternative formulation for the objective function of managers. In the lines of Azar (2016, 2017) and Brito *et al.* (2018a), we use a probabilistic voting model. But unlike prior literature, we assume that managers expect shareholders with higher financial stakes in their firm would (for example) incur more effort to become informed on the vote and thus could potentially have a larger preference for (or against) the challenger than other shareholders. Then, in equilibrium, the manager would choose the strategy of the firm to maximize a control-weighted sum of the firm's shareholders *relative* returns:

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \tilde{R}_k = \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \left(\sum\nolimits_{g\in\Im} \frac{\phi_{kg}}{\phi_{kj}} \pi_g \right) = \pi_j + \sum\nolimits_{g\in\Im, g\neq j} \sum\nolimits_{k\in\Theta_j} \frac{\gamma_{kj}\phi_{kg}}{\phi_{kj}} \pi_g, \quad (3)$$

where $\tilde{R}_k = \sum_{g\in\Im} \left(\phi_{kg}/\phi_{kj}\right) \pi_g$ denotes the *relative* returns of shareholder k, which are normalized by her financial rights in firm j. The intuition is as follows. The strategy proposals of the candidates impact the returns of the firm's shareholders, which in turn impacts their probability of voting for the candidates. Our key assumption implies that the latter impact is lower for shareholders with higher financial stakes since, having a larger preference for (or against) the challenger, they already have a larger probability of voting in one direction. As a consequence, candidates pay less attention to those shareholders than they would under the dominant formulation. They do so, by weighting not the absolute, but the relative returns of shareholders.[6] This proposed alternative formulation is similar in nature to the formulation in Crawford *et al.* (2018) who, to address property (vi), consider the relative returns of shareholder k to be normalized by her overall financial investments: $\tilde{R}'_k = \sum_{g\in\Im} \left(\phi_{kg}/\sum_{h\in\Im} \phi_{kh}\right) \pi_g$, but we microfound our proposal through a probabilistic voting model.

[6]This means that if a shareholder owns a portfolio that is equal to another shareholder's portfolio multiplied by α, the manager will consider they both have the same relative returns. Their control rights will naturally be different, but their relative returns will be the same. This makes the smaller shareholder more relevant in the manager's objective function because in the dominant formulation, this shareholder would have smaller control rights and also smaller absolute returns.

The weight w_{jg} that the manager of firm j assigns to rival firm g profits is, under this proposed alternative formulation, thus, given by $w_{jg} = \sum_{k \in \Theta_j} \gamma_{kj} \left(\phi_{kg} / \phi_{kj} \right)$.[7] This weight satisfies, as discussed in Appendix A, properties (i) to (iv) above. Again, similarly to the dominant formulation, it may not necessarily satisfy property (v) as we show in our illustrative example below. However, it does satisfy property (vi). In order to see why, note that if the rights of the horizontal shareholders in firms j and g coincide, the weight w_{jg} will solely approximate one, which reflects their exact interests in the two firms, when their voting rights do induce full control of the firm.

The remainder of the paper is organized as follows. Section 2 presents an illustrative example to examine how the two formulations compare. Section 3 presents the theoretical framework under which the proposed alternative objective function of managers is derived and Section 4 concludes.

2 An Illustrative Example

In order to examine how the two formulations compare, we now address an illustrative example, borrowed from Gramlich and Grundl (2017). We focus in properties (v) and (vi) since, as discussed in Appendix A, both formulations satisfy properties (i) to (iv). Consider a duopoly between firms j and g. Consider also that one shareholder, shareholder c, holds symmetric financial and voting rights in both firms: $\phi_{cj} = \upsilon_{cj} = \phi_{cg} = \upsilon_{cg} = \phi_c = \upsilon_c = x < 1$; and each of each firm's remaining n shareholders holds *smaller* equal financial and voting rights in solely one firm: $\phi_{kj} = \upsilon_{kj} = \phi_{kg} = \upsilon_{kg} = \phi_k = \upsilon_k = (1-x)/n < x$ for all $k \neq c$, which implies that $n > (1-x)/x$. Finally, consider that υ_c and υ_k induce γ_c and $\gamma_k = (1-\gamma_c)/n$ control rights, respectively, for the horizontal shareholder and each non-horizontal shareholder $k \neq c$, in each firm.

In this setting, the weight w_{jg} that the manager of firm j assigns to the profit of rival firm g is, under O&S's formulation, given by:

$$w_{jg} = \frac{\gamma_c x}{\gamma_c x + \sum_{k \neq c} \gamma_k \frac{(1-x)}{n}} = \frac{\gamma_c x}{\gamma_c x + (1-\gamma_c) \frac{(1-x)}{n}}, \tag{4}$$

and, under the proposed alternative formulation, by:

$$w_{jg} = \gamma_c. \tag{5}$$

[7] The weight w_{jg} is - for a given distribution of control rights - constant with respect to the dispersion of the ownership of non-horizontal shareholders. However, that does not imply that the dispersion of non-horizontal shareholders does not have an impact on w_{jg}. It does, but solely via the control rights which are induced by the distribution of the firm's voting rights.

In order to examine if the above weights satisfy properties (v) and (vi), we begin by computing the control rights of the horizontal shareholder γ_c. Under an assumption of proportional control, these rights are given by $\gamma_c^p = v_c = x$ while, under an assumption of Banzhaf control, are given by $\gamma_c^b = \lambda_c^p / \sum_{h \in \Theta_j} \lambda_h^p$, where λ_c^p denotes the number of subsets of Θ_j that can award victory to a candidate and in which shareholder c is pivotal. Appendix A computes the number of such subsets and shows that:

$$\gamma_c^b = \frac{\sum_{y=\left\lfloor \left(\frac{1}{2}-x\right)\frac{n}{1-x} \right\rfloor + 1}^{\left\lfloor \frac{n}{2(1-x)} \right\rfloor} C_y^n}{\sum_{y=\left\lfloor \left(\frac{1}{2}-x\right)\frac{n}{1-x} \right\rfloor + 1}^{\left\lfloor \frac{n}{2(1-x)} \right\rfloor} C_y^n + n\left(C_{\left\lfloor \frac{n}{2(1-x)} \right\rfloor}^{n-1} + C_{\left\lfloor \left(\frac{1}{2}-x\right)\frac{n}{1-x} \right\rfloor}^{n-1} \right)}, \tag{6}$$

where $\lfloor y \rfloor$ denotes the largest integer lower than y and C_y^n denotes the number of y-combinations from n.

