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Optimal debt adjustment in a nonlinear endogenous growth model

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About the author

Oscar Wilde once said that *"Man is least himself when he talks in his own person. Give him a mask, and he will tell you the truth"*. Since no mask can be put on and the author stands in line with his work, nothing besides his work can serve as a fair judgment.

To help in pursuing his work, the author obtained a MSc in Software Engineering, is currently submitting a Master Thesis to earn an MSc in Economics and is now a PhD student. The author has worked abroad for a private multinational company, did research in computer science and economics, co-founded a software company and travelled the world for a year carrying nothing but a backpack and a bunch of books on his back.

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But if these people made it possible and motivated me to write these lines, due appreciation must also be forever paid to the ones that instilled in me a passion for Economics without which the pages would stay blank. If it weren't for them, I would be following a different path, perhaps to a different equilibrium. Destiny, or chance, decided that I should stumble upon the works of the Austrian economists Friedrich von Hayek, Ludwig von Mises, Henry Hazlitt, along with the insightful work of Adam Smith and, preceding him, David Hume. It was a life-changing event. It alluded to me the fundamental role that Economics and the due understanding of simple economic concepts play in society. It then led me to read Karl Popper and refine my understanding of how philosophy and economics were so deeply ingrained. They were the foundation blocks that awakened my passion, while contemporary economists like Milton Friedman, Thomas Sowell, Robert Lucas, Robert Barro and Xavier Sala-i-Martin sedimented it.

My intellectual growth was a positive linear function of the work of these men, and for that I, but mostly we, will be forever in debt.

Abstract

This work is a twofold contribution to the economic literature. First, we provide an Optimal Control theorem to transform a nonlinear infinite-horizon economic growth model into an equivalent finite-horizon representation that can then be solved numerically. This approach has several benefits compared to the methods currently in practice: (a) the nonlinear form of the model can be solved directly, avoiding log-linearization qualitative and quantitative errors; (b) the dynamic system is solved at each point in time so studying the transitional dynamics is now considerably easier; (c) shocks can hit the system at any point in time and do not require the economy to be in a steady-state; (d) more complex models, once deemed analytically intractable, can now be solved numerically. On top of this, we developed a framework to exemplify the application of the theorem and demonstrate its potential use in the exogenous and endogenous economic growth literature by solving the original, nonlinear description of two standard models: the utility-maximizing Ramsey-Cass-Koopmans neoclassical growth model and the Lucas endogenous growth model with human capital. We exemplified how to simulate shocks to a system that might not be at its steady-state, a strong requirement still in use in the literature, as well as multiple, sequential shocks.

In the second part, we studied optimal debt adjustment by applying the numerical framework and developing a nonlinear endogenous growth model with public capital and public debt in order to study the optimal fiscal policy to curb large levels of public debt. Given that the model was solved in its nonlinear form, we were able to consider time-variant tax rates and a quantitative budget rule. It is shown that the optimal fiscal adjustment policy depends on the initial level of debt and the results suggest there is an inverted U-shape negative relationship between debt and growth. In the short-tun, the strategy consists in immediately cutting public expenditure, followed by a tax raise in the capital income tax. In the long-run, public expenditure resumes, capital income tax converges to zero and productive public spending is then fully supported by a tax on consumption.

Resumo

Este trabalho apresenta duas contribuções para a literatura económica. Em primeiro lugar, disponibilizamos um teorema de Controlo Óptimo que transforma um modelo de crescimento económico não-linear e de horizonte-infinito num problema de horizonte-finito equivalente que pode então ser resolvido numericamente. Esta abordagem tem vários benefícios comparativamente com os métodos atualmente usados: (a) a forma não-linear do modelo pode ser resolvida de forma numérica, evitando os erros qualitativos e quantitativos que podem emergir da linearização; (b) o sistema dinâmico é resolvido em cada instante pelo que analisar a dinâmica de transição torna-se consideravelmente mais fácil; (c) o sistema pode ser sujeito a choques em cada instante e não é necessário que a economia esteja num ponto de equilíbrio; (d) modelos mais complexos, analiticamente intratáveis, podem agora ser resolvidos numericamente. Adicionalmente, desenvolvemos uma *framework* para exemplificar a aplicação do teorema e demonstrar o seu uso potencial na literatura de crescimento económico exógeno e endógeno, para tal resolvendo a versão original, não-linear, de dois modelos *standard*: o modelo neoclássico de crescimento económico, Ramsey-Cass-Koopmans, e o modelo de crescimento endógeno e com capital humano de Lucas. Exemplificamos também como simular choques numa economia que poderá não estar no seu equilíbrio, uma consideração geralmente assumida na literatura, assim como choques múltiplos e sequenciais.

Na segunda parte, estudamos o ajustamento óptimo da dívida pública aplicando a framework numérica e construindo um modelo não-linear de crescimento endógeno com capital público e dívida pública, que permitirá estudar a política fiscal óptima para reduzir grandes níveis de dívida. Considerando que o modelo é resolvido na sua forma não linear, foi possível considerar taxas de imposto que podem variar com o tempo e ainda uma regra orçamental quantitativa. Mostramos que a política óptima de ajustamento fiscal depende no nível inicial de dívida e os resultados sugerem que existe uma relação em U invertido entre dívida e a taxa de crescimento. No curto-prazo, a estratégia consiste em reduzir imediatamente o investimento público, seguindo-se um aumento dos impostos sobre o capital e o trabalho. No longo prazo, ocorre uma retoma da despesa pública, os impostos sobre o capital e o trabalho converge para zero e o investimento público produtivo é então financiado recorrendo unicamente a imposto sobre o consumo.

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Nomenclature

BGP Balanced Growth Path

CPI Consumer Price Index

GDP Gross Domestic Product

MPK Marginal Productivity of Capital

MPL Marginal Productivity of Labour

NPV Net Present Value

PB Primary Balance

SFA Stock-Flow Adjustment

ZLB Zero Lower Bound

"The curious task of economics is to demonstrate to men

how little they really know about what they imagine they can design."

— Friedrich von Hayek

Chapter 1

Introduction

"There's no such thing as a free lunch"

– Milton Friedman

Unraveling the engine of economic growth is the key to prosperity. Homage is generally paid to the work of Adam Smith, heavily inspired by his scotisch compatriot David Hume, one of the first intellectuals to refute mercantilism and to show how prices in a country are strongly related to changes in the money supply — a first sketch of what we now call the monetarist theory quantitity of money. It goes without saying that the work of Hume and Smith set an important cornerstone in establishing contemporary economic thought, but the study of the causes of the wealth of nations did not commence nor end with Smith's seminal book. We have to go back a couple of centuries and recover the work of one of the greatest philosophers ever to be born, Aristotles. Not only did he provide a concise analysis of human motivation, Aristotles constructed the logical-deductive framework that permitted concise and valid scientific thought to be derived. In *Politics*, Aristotles explained the concept of money and how society evolved from barter to a monetary economy. The work of the greek philosopher would last for all the medieval period. Centuries later, an important school of thought emerged during Renaissance that would not only later inspire Hume to further develop the concept of natural law, but also to leave an important mark in the history of economic thought. It was the work of the Spanish scholastics, most notoriously the School of Salamanca, the first to systematize the origin of price as determined by its supply and demand. Their ideas on the *ex ante* mutual benefit of individuals to freely exchange goods and services and the importance of private property preceded Smith by almost two centuries. The work of these scholastics was so important that it left a strong influence on Carl Menger, one of the *marginalists* and the

founder of the Austrian school. In fact, well before the marginal revolution initiated by Menger, Jevons and Walras, the scholastic Francisco Garcia was almost brought to the brink of a marginal utility analysis of valuation by realizing that if bread were to outpace the quantity of meat and become abundant in comparison to meat, its price would drop in comparison to meat, which actually happened a few decades later. Despite the accurate and rigorous economic analysis of such prominent scholars, the economic principles of mercantilism would reign between the sixteenth and eighteenth centuries. Smith's contribution to the fading of mercantilism was quintessential, but not without the influence of one of the founders of modern economic thought, the mostly unknown Richard Cantillon.

None of these authors were able to decipher the fundamentals of economic growth. Hume thought that inflation could generate economic growth. Unless we consider the increase in the nominal accounting of prices to count as economic growth, he missed it. The economists that later followed, like Thomas Malthus, ignored the impact that advancements in technology can have in improving the general material welfare of the population, i.e., real economic growth, and drew a pessimist account of the future. Austrians went much further, authoring the "stages of production" approach and a comprehensive analysis of the creation of wealth. No price can be tagged to the work of Ludwig von Mises in deciphering the basis of human action and incorporating micro-foundations that, although conflicting with classical economics, were fundamental to understand the organic and distributed essence of the economy. His view on the interest rate as a time-preference discount rate — Mises witfully observed that we prefer present to future consumption — along with Hayek's explanation on the role of prices based on sound money to coordinate actions between producers and consumers and efficient allocation of resources was a breakthrough in the economics profession. At the same time, Keynes was performing a facelift in the economics profession and introducing the definition of aggregates based on statistical macroeconomic data. These were times of revolution in economics. Hayek and Keynes were at the forefront in one of the greatest intellectual battles ever. But perhaps because the Austrian school texts were mostly unavailable in English, the economic theory of the anglo-saxon world prevailed up to this day.

It was not until the twentieth century that economic growth theory witnessed a major contribution from an American economist, Robert Solow, that finally unveiled part of the origin of economic growth and prosperity. He realized the role played by capital, how it turns production into a more efficient process and how its accumulation benefits society in general, regardless of who owns it. Moreover, the importance of technological evolution and productivity enhancements was now systematized into a clear framework capable of providing insightful clues and suggestions on how to enact growth. The seed was planted. What emerged subsequently was a serious of seminal contributions to this

theory. Ramsey, Cass and Koopmans added more microeconomic foundations assumed to hold by the classical economists, that is, the maximization of the utility of consumption. Romer, Lucas and Barro later succeeded with incorporating some of these exogenous factors into the model, turning exogenous facets of the economy into endogenous factors that lead to growth. They showed the importance of public and human capital as well as R&D spillovers.

Nowadays, economic growth theory lost part of its appeal in opposition to neo-Keynesian models that, despite having more consistent micro-foundations than traditional Keynesian aggregate models, still fall short on incorporating a production function. The explanatory power of these models was derogated in favor of its potentially predictive capabilities. It might be conjectured that one of the reasons to be so was that the construction of these models was severely limited by the scope of the analytical tools. A simple endogenous growth model incorporating time-variant tax rates becomes analytically intractable due to the high degree of nonlinearity that is introduced. Adding to this, any modification to the models involves complex and time-consuming calculus that may demotivate the most perseverant of the researchers.

This brings us to the first contribution of this dissertation. With the natural tendency to conceive models ever more complex comes the necessity of being able to solve them. Analytical methods may no longer serve that purpose, despite their formal elegance and rigorousness. Therefore, we resort to powerful numerical tools capable of solving complex models. To meet this end, we have proved a theorem that explains how to transform an infinite-horizon economic growth model, exogenous or endogenous, into an equivalent finite-horizon form. This way, it is possible to apply state-of-the-art numerical tools to solve it. More specifically, we are able to solve an optimal control problem, the mathematical underpinning of most economic growth models, using a direct method that first discretizes the model and then transforms it into a Nonlinear Programming Problem (NLP) which can be solved using advanced solvers. This contrasts with the traditional approach of first applying Pontyagrin's Maximum Principle in order to obtain the Necessary Optimality Conditions (NOCs) and then, due to the impossibility of solving its nonlinear form, loglinearizing it around the steady state. Besides the problems posed by the loglinearization, which we will cover in the next sections, this approach also requires the system to part from equilibrium, which may be an inaccurate assumption. Moreover, the novel approach we propose opens the way to more complex models like those that, as previously mentioned, contemplate time-variant tax rates. We use the framework we propose to solve the Ramsey-Cass-Koopmans exogenous growth model and the Lucas endogenous growth model with public capital and we show how it can be used to simulate non steady-state single shocks, along with multiple sequential shocks.

In the second part of this work, we delve into one of the subjects most perplexing to contemporary economists and we use the framework previously developed to shed a light on the subject. The role of public debt or, more specifically, what is the impact of debt on growth. Do they go hand-in-hand? Or perhaps one may put the other in jeopardy? Or a mix of both, at different times? Starting with Barro and the introduction of public capital and followed by Futagami with the introduction of public debt, the economic growth literature has been prolific in trying to understand the role of public investment and how to finance it, which boils down to a financial decision on optimal capital structure. According to David Ricardo, it is irrelevant since households will compensate present borrowed consumption and investment with savings to accommodate the future burden arising from taxation. But that suggests that consumers will smooth their consumption whenever possible. It does not explain how debt can promote or affect growth in the long-run. The contributions of Futagami, Maebayashi and Greiner help in showing how to adjust the levels of debt, but more remains to be done.

We pick on these contributions and build upon them so as to concoct an endogenous growth model with public capital and public debt in order to study the optimal adjustment of debt. Thanks to our numerical framework, we are able to go one step further and consider time-variant tax rates and a quantitative budget rule that does not enforce an exogenous functional form between debt and growth. Also, we avoid log-linearization potential qualitative and quantitative errors by solving its nonlinear form. This provides with a consistent model to help studying the fiscal policy better suited to curb the levels of public debt, at the same time minimizing the impact in terms of welfare cost, i.e., that is optimal. It also allows us to derivate some interesting results that point to an inverted U-shape relationship between debt and growth.

We are confident that these contributions may provide a considerable insight into one of this age's most challenging economic puzzles.

Chapter 2

Optimal control of infinite-horizon growth models: a numerical framework

"It is a myth that poverty in Africa is falling because of aid or redistribution.
It is falling because of economic growth"

– Xavier Sala-i-Martin

2.1 Introduction

Optimal control theory has been extensively applied to the solution of economic problems since the pioneering article of Arrow (1968). Its main application in economics is usually found within dynamic macroeconomic theory, more specifically in the repertoire of both exogenous and endogenous economic growth models, like in Ramsey (1928); Uzawa (1965); Lucas (1988); Romer (1994), to name but just a few. At their core, such models typically incorporate microeconomic foundations and involve multiple distinct and decentralized optimization problems over an infinite-horizon. This formulation gives rise to a system of nonlinear differential equations describing the economy. Nonlinearities usually arise from the diminishing marginal utility of consumption and from the diminishing marginal productivity of the production factors, but can also be the result of incorporating R&D (Williams and Jones, 1995) or government spending (Barro, 1990).

The study of these dynamic growth models follows a standard procedure, which consists in applying Pontryagin's Maximum Principle and obtaining the necessary optimality conditions (NOC), along with the transversality condition. If we define initial values for the state variables we have a complete description of the system, enough to expound the economy around its steady state equilibrium. But the nonlinearity in the production and utility functions and the saddle-path stability introduced by the forward-looking assumption of agents, while not posing much of a problem in the characterization of the static short-run, can indeed affect the analysis of the transitional dynamics following a structural change or a policy shock, as Atolia et al. (2010) points out. According to the author, one way of overcoming this issue is to linearize the dynamic system around its (post-shock) steady state and then to study the properties of this linearized, thus simplified, version of the dynamic system as a proxy to the original nonlinear system.

Linearization can be extremely misleading, though. In fact, this mathematical stunt may yield specious predictions and lead to erroneous qualitative assessments, even when probed near the steady state. Wolman and Couper (2003) had already identified the problem. They point out three main pitfalls of excess of reliance in the linearization process: (i) *spurious nonexistence*, that is, results suggesting that there is no nonexplosive solution when global analysis shows that one in fact exists; (ii) *spurious existence*, i.e., a unique equilibrium found in the linearized version of the model while no equilibria indeed exists for a wide range of initial conditions; and (iii) *spurious uniqueness*, when linearization gives origin to a unique nonexplosive equilibrium when in fact there are multiple nonexplosive equilibria. Several studies have already shown how misleading the conclusions may be when linearization is used. For example, Futagami et al. (2008) devised a model to study the long-run growth effect of borrowing for public investment. If the public debt target is defined as a ratio to private capital ($\bar{b} \equiv B/K$), as they originally do, then the

model exhibits a multiplicity of balanced growth paths (BGP) in the long run and a possible indeterminacy of the transition path to the high-growth BGP. If, on the other hand, one follows Minea and Villieu (2012) and opts to define the public debt target as a ratio to output ($\bar{b} \equiv B/Y$) then the model exhibits a unique BGP and a unique adjustment path to equilibrium. This contrast in results highlights the fallacious influence linearization can have on the interpretation of a model.