Having computed the control rights of the horizontal shareholder, we now discuss if the above weights satisfy properties (v) and (vi). We begin by addressing them under O&S's formulation in combination with an assumption of proportional control, which yields that w_{jg} is given by:

$$w_{jg} = \frac{\gamma_c^p x}{\gamma_c^p x + (1-\gamma_c^p)\frac{(1-x)}{n}} = \frac{x^2}{x^2 + \frac{(1-x)^2}{n}}. \tag{7}$$

This weight, as discussed in Appendix A, increases in x and n. Since the relative size of the horizontal shareholder depends, in this setting, both on its absolute size x and on the number of smaller non-horizontal shareholders n, this implies that property (v) holds for increasing levels of horizontal shareholding with a constant number of shareholders and for a constant level of horizontal shareholding with an increasing number of shareholders.[8] Further, the above combination also yields that w_{jg} approximates one, for any given value of the horizontal shareholder's control rights x, as $n \to \infty$. This predicts that managers would engage in near-monopoly pricing when the non-horizontal shareholders are highly dispersed, even if the horizontal shareholder does not have full control. This implies that property (vi) does not hold.

If, however, we combine O&S's formulation with an assumption of Banzhaf control, w_{jg} is given by:

$$w_{jg} = \frac{\gamma_c^b x}{\gamma_c^b x + (1-\gamma_c^b)\frac{(1-x)}{n}}, \tag{8}$$

which, as discussed in Appendix A, increases in x but may decrease with n. This implies that property (v) holds for increasing levels of horizontal shareholding with a constant number of

[8] We interpret property (v) as requiring $\partial w_{jg}/\partial x \geq 0$ and $\partial w_{jg}/\partial n \geq 0$.

shareholders, but may or may not hold for a constant level of horizontal shareholding with an increasing number of shareholders. Further, the above combination also yields that w_{jg} approximates one, for any given value of the horizontal shareholder's control rights γ_c^b, as $n \to \infty$. This implies, again, that property (*vi*) does not hold. Table 1, Panel A illustrates these features.

We now discuss properties (*v*) and (*vi*) under our proposed alternative formulation. If we combine this formulation with an assumption of proportional control, w_{jg} is given by:

$$w_{jg} = \gamma_c^p = x, \tag{9}$$

which increases in x and does not vary (and thus does not decrease) in n. This implies that property (*v*) holds for increasing levels of horizontal shareholding with a constant number of shareholders and for a constant level of horizontal shareholding with an increasing number of shareholders. Further, the above combination also yields that w_{jg} approximates one solely when x tends to one. As such, the formulation does not predict that managers would engage in near-monopoly pricing, even when the non-horizontal shareholders are highly dispersed, if the horizontal shareholder does not have full control. This implies that property (*vi*) holds.

If, however, we combine our formulation with an assumption of Banzhaf control, w_{jg} is given by:

$$w_{jg} = \gamma_c^b, \tag{10}$$

which, as discussed in Appendix A, never decreases with x but may decrease with n. This implies that property (*v*) holds for increasing levels of horizontal shareholding with a constant number of shareholders, but may or may not hold for a constant level of horizontal shareholding with an increasing number of shareholders. Further, the above combination also yields that w_{jg} approximates one solely when γ_c^b tends to one. As such, the formulation does not predict that managers would engage in near-monopoly pricing, even when the non-horizontal shareholders are highly dispersed, if the horizontal shareholder does not have full control. This implies that property (*vi*) holds. Table 1, Panel B illustrates these features.

3 The Theoretical Framework

This section introduces the theoretical framework under which the proposed alternative objective function of managers is derived. The general setting combines features from Brito *et al.* (2014), Azar (2016, 2017) and Brito *et al.* (2018a).

Table 1

*Weight that each Manager Assigns to the Profit of the Rival Firm**

	Number of Non-Horizontal Shareholders					
	1	100	499	500	501	1000
Panel A: O&S's Formulation						
1% Horizontal Shareholding						
Proportional Control	0.000	0.010	0.048	0.049	0.049	0.093
	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
Banzhaf Control	0.000	0.052	0.296	0.312	0.296	0.597
	(0.000)	(0.051)	(0.077)	(0.082)	(0.077)	(0.128)
5% Horizontal Shareholding						
Proportional Control	0.003	0.217	0.580	0.581	0.581	0.735
	(0.050)	(0.050)	(0.050)	(0.050)	(0.050)	(0.050)
Banzhaf Control	0.000	0.222	0.687	0.703	0.686	0.885
	(0.000)	(0.051)	(0.077)	(0.082)	(0.077)	(0.128)
10% Horizontal Shareholding						
Proportional Control	0.012	0.552	0.860	0.861	0.861	0.925
	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)
Banzhaf Control	0.000	0.650	0.986	0.985	0.986	1.000
	(0.000)	(0.143)	(0.564)	(0.534)	(0.560)	(0.950)
20% Horizontal Shareholding						
Proportional Control	0.059	0.862	0.969	0.969	0.969	0.984
	(0.200)	(0.200)	(0.200)	(0.200)	(0.200)	(0.200)
Banzhaf Control	0.000	0.986	1.000	1.000	1.000	1.000
	(0.000)	(0.744)	(1.000)	(1.000)	(1.000)	(1.000)
Panel B: Proposed Alternative Formulation						
1% Horizontal Shareholding						
Proportional Control	0.010	0.010	0.010	0.010	0.010	0.010
	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
Banzhaf Control	0.000	0.051	0.077	0.082	0.077	0.128
	(0.000)	(0.051)	(0.077)	(0.082)	(0.077)	(0.128)
5% Horizontal Shareholding						
Proportional Control	0.050	0.050	0.050	0.050	0.050	0.050
	(0.050)	(0.050)	(0.050)	(0.050)	(0.050)	(0.050)
Banzhaf Control	0.000	0.051	0.077	0.082	0.077	0.128
	(0.000)	(0.051)	(0.077)	(0.082)	(0.077)	(0.128)
10% Horizontal Shareholding						
Proportional Control	0.100	0.100	0.100	0.100	0.100	0.100
	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)
Banzhaf Control	0.000	0.143	0.564	0.534	0.560	0.950
	(0.000)	(0.143)	(0.564)	(0.534)	(0.560)	(0.950)
20% Horizontal Shareholding						
Proportional Control	0.200	0.200	0.200	0.200	0.200	0.200
	(0.200)	(0.200)	(0.200)	(0.200)	(0.200)	(0.200)
Banzhaf Control	0.000	0.744	1.000	1.000	1.000	1.000
	(0.000)	(0.744)	(1.000)	(1.000)	(1.000)	(1.000)

* Control rights of the horizontal shareholder in parenthesis. The weights in Panel B exactly coincide with the control rights of the horizontal shareholder since - in our example - her financial rights in the two firms exactly cancel.

3.1 Setup

There are K shareholders, indexed by k and N single-product firms, indexed by j, whose total stock is composed of voting stock and non-voting (preferred) stock. Both give the holder the right to a share of the stream of profits, but only the former gives the holder the right to vote for the Board. The holdings of total stock of shareholder k in firm j, regardless of whether it be voting or non-voting stock, capture her *financial rights* to the firm's stream of profits. The holdings of voting stock of shareholder k in firm j, capture her *voting rights* in the firm. These voting rights may not necessarily coincide with her control rights in the firm, which refer to her rights to influence the decisions of firm j.