Atolia et al. (2010) conducted an extensive analysis on linearization pitfalls. Expectedly, they show that the further away the economy is from a steady-state equilibrium, the larger are the errors generated by linearization, both qualitatively and quantitatively. Furthermore, in models where government expenditure is introduced in the form of stock of public capital, the linearized model over-predicts consumption and welfare gains from an increase in public investment, providing an extremely erroneous signal to the policy maker. More worryingly, the errors can be of a qualitative nature. The authors give an example, where the linear proxy predicts a short-run increase in consumption, while the original nonlinear model solution shows a decline. Their conclusion is remarkably important — linearization is potentially quite misleading. Notwithstanding, linearization is still the dominant practice adopted in the macro-growth literature.

In fact, up until recently, and even with linearization at our disposal, the transitional dynamics of models with stiff differential equations or giving origin to a center manifold had hardly been investigated, as Trimborn et al. (2004) duly noted. Trimborn et al. (2004) confer added importance to the transition process in growth models, insofar as the positive and normative implications might differ dramatically depending on whether an economy converges towards a BGP or grows along it, as we have already noted. Moreover, conducting welfare comparisons between different policy regimes or instruments depends on studying this process. Consequently, some numerical procedures started emerging to overcome this problem, namely: "projection method" (Judd, 1992), the "discretization method" (Mercenier and Michel, 1994), the "shooting method" (Judd, 1998), the "time elimination method" (Mulligan and Sala-i Martin, 1991), the "backward integration" procedure (Brunner and Strulik, 2002) and the "relaxation procedure" (Trimborn et al., 2004). The latter on is now widely used in the economic growth literature.

Although more reliable than linearization, the aforementioned procedures rely on *indirect* methods to solve the intrinsic optimal control problem of the growth model. Indirect methods, which are based on the calculus of variations, start by first applying Pontryagin's Maximum Principle and introducing an adjoint variable for each state variable in order to obtain the NOCs (from the first order conditions of the Hamiltonian), along with the transversality condition. According to Betts (2010, Chapter 4.3), indirect methods might suffer from the following issues: a) the NOCs have to be computed explicitly, and

an explicit expression for the Euler-Lagrange equation might be hard to determine, especially in systems with singular arcs; this is also far from flexible, since a new derivation is required each time the problem is changed; b) problems with path inequalities require an estimation of the constrained-arc sequence, which can be a considerably cumbersome task; c) the basic method is not robust, requiring an initial guess for the adjoint variables, which even if done properly can lead to ill-conditioned adjoint equations. Economic models are usually conceived with this in mind and sometimes simplifications are forced on to them so that an analytical solution can be determined. Moreover, the transitional dynamics are always studied assuming that the economy departs from a steady state. Real-world economies have yet to reach the theoretical boundary set by a steady state[1], and the adjustment trajectories differ sharply for a shock that hits an economy that is at a non-steady state, as we will see in Section 2.4.

Given this, we propose a framework that transforms the infinite-horizon problem into an equivalent finite-horizon representation so that a *direct* method (control discretization) can be employed to solve the underlying infinite-horizon optimal control problem. The procedure incorporates the work first proposed by Fontes (2001). It is worth stressing that this approach has several advantages: (i) it solves a nonlinear programming problem (NLP) and not its linear approximation; (ii) it is capable of solving complex problems where the NOCs are hard to determine; (iii) allows for a non-steady state analysis of the transitional dynamics; (iv) has the capability of studying anticipated shocks without introducing discontinuities or reformulating the original problem; (v) allows for the study of multiple sequential shocks (either expected or not); (vi) is easy to use, as no *a priori* knowledge of the analytical trajectories is required and everything is handled numerically; (vii) it is extremely robust, as the optimal control problem is solved using well-established and tested NLP solvers.

The proposed method provides a significant contribution to the literature since it is an alternative to either the linearization approach and to the above mentioned numerical procedures. In addition to being more accurate, it allows for the study of the transitional dynamics for models in a non-steady state, while keeping intact all the properties inherent to a nonlinear model. Also, it permits the analysis of anticipated shocks like policy measures, which are usually announced well before being enforced. Finally, it provides a viable and straightforward way of examining the possible behaviors due to unexpected shocks.

The remainder of this paper is structured as follows. Section 2.2 introduces the models that will be used as a testbed for the framework, namely the neoclassical Ramsey

[1]It is interesting to ask whether infinite growth is possible on a world with finite resources. If it is not, economies will eventually reach a steady-state of no further growth, with capital growing just at the rate required to compensate for depreciations and to equip new borns.

growth model (Ramsey, 1928; Cass, 1965; Koopmans, 1963, – henceforth, RCK) and the endogenous two-sector growth model of (Uzawa, 1965; Lucas, 1988, – henceforth, UL). Section 2.3 describes the framework proposed, as well as its theoretical background. In this section we also provide the numerical solution in the selected growth models. Section 2.4 focus on the transitional dynamics of the models and shows the impulse responses upon expected shocks and multiple and sequential shocks. Also, it compares the effect of a shock when the economy is *not* at a steady state. Finally, a brief overview of our findings can be found in Section 2.5.

2.2 Models of economic growth

In order to illustrate the use of the framework we will employ the RCK growth model, a model exhibiting exogenous growth, and the UL endogenous growth model. The neoclassical growth model exhibits saddle-point stability, and thus a closed-form solution exists for a particular choice of parameters. This allow us to compare the accuracy of the numerical results we obtain with the analytical solution of the system of differential equations. The second model exhibits multi-dimensional stable manifold and is considerably more complex. In fact, the transition process of this growth model has hardly been investigated due to its intrinsic complexity. These models are a powerful workhorse for studying some of the mechanisms of growth.

2.2.1 Neoclassical growth model

We will consider the version of the model as defined in Barro and Sala-i Martin (2003). Our population grows according to $L(t) = L(0) \cdot e^{nt}$, normalized to unity at $t = 0$. The households wish to maximize their overall utility U by means of consumption, C. Also, we consider a current-value formulation with a discount factor ρ. Since all variables are time-dependent, for simplicity we will omit the subscripts. This can be summarized as follows.

$$U = \int_0^{\infty} u(C) \cdot e^{(n-\rho)t} \cdot dt, \quad C \in [0, +\infty). \tag{2.1}$$

Families hold assets b and obtain capital gains from assets, rb, and wages from working, w. Labor supply is inelastic and no unemployment exists. The budget constraint, in per capita terms, is then represented by

$$\dot{b} = (r-n)b + w - c. \tag{2.2}$$

Utility is given by a constant inter-temporal elasticity of substitution (CIES) function

$$u(c) = \frac{C^{(1-\theta)} - 1}{1-\theta}. \tag{2.3}$$

To avoid numerical errors from potential divisions by zero we can replace the CIES function by one given in Equation (3.20)

$$u(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta} & \theta \neq 1 \\ ln(C) & \theta = 1 \end{cases} \tag{2.4}$$

We work with per effective worker ratios, so we need to transform the variable consumption C. Henceforth, we consider "effective labor" to be given by $\hat{L} \equiv L \cdot X$, the product of raw labor and the level of technology. Let us denote c as the consumption per unit of effective labor such that $c \equiv C/\hat{L}$. We consider the technological progress to grow at rate x, such that $X = X(0) \cdot e^{xt}$. After normalizing $X(0)$ to unity we obtain:

$$\frac{C^{1-\theta}}{1-\theta} = \frac{e^{(1-\theta)xt} \cdot c^{1-\theta}}{1-\theta}. \tag{2.5}$$