We follow Azar (2016, 2017) in assuming a standard theory of probabilistic voting. We also assume, along Lindbeck and Weibull (1987), that the manager of each firm j is elected in a shareholder assembly between two potential candidates, an incumbent a_j and a challenger b_j. Shareholders and candidates are assumed to play the following two-stage game. In the first stage, the two candidates to each firm j compete for the voting rights of shareholders by simultaneously proposing a strategy $x_j \in \{x_{aj}, x_{bj}\}$ for the firm, where x_{aj} and x_{bj} denote the strategy proposals of the incumbent and the challenger for firm j, respectively. This strategy can refer to any decision variable - e.g., quantity, price, R&D investment, etc. - of the firm. Let Ω_j denote the strategy space available to the candidates of firm j and $\mathbf{x} = (x_1, \ldots, x_j, \ldots, x_N)^\top$ denote the $N \times 1$ vector of strategy proposals for all the firms in the industry. In the second stage of the game, the shareholder assemblies of all firms are simultaneously held and shareholders vote to elect the manager of each firm. The candidate that receives the majority of each firm j's voting rights is elected manager of the firm and her identity is denoted $m_j \in \{a_j, b_j\}$. Let $\mathbf{m} = (m_1, \ldots, m_j, \ldots, m_N)^\top$ denote the $N \times 1$ vector of elected managers for all the firms in the industry.

3.2 Owners Voting

We begin by addressing the second stage of the game. To do so, we make the following assumptions regarding the voting behavior of shareholders.

Assumption 1 *Owners are conditionally sincere.*

Assumption 1 implies, following Alesina and Rosenthal (1995), that shareholders are assumed to vote, in each firm's shareholder assembly, for the candidate whose strategy proposal maximizes their utilities, given the equilibrium strategy proposals of the candidates to the remaining firms, randomizing between the two in case of indifference.

We consider that the utility of each shareholder k is a function of the winning strategies for all firms in the industry and involves two elements, assumed additively separable, as follows:

$$\begin{aligned} u_k(\mathbf{x},\mathbf{m}) &= R_k(\mathbf{x}) + \sum\nolimits_{g\in\Im} 1(m_g = b_g)\,\xi_{kg} \\ &= \sum\nolimits_{g\in\Im} \phi_{kg}\pi_g(\mathbf{x}) + \sum\nolimits_{g\in\Im} 1(m_g = b_g)\,\xi_{kg}. \end{aligned} \tag{11}$$

The first utility element follows from O&S and captures (assuming a linear utility function) the utility associated to the returns from shareholder k's financial rights holdings. The second utility element follows from Kramer (1983) and captures the utility associated to the credibility (or lack of credibility) attached to the challenger' strategy proposal, where $1(m_g = b_g)$ denotes a dummy variable that takes the value 1 if the challenger is elected manager of firm g and ξ_{kg} denotes the utility that shareholder k obtains from such event, which reflects her preference for (or against) the challenger. These two utility elements imply that the shareholder's choice is deterministic and it is a discontinuous function of the difference in the utilities obtained from the strategy proposals of each candidate. That is, each shareholder k will vote for firm j's incumbent with probability 1 if $u_k(\mathbf{x}_a,\mathbf{m}_a) > u_k(\mathbf{x}_b,\mathbf{m}_b)$, will vote for firm j's challenger with probability 1 if $u_k(\mathbf{x}_a,\mathbf{m}_a) < u_k(\mathbf{x}_b,\mathbf{m}_b)$, and will vote for the two candidates with probability 1/2 if $u_k(\mathbf{x}_a,\mathbf{m}_a) = u_k(\mathbf{x}_b,\mathbf{m}_b)$, where $\mathbf{x}_a = (x_1,\ldots,x_{aj},\ldots,x_N)^\top$, $\mathbf{x}_b = (x_1,\ldots,x_{bj},\ldots,x_N)^\top$, $\mathbf{m}_a = (m_1,\ldots,a_j,\ldots,m_N)^\top$ and $\mathbf{m}_b = (m_1,\ldots,b_j,\ldots,m_N)^\top$.

3.3 Candidates Strategy Proposals

Having described the second stage of the game, we now address the first stage, in which candidates simultaneously choose strategy proposals. To do so, we follow Lindbeck and Weibull (1987) in assuming that the utility associated to the credibility (or lack of credibility) attached to the challenger' strategy proposal, while known to voters, is unobserved by candidates, which treat it as a random utility shock. Further, we assume that these random utility shocks are independently distributed across firms and shareholders according to a symmetric probability distribution with mean zero and cumulative distribution $G_{kj}(\cdot)$.[9] As a consequence, from the perspective of the candidates, voting by shareholders is probabilistic.

We make the following alternative assumptions regarding the candidates objective function when deciding their strategy proposals.

[9] This contrasts with Brito *et al.* (2018a), who assume that random utility shocks are also *identically* distributed across firms and shareholders.

Assumption 2a *Candidates choose strategy proposals to maximize the expected utility from corporate office.*

Assumption 2b *Candidates choose strategy proposals to maximize their vote share.*

Assumption $2a$ implies that candidates choose strategy proposals to maximize the product of the probability that they are elected in the second stage and the utility obtained from the rent associated with corporate office they expect to accrue conditional upon being elected, which we denote Ξ. In turn, Assumption $2b$ implies that candidates choose strategy proposals to maximize the sum, across all shareholders, of the product of the probability that each shareholder votes for the candidate.

Since the maximization problem of the two candidates to firm j is symmetric, we describe - for simplicity of exposition - solely the incumbent's problem. Under Assumption $2a$, the incumbent chooses x_{aj} so to solve:

$$\max_{x_{aj}} \Pr\left(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b\right) \Xi, \tag{12}$$

while, under Assumption $2b$, the incumbent chooses x_{aj} so to solve:

$$\max_{x_{aj}} \sum\nolimits_{k \in \Theta_j} \Pr\nolimits_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) \upsilon_{kj}, \tag{13}$$

where $\Pr\left(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b\right)$ denotes the probability with which the incumbent is, from the candidates perspective, elected manager of the firm in the second stage and $\Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)$ denotes the probability that shareholder k votes for the incumbent.

In order to solve the above maximization problem - in its two versions - we must beforehand derive $\Pr\left(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b\right)$ and - for each shareholder k - $\Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)$. We begin by deriving the latter. Under the assumptions described above for the second utility element, the probability that shareholder k votes for the incumbent is given by:

$$\begin{aligned} \Pr\nolimits_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) &= \Pr\left(u_k\left(\mathbf{x}_a, \mathbf{m}_a\right) > u_k\left(\mathbf{x}_b, \mathbf{m}_b\right)\right) \\ &= \Pr\left(R_k\left(\mathbf{x}_a\right) > R_k\left(\mathbf{x}_b\right) + \xi_{jg}\right) \\ &= \Pr\left(\xi_{jg} < R_k\left(\mathbf{x}_a\right) - R_k\left(\mathbf{x}_b\right)\right) \\ &= G_{kj}\left(R_k\left(\mathbf{x}_a\right) - R_k\left(\mathbf{x}_b\right)\right), \end{aligned} \tag{14}$$

where the second equality makes use of the fact that the term $\sum_{g \in \Im, g \neq j} 1\left(m_g = b_g\right) \xi_{kg}$ enters the utility obtained from both strategy proposals. We now derive the former. To do so, let ℓ_j denote the number of shareholders with voting rights in firm j, $\wp_j$ denote all the $2^{\ell_j - 1}$

possible subsets of those shareholders that can award the majority of votes to a candidate and $\Theta_j^\imath \in \wp_j$ denote a particular subset of those shareholders. Given that her election is ensured with the votes of all shareholders in each subset in $\wp_j$, we have that the probability $\Pr(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b)$ with which the incumbent is elected manager of firm j just sums the probabilities $\Pr\left(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b, \Theta_j^\imath\right)$ with which she is elected in each subset $\Theta_j^\imath$, as follows:

$$\begin{aligned} \Pr(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b) &= \sum\nolimits_{\Theta_j^\imath \in \wp_j} \Pr\left(m_j = a_j | \mathbf{x}_a, \mathbf{x}_b, \Theta_j^\imath\right) \\ &= \sum\nolimits_{\Theta_j^\imath \in \wp_j} \prod\nolimits_{k \in \Theta_j^\imath} \Pr_{ka_j}(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b) \prod\nolimits_{k \notin \Theta_j^\imath} \left(1 - \Pr_{ka_j}(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b)\right) \\ &= \sum\nolimits_{\Theta_j^\imath \in \wp_j} \prod\nolimits_{k \in \Theta_j^\imath} G_{kj}\left(R_k(\mathbf{x}_a) - R_k(\mathbf{x}_b)\right) \prod\nolimits_{k \notin \Theta_j^\imath} G_{kj}\left(R_k(\mathbf{x}_b) - R_k(\mathbf{x}_a)\right), \end{aligned} \tag{15}$$

where the last equality makes use of the fact that the probability distribution of the random utility shocks is symmetric.

Substituting the probabilities (14) and (15) into problems (12) and (13), we can rewrite the incumbent's problem. Under Assumption $2a$, we have that:

$$\max_{x_{aj}} \sum\nolimits_{\Theta_j^\imath \in \wp_j} \prod\nolimits_{k=1}^{\ell_j} G_{kj}\left((2d_k - 1)\left(R_k(\mathbf{x}_a) - R_k(\mathbf{x}_b)\right)\right) \Xi, \tag{16}$$

while, under Assumption $2b$, we have that:

$$\max_{x_{aj}} \sum\nolimits_{k \in \Theta_j} G_{kj}\left(R_k(\mathbf{x}_a) - R_k(\mathbf{x}_b)\right) \upsilon_{kj}. \tag{17}$$

where d_k takes the value one if shareholder $k \in \Theta_j^\imath$ and takes the value zero otherwise.

We make the following technical assumptions regarding the strategy space Ω_j available to the candidates of each firm j, the return $R_k(\mathbf{x})$ of each shareholder k, and the distribution of the random utility attached by shareholder k to the challenger' strategy proposal ξ_{kj} for firm j.

Assumption 3 *The strategy space Ω_j available to the candidates of each firm j* is a *nonempty compact subset of $\Re$.*

Assumption 4 *The return $R_k(\mathbf{x})$ of shareholder k is (a) continuous and twice differentiable in $\mathbf{x}$, with continuous second derivatives; and (b) strictly concave in firm j's strategy $x_j \in \{x_{aj}, x_{bj}\}$, conditional on the strategies of the remaining firms.*

Assumption 5 *The random utility associated to the credibility (or lack of credibility) attached by shareholder k to the challenger' strategy proposal ξ_{kj} for firm j is distributed uniformly on $\left(-\frac{1}{2}\tau_j\phi_{kj}, \frac{1}{2}\tau_j\phi_{kj}\right)$.*

The key, distinctive technical assumption is Assumption 5. It implies that managers expect that shareholders with higher financial stakes in the firm would take more interest on their actions and could therefore potentially have a larger or smaller preference towards or against the challenger than other shareholders. This is consistent with a significant literature that has examined the incentives of large shareholders to undertake costly monitoring of the firm and intervene to correct the manager's suboptimal decisions (see, e.g., Shleifer and Vishny, 1986; Chidambaran and John, 2003). Finally, it implies also that the utility associated to a firm in which a shareholder does not hold financial stakes is null.

Assumptions 1 to 5 ensure the existence of a pure-strategy Nash equilibrium for the candidates strategy proposals' game $(x_{a1}, x_{b1}, \ldots x_{aj}, x_{bj}, \ldots, x_{aN}, x_{bN})$, characterized as follows.

Proposition 1 *Under Assumptions* 1, 2*a or* 2*b*, 3, 4 *and* 5, *there exists a pure-strategy Nash equilibrium for the candidates strategy proposals' game where each candidate maximizes:*

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \tilde{R}_k \tag{18}$$

Under Assumption 2*a*, γ_{kj} *is measured by the normalized Banzhaf (*1965*) power index of shareholder* k *in firm* j*:* $\gamma_{kj} = \lambda^p_{kj} / \sum_{h\in\Theta_j} \lambda^p_{hj}$, *where* λ^p_{kj} *denotes the number of subsets of* Θ_j *that can award victory to a candidate in which shareholder* k *is pivotal. Under assumption* 2*b*, γ_{kj} *is measured by the voting rights of shareholder* k *in firm* j*:* $\gamma_{kj} = \upsilon_{kj}$.

Proof. See Appendix B.

Proposition 1 establishes, in contrast with O&S, that the manager would decide the strategy of the firm to maximize a weighted sum of the firm's shareholders relative returns. The weights γ_{kj} (that are non-negative and sum up to one) capture the importance (and, thus, influence) of each shareholder over the decision-making of the firm and can, thus, be interpreted as a measure of their control in the firm. This implies that the manager of each firm j maximizes a weighted sum of the operating profits of (potentially) all the firms in the industry:

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \tilde{R}_k (\mathbf{x}) = \pi_j (\mathbf{x}) + \sum\nolimits_{g\in\Im, g\neq j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \frac{\phi_{kg}}{\phi_{kj}} \pi_g (\mathbf{x}), \tag{19}$$

where the weights assigned to a rival firm's profits $w_{jg} = \sum_{k\in\Theta_j} \gamma_{kj} (\phi_{kg}/\phi_{kj}) \geq 0$ for any $j, g \in \Im$ and $j \neq g$, are non-negative.[10] This constitutes our proposed new, alternative formulation of the objective function of the manager.

[10] In order to see why the weights w_{jg} are non-negative, note that $\gamma_{kj} \geq 0$, $\phi_{kj} > 0$ and $\phi_{kg} \geq 0$ for all $k \in \Theta_j$ and all $j, g \in \Im$.