This results in the objective function, rewritten in consumption per effective worker quantities, given in Equation (2.6)

$$U = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} \cdot e^{(n+(1-\theta)x-\rho)t} dt. \tag{2.6}$$

The goods to be consumed are produced by firms by employing labor and capital. We assume a standard Cobb-Douglas production function $Y \equiv F(K, \hat{L}) = AK^{\alpha}\hat{L}^{1-\alpha}$ with $0 < \alpha < 1$ and A the level of technology. We include labor-augmenting technological progress at a constant rate, which we already know to grow at rate x. As done for consumption, we will express all variables in quantities per unit of effective labor, $y \equiv Y/\hat{L}$ and $k \equiv K/\hat{L}$. The output of the economy can then be expressed in the intensive form as

$$y = f(k) = Ak^{\alpha},\ f(0) = 0. \tag{2.7}$$

Goods and labor markets clear. From this assumption we know that supply and demand quantities meet. This implies that the supply of loans b is met by the demand of capital, k. With this in mind we can write the resource constraint for the overall economy, expressed in units of effective labor, as given in Equation (2.8)

$$\dot{k} = f(k) - c - (x+n+\delta)k, \quad k(0) = k_0,\ k \geq 0. \tag{2.8}$$

Equations (2.6), (2.7) and (2.8) sum up the interactions between agents in the Ramsey growth model.

2.2.2 Uzawa-Lucas endogenous growth model

The neoclassical growth model falls short of explaining the engine of long-term growth in income per capita observed in developed countries. The introduction of technological progress causes such phenomenon to occur but provides no explanation on its origin. Endogenous growth models were conceived as an attempt to overcome such theoretical fragility and to give a consistent account to what causes economies to keep on growing. One prominent endogenous growth model was developed by Uzawa (1965) and later used by Lucas (1988). We follow the formulation laid by Barro and Sala-i Martin (2003).

The Uzawa-Lucas model introduces human capital h, another productive input of the economy that is produced by a different technology than that of physical capital K. Also, labor L can be partly employed on the final output production, μ, with the remaining share $1-\mu$ dedicated to formal education. This model provides for a very comprehensive assessment of the capabilities of our framework. On the one hand, it exhibits steady-state growth, meaning that consumption and capital (physical and human) are unbounded. On the other hand, the introduction of human capital and specialized labour add another state and control variables to the system, respectively, which results in increased complexity. In fact, the transition process of this model is still unclear, since the indirect methods for solving the underlying optimal control problem employed by researchers give origin to stiff ordinary differential equations, again adding an extra burden to the task of the analyst, as Trimborn et al. (2004) notes.

This model uses the Ramsey consumption-optimizer framework specified in Section 2.2.1. As before, households try to maximize their utility by consuming according to a standard CIES function $u(C)$.

$$U = \int_0^\infty u(C) \cdot e^{-\rho t} dt. \tag{2.9}$$

Goods are produced according to the following production function given by Equation (2.10)

$$Y = AK^\alpha (\mu h L)^{1-\alpha}. \tag{2.10}$$

Physical capital K and human capital H growth follow the laws of motion stated in Equation (2.11)

$$\begin{aligned} \dot{K} &= Y - C - \delta_K K \\ \dot{h} &= B(1-\mu)h - \delta_H h \end{aligned}, \tag{2.11}$$

where $B > 0$ is a constant reflecting productivity of quality adjusted effort in education and δ_H $(0 \leq \delta_H < B)$ is the rate of depreciation of human capital, which is set to zero.

Considering per-capita variables $k \equiv K/L$, $y \equiv Y/L$ and $c \equiv C/L$ and no population growth $(n = 0)$ we obtain the description of the economy

$$\max U = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad s.t. \tag{2.12}$$

$$\begin{aligned} c &> 0, & 0 \leq \mu \leq 1 \\ \dot{k} &= Ak^\alpha (\mu h)^{1-\alpha} - c - \delta_k k \\ \dot{h} &= B(1-\mu)h \\ k(0) &= k_0 & k \geq 0,\ \forall t > 0 \\ h(0) &= h_0 & h \geq 0,\ \forall t > 0 \end{aligned} \tag{2.13}$$

with the social planner or household choosing an allocation $(c, \mu)_{t=0}^\infty$ that maximizes U.

2.3 A framework for infinite-horizon models

The framework we present is inspired by the work on model predictive control (MPC), originally conceived to address industrialized optimal control problems with infinite-horizon objectives, but for which only finite-horizon computations are feasible. In particular, we refer to the quasi-infinite horizon approach of Chen and Allgöwer (1998) and the general framework of Fontes (2001). The procedure is as follows: we transcribe the infinite horizon problem into a finite dimensional, NLP representation of the initial problem, which we prove to be an equivalent representation of the original. We then use a state-of-the-art NLP solver to find the optimal trajectories. We are in fact first discretizing and then optimizing, inverting the process used by indirect methods.

This approach overcomes the problems already identified in Section 2.1 while introducing several degrees of freedom. The optimal trajectories can be numerically determined without requiring the linearization of the differential equations. This, in itself, avoids the problems posed by a change of base or any other supposedly neutral manipulations. Also, it allows for the study of the transition process without having the system depart from or be at a steady state. Actually, the economy can depart from any given state. Furthermore, it is a powerful tool to study complex phenomena like anticipated or multiple, sequential shocks.

Indeed, one does not need to derive or even know any necessary optimality conditions (NOCs), which can be especially useful for problems whose adjoint functions are hard to determine. In fact, this is the current method of choice in the field of optimal control for engineering applications due to its easy applicability and robustness.

We start by showing the theorem and the proof and then we do a numerical implementation and solution of two mainstream economic growth models to serve as an example.

Theorem 1. *Consider the following generic optimal control problem.*

$$P_\infty : \max \int_0^\infty L(t,x(t),u(t)) \cdot dt, \tag{2.14}$$

subject to:

$\dot{x} = f(x,u) \quad a.e.$
$x(0) = x_0,$
$x(t) \in \Gamma(t),$
$u(t) \in \Omega(t)$

for which we assume there is a finite solution. Assume additionally that after some time T, the state is within some invariant set S (that is, $x(t) \in S,\ S \subset \Gamma(t)$, for all $t \geq T$) for which the problem still has a finite solution. Then, there exists a terminal cost function W, such that the problem is equivalent to the finite horizon problem

$$P_T : \max \int_0^T L(t,x(t),u(t)) \cdot dt + W(x(T)), \tag{2.15}$$

subject to:

$\dot{x} = f(x,u) \quad a.e.$
$x(0) = x_0,$
$x(t) \in \Gamma(t),$
$u(t) \in \Omega(t),$
$x(T) \in S.$

Proof. Consider problem P_∞ with the additional assumption, $x(t) \in S$ for $t \geq T$, i.e., redefine $\Gamma(t) = S$ for all $t \geq T$. By Bellman's optimality principle the value for P_∞ is

$$V(0,x_0) = \max \left\{ \int_0^T L(t,x,u(t)) \cdot dt + V(T,x(T)) \right\}.$$

We can now define $W(x) \subset V(T,x)$ and since $x(t) \in S$ for all $t \geq T$we have

$$W(x(T)) = V(T,x(T)) = \max_{x \in S} \left\{ \int_T^{+\infty} L(t,x,u) \cdot dt \right\}.$$

We can then rewrite the problem as P_T. □

2.3.1 Application

The above theorem is useful only for the case that the set S is such that a characterization of the solution to the problem

$$W(x(T)) = \max_{x \in S} \left\{ \int_T^{+\infty} L(t,x,u) \cdot dt \right\}$$

is possible (via NCO or other means) and therefore we can explicitly compute $W(x)$ for any $x \in S$.

One such example is when economies are at a balanced growth path (BGP), either because the characteristics of the problem always lead to it, or because we assume (or impose) that to be the case. We will exemplify how to use the framework on two of such models, previously described in Section 2.2.1 and in Section 2.2.2.