3.4 Cross-Ownership

This proposed alternative formulation of the objective function of managers can also cope with cross-ownership among rival firms. In those cases, the weights $w_{jg} = \sum_{k\in\Theta} \gamma_{kj}^{u}(\phi_{kg}^{u}/\phi_{kj}^{u}) \geq 0$ for any $j, g \in \Im$ and $j \neq g$, where ϕ_{kj}^{u} and γ_{kj}^{u} denote the *ultimate* financial and control rights, respectively, of *external* shareholder k in firm j, which can be computed following the algorithm in Brito *et al.* (2018a).

3.5 Unilateral and Coordinated Effects

This proposed alternative formulation of the objective function of managers can also be straightforwardly incorporated into (a) the generalized HHI and GUPPI proposed in Brito *et al.* (2018a) to screen the unilateral effects of partial horizontal acquisitions; and (b) the structural empirical methodologies proposed in Brito *et al.* (2014, 2018b) to quantify the unilateral and coordinated effects of partial horizontal acquisitions.

4 Conclusions

We propose an alternative formulation to model the objective function of managers in the presence of horizontal shareholding. In this alternative formulation, managers would decide the strategy of the firm by maximizing a weighted sum of the firm's shareholders relative returns. We do not claim it to be preferred to O&S's formulation. We solely propose it as microfounded alternative which avoids an allegedly unattractive feature of the O&S's formulation: that if non-horizontal shareholders are highly dispersed and the interests of horizontal shareholders in the different firms coincide, managers would mimic the interests of horizontal shareholders even if they own a share of the firm that does not induce full control. Future empirical testing might help establish which formulation more accurately predicts firm behavior.

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Appendix A

In this appendix, we present the proofs of the mathematical results discussed in the main text. We do so in five parts.

In part (a), we examine the dominant formulation for the objective function of managers regarding properties (i) to (iv). First, absent horizontal shareholding, the manager of each firm j would maximize own-profit π_j, since $\phi_{kg} = 0$ for the subset of shareholders k who hold financial rights in firm j and all $j, g \neq j$. This implies $w_{jg} = 0$ for all j,$g \neq j$ and, thus, that property (i) holds. Second, with horizontal shareholding, the manager of each firm j would internalize the impact of her firm's strategy on rival firm g profits when her firm's controlling shareholders have financial rights in the rival firm, since if $\gamma_{kj} \neq 0$ and

$\phi_{kg} \neq 0$ for at least one shareholder k, we have that $w_{jg} \neq 0$ for all j,$g \neq j$. This implies that the manager of each firm j would maximize $\pi_j + w_{jg}\pi_g$ and, thus, that property (ii) holds. Third, the weight w_{jg} that the manager of each firm j assigns to rival firm g profits is continuous in ϕ_{kj}, γ_{kj} and ϕ_{kg} for the subset of shareholders k with control rights in firm j, since the product, sum and quotient, respectively, of continuous functions is continuous. This implies that property (iii) holds. Finally, the manager of each firm j would maximize industry profit when all controlling shareholders are fully diversified across rivals, since if those shareholders are fully diversified, for the subset of shareholders k who hold financial rights in firm j, we have $\phi_{kj} = \phi_{kg} = \phi_k$ and $\gamma_{kj} = \gamma_{kg} = \gamma_k$ for all j,$g \neq j$. This implies $w_{jg} = \sum_{k\in\Theta_j} \gamma_k\phi_k / \sum_{k\in\Theta_j} \gamma_k\phi_k = 1$ for all j,$g \neq j$ and that the manager of each firm j would maximize $\pi_j + \sum_{g\in\Im,g\neq j} \pi_g$. As such, property ($iv$) holds.

In part (b), we examine the proposed alternative formulation for the objective function of managers regarding properties (i) to (iv). First, absent horizontal shareholding, the manager of each firm j would maximize own-profit π_j, since $\phi_{kg} = 0$ for the subset of shareholders k who hold financial rights in firm j and all $j, g \neq j$. This implies $w_{jg} = 0$ for all j,$g \neq j$ and, thus, that property (i) holds. Second, with horizontal shareholding, the manager of each firm j would internalize the impact of her firm's strategy on rival firm g profits when her firm's controlling shareholders have financial rights in the rival firm, since if $\gamma_{kj} \neq 0$ and $\phi_{kg} \neq 0$ for at least one shareholder k, we have that $w_{jg} \neq 0$ for all j,$g \neq j$. This implies that the manager of each firm j would maximize $\pi_j + w_{jg}\pi_g$ and, thus, that property (ii) holds. Third, the weight w_{jg} that the manager of each firm j assigns to rival firm g profits is continuous in ϕ_{kj}, γ_{kj} and ϕ_{kg} for the subset of shareholders k with control rights in firm j, since the product, sum and quotient, respectively, of continuous functions is continuous. This implies that property (iii) holds. Finally, the manager of each firm j would maximize industry profit when all controlling shareholders are fully diversified across rivals, since if those shareholders are fully diversified, for the subset of shareholders k who hold financial rights in firm j, we have $\phi_{kj} = \phi_{kg} = \phi_k$ and $\gamma_{kj} = \gamma_{kg} = \gamma_k$ for all j,$g \neq j$. This implies $w_{jg} = \sum_{k\in\Theta_j} \gamma_k(\phi_k/\phi_k) = \sum_{k\in\Theta_j} \gamma_k = 1$ for all j,$g \neq j$ and that the manager of each firm j would maximize $\pi_j + \sum_{g\in\Im,g\neq j} \pi_g$. As such, property ($iv$) holds.

In part (c), we examine, for our illustrative example, the number of subsets of shareholders that can award victory to a candidate. Consider initially those subsets of shareholders that aggregate more than 50% of the voting rights and that do not include the horizontal shareholder. Each subset must include z smaller non-horizontal shareholders such that $z(1-x)/n > 0.5$ which is equivalent to $z \geq \lfloor n/2(1-x) \rfloor + 1$, where $\lfloor y \rfloor$ denotes the largest integer lower than y. Any single smaller non-horizontal shareholder is pivotal in one of these subsets if $(z-1)(1-x)/n < 0.5$ which is equivalent to $z \leq \lfloor n/2(1-x) + 1 \rfloor$.[11] Therefore, any small non-horizontal shareholder is pivotal in all subsets of $z = \lfloor n/2(1-x) + 1 \rfloor$ small shareholders in which she is present. There are $C^n_z = n!/(n-z)!z!$ different subsets with z smaller non-horizontal shareholders. Therefore, the number of subsets that do not include the horizontal shareholder in which any small non-horizontal shareholder is pivotal is $C^{n-1}_{\lfloor n/2(1-x) \rfloor + 1 - 1}$. Consider now those subsets of shareholders that aggregate more than 50% of the voting rights and that include the horizontal shareholder. Each subset must include z smaller non-horizontal shareholders such that $z(1-x)/n + x > 0.5$ which is equivalent to $z \geq \lfloor (0.5-x)n/(1-x) \rfloor + 1$. Any single smaller non-horizontal shareholder is pivotal in one of these subsets if $(z-1)\frac{1-x}{n} + x < 0.5$ which is equivalent to $z \leq \lfloor (0.5-x)\, n/(1-x) + 1 \rfloor$. Therefore, any small shareholder is pivotal in all subsets that include the horizontal shareholder and $z = \lfloor (0.5-x)n/(1-x) + 1 \rfloor$ small shareholders in which she is present. The number of subsets that include the horizontal shareholder in which any small non-horizontal shareholder is pivotal is $C^{n-1}_{\lfloor (0.5-x)n/(1-x) \rfloor + 1 - 1}$. In turn, the horizontal shareholder is pivotal in those subsets that include her and z small non-horizontal shareholders if $z(1-x)/n < 0.5$ which is equivalent to $z \leq \lfloor n/2(1-x) \rfloor$. Therefore, the horizontal shareholder is pivotal in all subsets that include her and z

[11] Please see Appendix B for the formal definition of pivotal.

small non-horizontal shareholders, with:

$$\left\lfloor (\frac{1}{2} - x)\frac{n}{1-x} \right\rfloor + 1 \leq z \leq \left\lfloor \frac{n}{2(1-x)} \right\rfloor, \tag{20}$$

which implies the number of sets in which the horizontal shareholder is pivotal is:

$$\sum_{y=\left\lfloor (\frac{1}{2}-x)\frac{n}{1-x} \right\rfloor + 1}^{\left\lfloor \frac{n}{2(1-x)} \right\rfloor} C_y^n. \tag{21}$$

Using the information above, by definition, the Banzhaf power index of the horizontal shareholder is given by:

$$\gamma_c^b = \frac{\sum_{y=\left\lfloor (\frac{1}{2}-x)\frac{n}{1-x} \right\rfloor + 1}^{\left\lfloor \frac{n}{2(1-x)} \right\rfloor} C_y^n}{\sum_{y=\left\lfloor (\frac{1}{2}-x)\frac{n}{1-x} \right\rfloor + 1}^{\left\lfloor \frac{n}{2(1-x)} \right\rfloor} C_y^n + n\left(C_{\left\lfloor \frac{n}{2(1-x)} \right\rfloor}^{n-1} + C_{\left\lfloor (\frac{1}{2}-x)\frac{n}{1-x} \right\rfloor}^{n-1} \right)}. \tag{22}$$

In part (d), we examine, for our illustrative example, how the weight that the manager of firm j assigns to the profit of rival firm g when we combine the dominant formulation with an assumption of proportional control, is impacted by x and n. Under this setting, this weight is given by $w_{jg} = x^2/(x^2 + (1-x)^2/n)$, which increases in x and n:

$$\begin{aligned} \frac{\partial w_{jg}}{\partial x} &= 2nx\frac{(1-x)}{(nx^2 - 2x + x^2 + 1)^2} > 0 \\ \frac{\partial w_{jg}}{\partial n} &= x^2\frac{(x-1)^2}{(nx^2 - 2x + x^2 + 1)^2} > 0. \end{aligned} \tag{23}$$

In part (e), we examine, for our illustrative example, how the weight that the manager of firm j assigns to the profit of rival firm g when we combine the dominant formulation with an assumption of Banzhaf control, is impacted by x and n. Under this setting, this weight is given by $w_{jg} = \gamma_c^b x/(\gamma_c^b x + (1-\gamma_c^b)(1-x)/n)$, which would increase in x and n if:

$$\begin{aligned} \frac{dw_{jg}}{dx} &= \frac{\partial w_{jg}}{\partial x} + \frac{\partial w_{jg}}{\partial \gamma_c^b}\frac{\partial \gamma_c^b}{\partial x} > 0 \\ \frac{dw_{jg}}{dn} &= \frac{\partial w_{jg}}{\partial n} + \frac{\partial w_{jg}}{\partial \gamma_c^b}\frac{\partial \gamma_c^b}{\partial n} > 0. \end{aligned}$$

We have that $\partial w_{jg}/\partial n > 0$, $\partial w_{jg}/\partial x > 0$ and $\partial w_{jg}/\partial \gamma_c^b > 0$, as follows:

$$\begin{aligned} \frac{\partial w_{jg}}{\partial n} &= \frac{x\gamma_c^b(1-\gamma_c^b)(1-x)}{(-x - \gamma_c^b + x\gamma_c^b + nx\gamma_c^b + 1)^2} > 0 \\ \frac{\partial w_{jg}}{\partial x} &= \frac{n\gamma_c^b(1-\gamma_c^b)}{(-x - \gamma_c^b + x\gamma_c^b + nx\gamma_c^b + 1)^2} > 0 \\ \frac{\partial w_{jg}}{\partial \gamma_c^b} &= \frac{nx(1-x)}{(-x - \gamma_c^b + x\gamma_c^b + nx\gamma_c^b + 1)^2} > 0, \end{aligned} \tag{24}$$

which implies that in order to evaluate the sign of dw_{jg}/dx and dw_{jg}/dn, we just have to evaluate the sign of $\partial\gamma_c^b/\partial x$ and $\partial\gamma_c^b/\partial n$. We begin by the latter. It is possible to show with examples that $\partial\gamma_c^b/\partial n$ may take any sign. Consider, for example, the case with $x = 0.05$. When n increases from 499 to 500, γ_c^b increases by 0.016 whereas when n increases from 500 to 501, γ_c^b

decreases by 0.016. The reason being that the number of subsets in which the horizontal shareholder is pivotal can decrease as the number of non-horizontal shareholders increase. This implies that $\partial\gamma_c^b/\partial n$ may take any sign and, thus, can decrease with n. We now address the former. As $\partial\left((0.5-x)n/(1-x)\right)/\partial x = -n/2(1-x)^2 < 0$ and $\partial\left(n/2(1-x)\right)/\partial x = n/2\left(1-x\right)^2 > 0$, the number of terms in the summation term that determines the number of subsets in which the horizontal shareholder is pivotal cannot decrease with x. We now consider the effect of x in the number of subsets in which any small shareholder is pivotal, starting with $C^{n-1}_{\lfloor(0.5-x)n/(1-x)\rfloor}$. By construction, C^{n-1}_p increases in p until $p=(n-1)/2$ and then decreases. As $((n-1)/2-(0.5-x)n)/(1-x) = (x\,(n+1)-1)/2(1-x) > 0$ for all x (otherwise, the horizontal shareholder is smaller than the others), we can conclude that increasing x may maintain or decrease $\lfloor(0.5-x)n/(1-x)\rfloor$, which never increases $C^{n-1}_{(0.5-x)n/(1-x)}$. We now turn to $C^{n-1}_{n/2(1-x)}$. As $(n-1)/2-n/2(1-x) = -(nx+1-x)/2(1-x) < 0$ for all x, increasing x may maintain or increase $\lfloor n/2(1-x)\rfloor$ which never increases $C^{n-1}_{\lfloor n/2(1-x)\rfloor}$. This implies that increases in x cannot lead to a decrease in γ_c^b and, hence, $dw_{jg}/dx > 0$.