The procedure is as follows:

1. Transcribe the infinite-horizon problem into an equivalent finite-horizon problem by applying Theorem 1;
2. Add the necessary boundary conditions that ensure that the set S is invariant (and so $W(x)$ can be computer explicitly for any $x \in S$);
3. Use a NLP solver to determine the trajectories of the control and state variables.

2.3.1.1 Neoclassical growth model

Consider the Ramsey-Cass-Koopmans growth problem described in Section 2.2.1. If we impose that $\rho > n + (1-\theta)x$ so that the model converges to a BGP, with $k(t) = k^*$ constant for $t \geq T$, then define

$$\begin{aligned} &S = \{k : \ \dot{k} = 0\} \Leftrightarrow \\ &S = \{k : \ A(k^*)^\alpha - c^* - (\delta + n + x)k^* = 0\}, \quad k(T) \in S \end{aligned} \tag{2.16}$$

where k^* and c^* are given by the values obtained at instant T, i.e., $k^* = k(T)$ and $c^* = c(T)$. Equation (2.16) is the invariant set S required by Theorem 1. Using the definition above, we can define

$$c^*(k^*) = A(k^*)^\alpha - (\delta + n + x)k^*. \tag{2.17}$$

We are now in a position to compute W, the *boundary cost*. From Theorem 1 we know that

$$W(k^*) = \max \int_T^{+\infty} L(t,k^*,c^*)dt$$

with $L(t,k^*,c^*) = u(c^*(k^*)) \cdot e^{-\rho t}$. We then have

$$W(k^*) = \max \int_T^{+\infty} u(c^*(k^*)) \cdot e^{-\rho t} = U(\cdot,\infty) - U(\cdot,T) \tag{2.18}$$

We apply a discount factor ρ and ensure that $\rho > n + (1-\theta)x$ must hold true for all t, so $U(\cdot,\infty)$ is 0. Hence, W will be equal to $-U(\cdot,T)$.

Since U is defined by Equation (2.6), it is trivial to integrate Equation (2.18) and obtain the following value for the boundary cost

$$W(k^*) = -\frac{e^{(n+(1-\theta)x-\rho)t}}{\rho - n - (1-\theta)x} \cdot \frac{c^*(k^*)^{(1-\theta)}}{1-\theta}. \tag{2.19}$$

The problem is then to

$$\max \int_0^T u(c)e^{-\rho t}dt + W(k(T)), \tag{2.20}$$

subject to (2.8) and the boundary condition

$$k(T) \in S \Leftrightarrow \dot{k}(T) = 0 \Leftrightarrow Ak(T)^\alpha - c(T) - (n+x+\delta)k(T) = 0 \tag{2.21}$$

2.3.1.1.1 Numerical solution

The next step is the implementation of the transcribed finite-horizon problem as described in the previous section on a NLP solver. In order to numerically solve this problem we will make use of the Imperial College London Optimal Control Software[2] (ICLOCS). As a general guideline, we also show how to specify the model as required by ICLOCS. Of course, another interface to an NLP solver could be used instead.

To run a simulation we will use the following parameterization. We will consider a fixed 300 year timespan (`tmin=300`, `tmax=300`), enough for the model to grow and converge, since it exhibits saddle-point stability. Without loss of generality, we will define the initial stock of private capital k to be ten percent of its steady state value ($k^* \simeq (\frac{\alpha A}{\delta+\rho+\theta x})^{1/(1-\alpha)}$, $k_0 = 0.1k^*$). State bounds are also fairly loose. k can assume any non-negative value since there is no economic definition of negative capital stock ($0 \leq k < \infty$) and the corner solution ($k = 0$) is non-optimal. In the end, capital should approach its steady state value ($k(T) \simeq k^*$). As for the input boundaries, we will just say that consumption per unit of effective labor c is limited between zero and infinity ($0 \leq c < \infty$). Again, a negative consumption has no meaning in economic terms.

For the parameterization of the model we will strictly follow the benchmark set by Barro and Sala-i Martin (2003), reported in Table 2.1. These are standard values.

[2]http://www.ee.ic.ac.uk/ICLOCS/ — This software solves optimal control problems with general path and boundary constraints and free or fixed final time. It uses another intermediary piece of software called Interior Point Optimizer (IPOPT) to solve the transformed NLP problem.

Parameter	α	θ	ρ	n	x	δ
Value	0.3/0.6	3	0.02	0.02	0.02	0.02

Table 2.1: Calibrated parameters for the numerical simulation of the RCK model. Values taken from (Barro and Sala-i Martin, 2003).

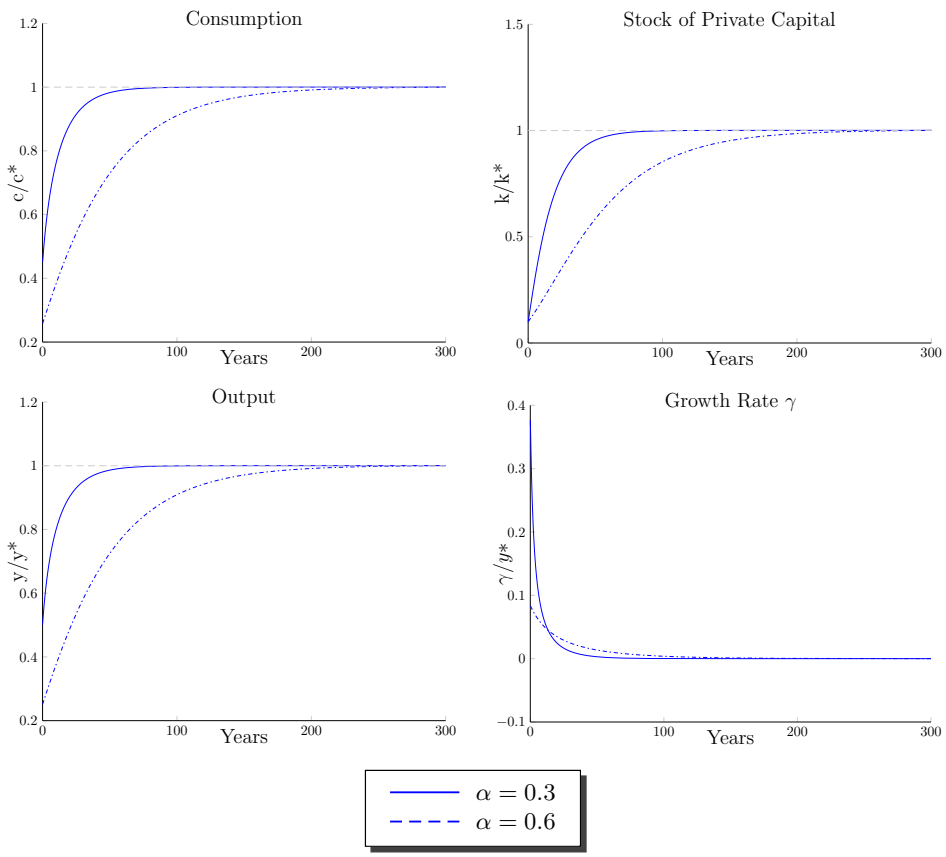

Figure 2.1: Numerical trajectories for the Ramsey-Cass-Koopmans model, using Barro and Sala-i Martin (2003) benchmark.

Figure 2.1 depicts the results obtained. As expected, the stock of private capital k, the consumption c and the output of the economy y in units of effective labor all converge to their steady state values k^*, c^*, y^*. These results are fully in line with the ones obtained by Barro and Sala-i Martin (2003). In the end, we will compare consumption to its steady state value $(c^* \simeq A(k^*)^\alpha - (n+x+\delta)k^*,\ c(T) \simeq c^*)$. Likewise, the same will be done with the output of the economy, which should converge to its steady state value $(y^* \simeq A(k^*)^\alpha)$.

2.3.1.2 Uzawa-Lucas endogenous growth model

As before, consider the Uzawa-Lucas growth model described in Section 2.2.2 by equations (2.12) and (2.13).

We know that in balanced growth the following holds true for our specification of the

model

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} = \frac{\dot{y}}{y} = \gamma$$

for some constant $\gamma > 0$. If we define $\omega \equiv k/h$ and $\chi \equiv c/k$, we know that $\dot{\omega} = 0$ and $\dot{\chi} = 0$ holds true in the steady state, for $t \geq T$ (see Barro and Sala-i Martin (2003) for details). Define the control policy $c(t)$ and $u(t)$ such that

$$\begin{aligned} c(t) &= \chi k(t) \\ k(t) &= \omega h(t) \end{aligned} \tag{2.22}$$

and χ, ω are both equal to a given positive constant.

We have that

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}(uh)^{1-\alpha} - (\chi+\delta) = (\frac{uh}{Ak})^{1-\alpha} - (\chi+\delta) = (\frac{u}{A\omega})^{1-\alpha} - (\chi+\delta) = \gamma,$$

and

$$\frac{\dot{h}}{h} = \frac{\chi\dot{k}}{\chi k} = \frac{\dot{k}}{k} = \gamma.$$

On the other hand,

$$\frac{\dot{c}}{c} = \frac{\chi\dot{k}}{\chi k} = \frac{\dot{k}}{k} = \gamma$$

and

$$\frac{\dot{k}}{k} = \frac{\omega\dot{h}}{\omega h} = B(1-u) = \gamma.$$

So (2.22) is achieved with $u(t) = u$ constant satisfying

$$B(1-u) = (\frac{u}{A\omega})^{1-\alpha} - (\chi+\delta).$$

Moreover we have

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} = B(1-u),$$

which implies that $\dot{c} = \gamma c$ in a BGP. At the end time $(t = T)$ consumption c will continue to grow at rate γ, according to

$$c(t) = c(T) \cdot e^{\gamma(t-T)},\ t \in [T, +\infty) \tag{2.23}$$

which enable us to compute W. Utility will be bounded as long as $\rho > \gamma(1-\theta)$, meaning that $U(\cdot, \infty) = 0$. The boundary cost is then given by integrating equation (2.12) incorporating the definition of $c(t)$ from Equation (2.23).

$$W = -\int_T^{\infty} \frac{(c(T)e^{\gamma(t-T)})^{1-\theta}}{1-\theta} e^{-\rho t} dt,$$

which we know to be equal to

$$W = -\frac{e^{\gamma(1-\theta)-\rho}}{\rho-\gamma(1-\theta)} \cdot \frac{[c(T)e^{-\gamma T}]^{1-\theta}}{1-\theta}. \tag{2.24}$$

The boundary condition becomes

$$S = \{(k,h) \in \mathbb{R}^2 : \frac{\dot{k}}{k} - \frac{\dot{h}}{h} = 0\} \quad (k(T),\, h(T)) \in S. \tag{2.25}$$

2.3.1.2.1 Numerical solution

The system as defined by Equation (2.13) along with the boundary cost (2.24) and the boundary condition (2.25) is all that is required to solve the model numerically. The model was run with the parameters set to those of Table 2.2.

Parameter	A	B	α	θ	ρ	δ_K	δ_H
Value	1	0.136	0.3	3	0.03	0.05	0

Table 2.2: Calibrated parameters for the Uzawa-Lucas model.

Figure 2.2 depicts the optimization results obtained upon running the model from an arbitrary starting point $(k_0,\ h_0)$. As can easily be seen and is expected, $c,\ k,\ h$ exhibit constant growth when the system is in equilibrium. On the contrary, ω and χ converge to a stationary state, as expected. Also, we see that in this economy approximately $2/3$ of labor will be employed in producing final goods, while the remaining $1/3$ will be developing human capital.

2.3.2 Evaluation of numerical results

In order to assess the quality of the numerical results obtained we will follow a now standard approach in the literature to measure the accuracy of numerical methods. This is the procedure conducted in other studies like those of Aruoba et al. (2006) or Heer and Maußner (2008). We will calculate the residual against a closed-form linear analytical solution of the RCK model. Since the other numerical methods use an indirect approach, the residual they calculate is for the Euler equation, the ordinary differential equation that describes how consumption evolves over time. The Euler residual provides a (unit-free) measure of the percentage error in the consumption trajectory of the household. In our case, we can directly compare the trajectory for consumption $c(t)$ we obtain numerically against the one determined analytically. This is in fact a more robust comparison, since the Euler equation is mostly concerned with the asymptotic properties of the accuracy of the numerical solutions. As Atolia et al. (2010) duly point out, this might be of interest to

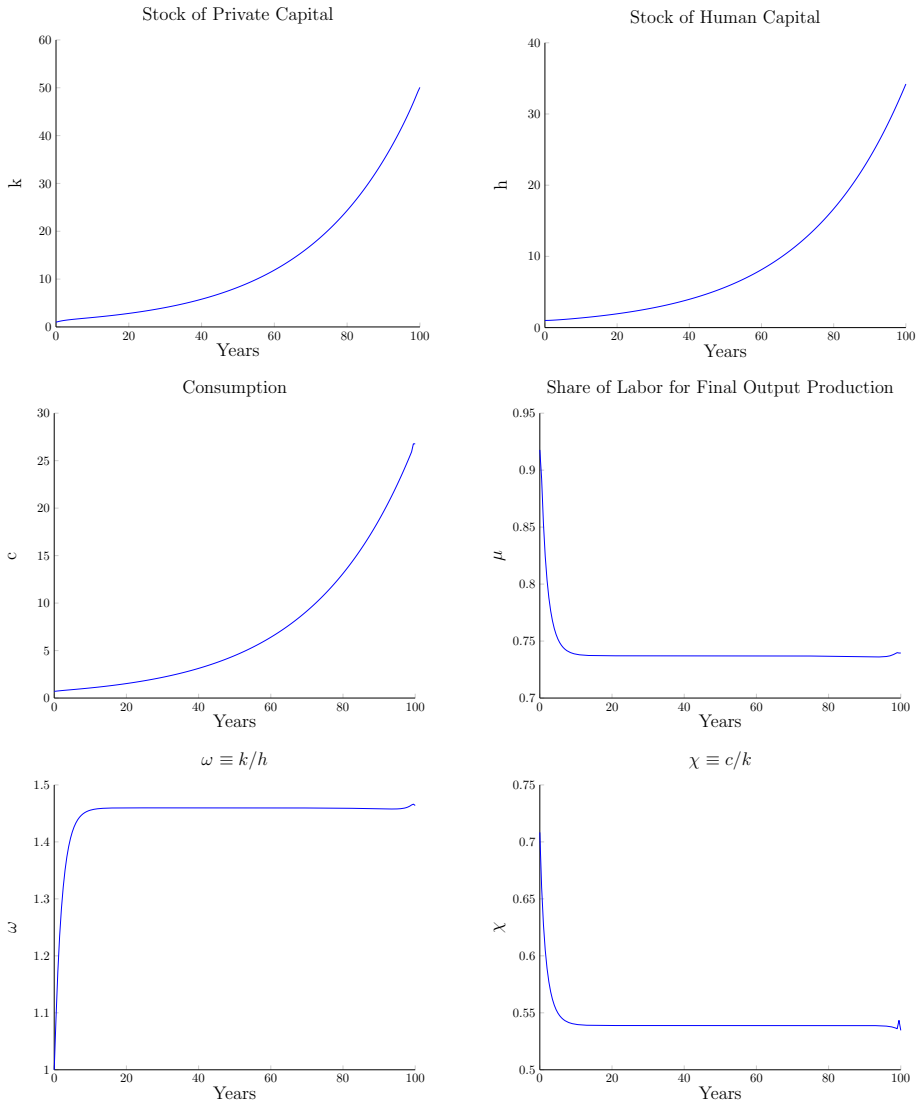

Figure 2.2: Numerical optimization of the Uzawa-Lucas endogenous growth model.

the dynamic stochastic general equilibrium (DSGE) literature, but not so much to growth theory, since we are concerned with the complete transitional path.