In part (e), we examine, for our illustrative example, how the weight that the manager of firm j assigns to the profit of rival firm g when we combine our proposed alternative formulation with an assumption of Banzhaf control, is impacted by x and n. Under this setting, this weight is given by $w_{jg}=\gamma_c^b$. As discussed in part (d), $\partial\gamma_c^b/\partial x \geq 0$, which implies it never decreases with x, but $\partial\gamma_c^b/\partial n \lesseqgtr 0$, which implies it may decrease with n.

Appendix B

In this appendix, we present the proof of Proposition 1. We divide it in two proofs. We first present the proof under Assumptions 1, 2a, 3, 4 and 5. We then present the proof under Assumptions 1, 2b, 3, 4 and 5.

Proof of Proposition 1 under Assumptions 1, 2a, 3, 4 and 5.

Let $\varpi_{aj} = \Pr\left(m_j = a_j|\mathbf{x}_a,\mathbf{x}_b\right)\Xi$ denote the objective function of the incumbent under Assumption 2a. Assumptions 1, 3, 4 and 5 imply that this objective function is strictly concave conditional on rival firm strategies and, therefore, has a unique maximum. In order to see why, note that, under Assumption 1, shareholders are conditionally sincere, which implies that the incumbent candidate to firm j can choose her strategy proposal taking the strategies of the candidates to the remaining firms as given. The first order condition of this problem is, thus, given by:

$$\frac{\partial\varpi_{aj}}{\partial x_{aj}} = \sum\nolimits_{k\in\Theta_j}\frac{\partial\Pr\left(m_j=a_j|\mathbf{x}_a,\mathbf{x}_b\right)}{\partial\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)}\frac{\partial\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)}{\partial x_{aj}}, \tag{25}$$

where, using probability (15), we have that:

$$\begin{aligned}\frac{\partial\Pr\left(m_j=a_j|\mathbf{x}_a,\mathbf{x}_b\right)}{\partial\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)} = {} & \sum\nolimits_{\Theta_j^\imath\in\wp_j,k\in\Theta_j^\imath}\prod\nolimits_{h\in\Theta_j^\imath,h\neq k}\Pr_{ha_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)\prod\nolimits_{h\notin\Theta_j^\imath}\Big(1-\Pr_{ha_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)\Big) \\ & -\sum\nolimits_{\Theta_j^\imath\in\wp_j,k\notin\Theta_j^\imath}\prod\nolimits_{h\in\Theta_j^\imath}\Pr_{ha_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)\prod\nolimits_{h\notin\Theta_j^\imath,h\neq k}\Big(1-\Pr_{ha_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)\Big),\end{aligned} \tag{26}$$

which (a) is, by definition, non-negative since increasing $\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)$ for any k can not have a negative impact on $\Pr\left(m_j=a_j|\mathbf{x}_a,\mathbf{x}_b\right)$; and ($b$) does not depend on $\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)$ since $\Pr\left(m_j=a_j|\mathbf{x}_a,\mathbf{x}_b\right)$ is linear in $\Pr_{ka_j}\left(\mathbf{x}_a,\mathbf{m}_a,\mathbf{x}_b,\mathbf{m}_b\right)$ for any k taking the corresponding probabilities of the remaining shareholders as given. The second order condition of the incumbent's problem, in turn, is given by:

$$\begin{aligned}
\frac{\partial^2 \varpi_{aj}}{\partial x_{aj}^2} &= \sum\nolimits_{k\in\Theta_j} \frac{\partial^2 \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)}{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)\partial x_{aj}} \frac{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)}{\partial x_{aj}} \\
&\quad + \sum\nolimits_{k\in\Theta_j} \frac{\partial \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)}{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)} \frac{\partial^2 \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)}{\partial x_{aj}^2} \\
&= \sum\nolimits_{k\in\Theta_j} \frac{\partial^2 \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)}{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)^2} \left(\frac{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)}{\partial x_{aj}}\right)^2 \\
&\quad + \sum\nolimits_{k\in\Theta_j} \frac{\partial \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)}{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)} \frac{\partial^2 \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)}{\partial x_{aj}^2} \\
&= \sum\nolimits_{k\in\Theta_j} \frac{\partial \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)}{\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)} \frac{\partial^2 \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)}{\partial x_{aj}^2},
\end{aligned} \tag{27}$$

where the last equality makes use of the fact that $\partial^2 \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)/\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)^2 = 0$ for all k, since it does not depend on $\Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)$. Using probability (15), we have that the objective function of the manager is strictly concave in x_{aj}, conditional on the strategies of the remaining firms, since (a) under Assumption 5 we have that $G_{kj}(R_k(\mathbf{x}_a) - R_k(\mathbf{x}_b)) = 0.5 + \left(1/\tau_j\phi_{kj}\right)\left(R_k(\mathbf{x}_a) - R_k(\mathbf{x}_b)\right)$ and, thus, $\partial^2 \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)/\partial x_{aj}^2 = \left(1/\tau_j\phi_{kj}\right)\partial^2 R_k\left(\mathbf{x}_a\right)/\partial x_{aj}^2$, which is negative under Assumption 4; and (b) $\partial \Pr\left(m_j = a_j|\mathbf{x}_a, \mathbf{x}_b\right)/\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) > 0$ for at least a shareholder k. Finally, given that the strategy proposal x_{aj} is, under Assumption 3, defined in a convex set, we have that the incumbent's maximization problem has an unique maximum.

Given the symmetry of the maximization problem of the challenger candidate to firm j, we have that the two candidates will choose the same best-response function, i.e., the same strategy proposal for the firm, conditional on the strategies of the candidates to the remaining firms. We now show that this best-response function is the same as the best-response function that arises while maximizing a weighted average of the relative returns of the firm's shareholders conditional on the strategies of the candidates to the remaining firms, with normalized Banzhaf power indices as weights. To do so, note that since the two candidates will choose the same best-response function, in equilibrium, we have $R_k(\mathbf{x}_a) = R_k(\mathbf{x}_b) = R_k(\mathbf{x})$ for all k. This implies that the first-order condition reduces to:

$$\frac{1}{2^{\ell_j - 1}} \sum\nolimits_{\Theta_j^\imath \in \wp_j} \sum\nolimits_{k\in\Theta_j^\imath} \frac{1}{\tau_j\phi_{kj}} \frac{\partial R_k\left(\mathbf{x}\right)}{\partial x_j} - \frac{1}{2^{\ell_j - 1}} \sum\nolimits_{\Theta_j^\imath \in \wp_j} \sum\nolimits_{k\notin\Theta_j^\imath} \frac{1}{\tau_j\phi_{kj}} \frac{\partial R_k\left(\mathbf{x}\right)}{\partial x_j} \leqslant 0, \tag{28}$$

which makes use of the fact that $\partial \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right)/\partial x_{aj} = \left(1/\tau_j\phi_{kj}\right)\partial R_k\left(\mathbf{x}_a\right)/\partial x_{aj}$ and $\Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) = \Pr_{kb_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) = 1/2$ when $\mathbf{x}_a = \mathbf{x}_b$, both for all k. This first-order condition can, in turn, be rewritten as:

$$\frac{1}{2^{\ell_j - 1}} \sum\nolimits_{k\in\Theta_j} \left(\lambda_{jk} \frac{1}{\tau_j\phi_{kj}} \frac{\partial R_k\left(\mathbf{x}\right)}{\partial x_j} - \left(2^{\ell_j - 1} - \lambda_{jk}\right) \frac{1}{\tau_j\phi_{kj}} \frac{\partial R_k\left(\mathbf{x}\right)}{\partial x_j}\right) \leqslant 0, \tag{29}$$

where λ_{jk} denotes the number of subsets in $\wp_j$ in which shareholder k enters and $\left(2^{\ell_j - 1} - \lambda_{jk}\right)$ denotes the number of subsets in $\wp_j$ in which shareholder k does not enter. Finally, consider that λ_{jk} can be divided in two terms: the number of subsets in $\wp_j$ in which shareholder k enters and is pivotal, λ_{jk}^{p}, and the number of subsets in $\wp_j$ in which shareholder k enters and is not pivotal, $\lambda_{jk}^{\tilde{p}}$.[12] The latter is, by construction, equal to the number of subsets in $\wp_j$ in which shareholder k does not enter. This

[12] Shareholder k is pivotal if for some subset $\Theta_j^\imath$ which does not include shareholder k, we have $\sum_{h\in\Theta_j^\imath} \upsilon_{hj} \leq 0.5$, but if we include shareholder k, $\sum_{h\in\Theta_j^\imath} \upsilon_{hj} > 0.5$.

implies that $\lambda_{jk}^{\tilde{p}} = \left(2^{\ell_j - 1} - \lambda_{jk}\right)$ and that the first-order condition can be rewritten as:

$$\sum\nolimits_{k\in\Theta_j} \left(\frac{\lambda_{jk}^{p}}{2^{\ell_j-1}}\right) \frac{1}{\tau_j \phi_{kj}} \frac{\partial R_k\left(\mathbf{x}\right)}{\partial x_j} \leqslant 0, \tag{30}$$

where $\lambda_{jk}^{p}/2^{\ell_j-1}$ denotes the Banzhaf power index associated to shareholder k in firm j. This establishes that, in equilibrium, the candidates to each firm converge to the same strategy, which also maximizes the following weighted average of the relative returns of the shareholders with voting rights in the firm, conditional on the strategies of the candidates to the remaining firms:

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \tilde{R}_k\left(\mathbf{x}\right), \tag{31}$$

where $\gamma_{kj} = \left(\lambda_{jk}^{p}/2^{\ell_j-1}\right) / \sum_{k\in\Theta}\left(\lambda_{jk}^{p}/2^{\ell_j-1}\right) = \lambda_{jk}^{p} / \sum_{k\in\Theta_j} \lambda_{jk}^{p}$ denotes the weight assigned by firm j's manager to the return of shareholder k, measured by the normalized Banzhaf power index of shareholder k in firm j.

Finally, given that the strategy proposal of each candidate to the different firms is, under Assumption 3, defined in a convex set and $R_k(\mathbf{x})$ is, under Assumption 4, continuous, the best-response functions of the candidates to the different firms are guaranteed to be upper-hemicontinuous, which implies that we can apply Kakutani's fixed point theorem to ensure that the Nash equilibrium exists.

Proof of Proposition 1 under Assumptions 1, 2b, 3, 4 and 5.

Let $\varpi_{aj} = \sum_{k\in\Theta_j} \Pr_{ka_j}\left(\mathbf{x}_a, \mathbf{m}_a, \mathbf{x}_b, \mathbf{m}_b\right) \upsilon_{kj}$ denote the objective function of the incumbent under Assumption 2*b*. Assumptions 1, 3, 4 and 5 imply that this objective function is strictly concave conditional on rival firm strategies and, therefore, has a unique maximum. In order to see why, note that, under Assumption 1, shareholders are conditionally sincere, which implies that the incumbent candidate to firm j can choose her strategy proposal taking the strategies of the candidates to the remaining firms as given. The first order condition of this problem is, thus, given by:

$$\frac{\partial \varpi_{aj}}{\partial x_{aj}} = \sum\nolimits_{k\in\Theta_j} \frac{1}{\tau_j \phi_{kj}} \frac{\partial R_k\left(\mathbf{x}_a\right)}{\partial x_{aj}} \upsilon_{kj}, \tag{32}$$

which makes use of the fact that, under Assumption 5, $G_{kj}((R_k\left(\mathbf{x}_a\right) - R_k\left(\mathbf{x}_b\right))) = (1/2) + \left(1/\tau_j\phi_{kj}\right)\left(R_k\left(\mathbf{x}_a\right) - R_k\left(\mathbf{x}_b\right)\right)$. In turn, the second order condition is given by:

$$\frac{\partial^2 \varpi_{aj}}{\partial x_{aj}^2} = \sum\nolimits_{k\in\Theta_j} \frac{1}{\tau_j \phi_{kj}} \frac{\partial^2 R_k\left(\mathbf{x}_a\right)}{\partial x_{aj}^2} \upsilon_{kj}, \tag{33}$$

which implies, given Assumption 4, that the objective function of the manager is strictly concave in x_{aj}, conditional on the strategies of the remaining firms. Finally, given that the strategy proposal x_{aj} is, under Assumption 3, defined in a convex set, we have that the incumbent's maximization problem has an unique maximum.

Given the symmetry of the maximization problem of the challenger candidate to firm j, we have that the two candidates will choose the same best-response function, i.e., the same strategy proposal for the firm, conditional on the strategies of the candidates to the remaining firms. This establishes that, in equilibrium, the candidates to each firm converge to the same strategy, which also maximizes the following weighted average of the relative returns of the shareholders with voting rights in the firm, conditional on the strategies of the candidates to the remaining firms:

$$\max_{x_j} \sum\nolimits_{k\in\Theta_j} \gamma_{kj} \tilde{R}_k\left(\mathbf{x}\right), \tag{34}$$

where $\gamma_{kj} = \upsilon_{kj}$ denotes the weight assigned by firm j's manager to the return of shareholder k, measured by the voting rights of shareholder k in firm j.

Finally, given that the strategy proposal of each candidate to the different firms is, under Assumption 3, defined in a convex set and $R_k(\mathbf{x})$ is, under Assumption 4, continuous, the best-response functions of the candidates to the different firms are guaranteed to be upper-hemicontinuous, which implies that we can apply Kakutani's fixed point theorem to ensure that the Nash equilibrium exists.