For the closed-form analytical solution of the RCK model we will follow Brunner and Strulik (2002). They show that for the particular case when $(\alpha\delta)/(\delta+\rho) = 1/\theta$ holds true, the consumer will select a constant savings rate of $s = 1/\theta$ and the solution of the model is

$$k(t) = [\frac{s}{\delta} + (k_0^{1-\alpha} - \frac{s}{\delta})e^{-\delta(1-\alpha)t}]^{\frac{1}{1-\alpha}} \tag{2.26}$$

and $c(t) = (1-s)k(t)^{\alpha}$. For this particular case the authors assume no technological progress and no population growth. Accordingly, the parameters were set as reported in Table 2.3 and the residuals of the trajectories for the consumption $c(t)$ were calculated for the interval $[0.1k^*, 2k^*]$.

Parameter	α	θ	ρ	n	x	δ
Value	0.3	4.6(7)	0.02	0	0	0.05

Table 2.3: Calibrated parameters as set byBrunner and Strulik (2002) for the numerical simulation of a particular closed-form solution to the RCK exogenous growth model.

Figure 2.3 depicts the logarithm of the consumption's trajectory $c(t)$ residual against the stock of private capital, k. Like Aruoba et al. (2006) and Ambler and Pelgrin (2010), we opt for reporting the absolute errors in base 10 logarithms as it facilitates the economic interpretation. A value of -3 means a \$1 mistake for each \$1000 spent, a value of -4 a \$1 mistake for each \$10 000, and so on. These results go in line with Euler residuals obtained for other numerical procedures, most of them identified in this paper and summarized in Aruoba et al. (2006). It is worth noting that near the steady state $(k^* \simeq 7.96)$ the log residual error of -16 is most neglectable.

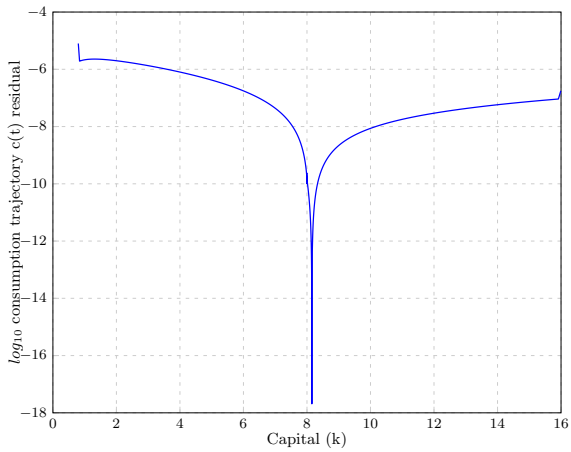

Figure 2.3: log_{10} of the residual of the consumption trajectory.

Moreover, in Table 2.4 we present maximum absolute errors for the trajectories of consumption $c(t)$, stock of private capital $k(t)$ and output of the economy $y(t)$. We also show a measure of the errors as a percentage of the initial pre-shock equilibrium value, namely by calculating the ratio $\frac{x_A(T)-x_N(T)}{x_0}$, a procedure also followed by Atolia et al. (2010).

The results obtained, along with the residual for the consumption trajectory, are extremely satisfactory when in comparison to all the other available procedures, according to the results published by Aruoba et al. (2006).

2.4 Transitional dynamics

The framework we propose is especially useful for the analysis of the transitional dynamics arising from policy or structural shocks. Without any reformulation of the problem,

	$\alpha = 0.3 \;\; \theta = 4.66(7)$			
	Max Abs Error		$(x_A(T) - x_N(T))/x_0$ Error	
	$k_0 = 0.1k^*$	$k_0 = 2k^*$	$k_0 = 0.1k^*$	$k_0 = 2k^*$
c	0.00610	0.00117	$0.88182e^{-4}$	$0.2549e^{-4}$
k	0.01638	0.00271	0.00353	$0.1653e^{-4}$
y	0.00208	$0.1895e^{-3}$	$0.21159e^{-3}$	$0.8056e^{-4}$
	$\alpha = 0.5 \;\; \theta = 2.8$			
	Max Abs Error		$(x_A(T) - x_N(T))/x_0$ Error	
	$k_0 = 0.1k^*$	$k_0 = 2k^*$	$k_0 = 0.1k^*$	$k_0 = 2k^*$
c	0.01097	0.00512	0.00115	$0.37462e^{-4}$
k	0.08491	0.02317	0.00713	$0.16750e^{-3}$
y	0.00828	0.00162	0.00113	$0.11841e^{-3}$

Table 2.4: Maximum absolute errors and relative errors as a percentage of the initial pre-shock equilibrium value for two distinct benchmarks, one using the values $\alpha = 0.3,\ \theta = 4.66(7)$ and the other using $\alpha = 0.5,\ \theta = 2.8$.

one can easily study expected or unexpected shocks, either departing from a steady state or not. Moreover, it is also extremely easy to study a sequence of multiple shocks. The innovation is that the economy does not have to converge to a new steady-state before a new shock can be applied. Shocks can occur at any given time.

In order to exemplify how to use the framework to study the transition dynamics we will extend the RCK model from Section 2.2.1 with the introduction of proportional taxes on wage income, τ_w, private asset income, τ_r, and consumption, τ_c. We follow Barro and Sala-i Martin (2003). This time we assume no technological progress, with no loss of generality.

This extension requires a change to equation (2.2), the budget constraint of the households. The budget constraint will then become

$$\dot{b} = (1-\tau_w)w + (1-\tau_r)rb - (1+\tau_c)c - nb \tag{2.27}$$

with $r = \alpha k^{\alpha-1} - \delta$ and $w = (1-\alpha)k^{\alpha}$. Since markets clear with $b = k$, equation (2.27) is also the global constraint of the economy, which assumes the following form

$$\dot{k} = (1-\tau_w)(1-\alpha)k^{\alpha} + (1-\tau_r)\alpha k^{\alpha} - (1+\tau_c)c - (n+\delta)k \tag{2.28}$$

These modifications allow us to introduce exogenous shocks by manipulating the policy variables $\{\tau_w, \tau_r, \tau_c\}$ and therefore study how the economy copes with a certain expected or unexpected change of policy.

For the parameterization we will consider the values specified in Table 2.5.

Parameter	α	θ	ρ	n	x	δ	τ_w	τ_r	τ_c
Value	0.3	2	0.02	0.01	0	0.03	0.4	0.3	0.1

Table 2.5: Calibrated parameters.

2.4.1 Expected shocks

In this particular case, the agents show perfect foresight, i.e., it is assumed that at the point in time when the maximization occurs, the maximizing agent is aware of the whole set of information. If that holds true, then it is also true that it knows the future time path of variables exogenous to the model, like those of the tax rates. We will first consider a simulation of an expected shock to the RCK model and then to the UL model.

To study such shocks, authors like Trimborn (2007) suggest a reformulation of the optimization problem, namely by decomposing the functional form of the objective function from $f^{(1)}$ to $f^{(2)}$ and the state equations from $g^{(1)}$ to $g^{(2)}$ at time $\tilde{t}$, when the shock occurs. The necessary optimality conditions would have to be augmented with the conditions derived from the interior boundary condition, that for this case are $\psi[\tilde{t}] = \tilde{t} - t_{jump} = 0$. Moreover, the adjoint variable functions introduced by the Maximum Principle have to attend to a continuity requirement, also known as the Weierstrass-Erdmann corner condition. For further details, Bryson and Ho (1975) provide an exhaustive explanation.

We do not require any reformulation of the RCK model. The only step needed is to set a change of τ_c to 20% at time $t = 20$. Again, we will depart from the steady state, with $k_0 = k^*$.

In Figure 2.4 we present the responses when an expected policy shock takes place. The results come as no surprise and are in line with the ones obtained by Trimborn (2007).

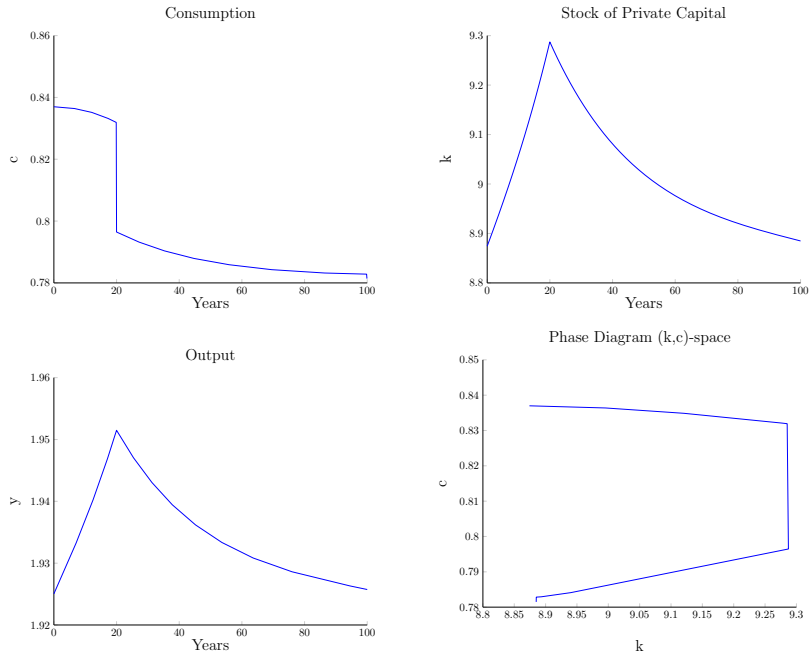

Figure 2.4: Impulse responses to an anticipated increase in τ_c at $t = 20$.

We also show how the UL model reacts to an anticipated shock. We will consider the scenario of an increase in the elasticity of physical capital from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$. This means that the marginal productivity of capital increases, so it will become more attractive to work in the production of final goods.

Figure 2.5 presents the responses of the model to such shock. From a quantitative point of view, we have a welfare increase of 0.132% (welfare with no shock is $U_0 = -9.1678$ and with the aforementioned shock it raised to $U_s = -9.1557$). But this analysis is particularly interesting from a qualitative point of view. Expectedly, we can see a surge in the share of labor dedicated to production in the final goods sector, reaching $\mu = 1$. Since there is no capital attrition, swapping labor between producing final goods and developing human capital comes at no extra cost, but in a more real case scenario it would probably have a greater effect on the evolution of human capital.

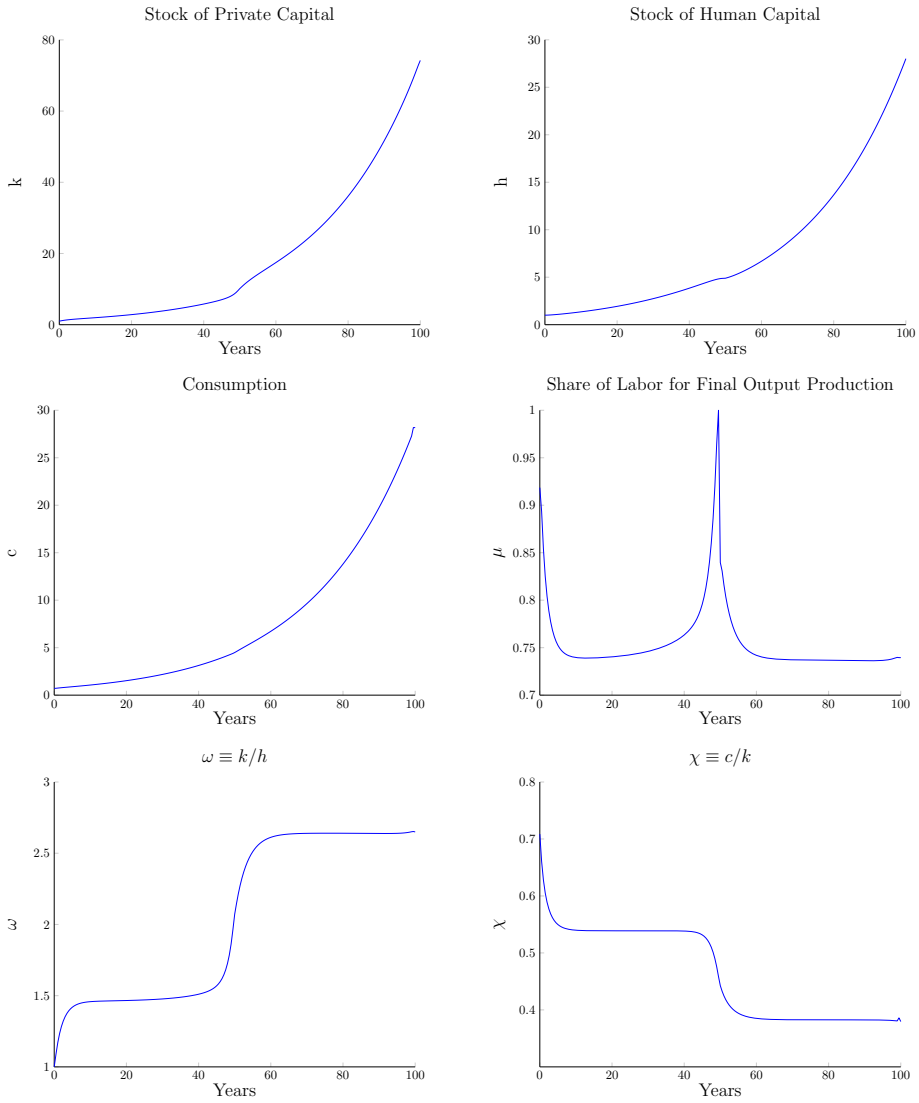

Figure 2.5: Numerical optimization of the Uzawa-Lucas endogenous growth model when subject to an anticipated capital elasticity increase from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$.

2.4.2 Non-steady state shocks

The available numerical approaches assume that the economy departs from a steady state prior to being hit by a shock. Indeed, such information is usually an input of the procedure. To be more precise, some methods (like Trimborn et al., 2004) do not require to start from a steady state, but rather calculate the state of the system at its equilibrium prior to applying a shock, which is conceptually the same.

In our framework there is no requirement to start from a steady state. In fact, a non-steady state analysis is more realistic in the sense that no empirical studies have consistently reported a real world economy to be at its long-term equilibrium state.

Consider the same shock as above, but now taking place for three different values of

$k_0 = \{0.5k^*, k^*, 1.5k^*\}$ (since the shock takes place at $\tilde{t} = 20$, it is closer to its steady state but still distant enough to serve as a viable example). In the first case, where $k_0 = 0.5k^*$, the initial value for the stock of private capital is set to half of its equilibrium value. In the second case, with $k_0 = k^*$, we are starting from an equilibrium state that goes fully in line with the results already obtained and represented in Figure 2.4 and also reported by Trimborn, 2007. In the third case, the value for k_0 is set to 50% higher than its steady state, with $k_0 = 1.5k^*$.

As can be seen from Figures 2.6, the outcome is substantially different. From a qualitative point of view, the trajectory of consumption $c(t)$ manifests a widely different behavior depending on its starting point. In the case where the economy is way over its steady state (red dashed line), there is a sharp drop in the consumption after the shock, as expected, and it continues to converge to its steady state. The adjustment trajectory is quite similar to the case when the economy departs from its steady state (blue straight line). But when the economy departs from a state considerably lower than its equilibrium value (green dashed line) the trajectory is considerably different. Instead of showing a continuous drop in the consumption, it can be seen that after the shock a sudden drop occurs but it is partially mitigated by a subsequent increase up until the new steady-state. The savings rate also exhibits a contrasting effect. Instead of rising (agents will necessarily consume less when their budget decreases), the savings rate will actually *decrease* for the case when the economy departs from a state over its equilibrium value, with a sudden increase at the time the tax policy comes into effect.

Actually, the behavior of consumption is not the only exhibiting such sharp differences in the adjustment trajectory. Also, there is no overshooting of the investment in stock of private capital, as occurs when the economy departs from a steady-state. Same happens with the output of the economy.

From a quantitative point of view, a welfare analysis shows also a difference in costs of adjustment, albeit with a major difference in qualitative and quantitative terms after the shock. Looking at the whole horizon, consumption decreases and so does the welfare. But if we look only to the period after the shock, we can clearly see a welfare decrease for the economy departing from and above the steady state, but a welfare *increase* for the economy departing from below its equilibrium. From a qualitative standpoint, it has a far better acceptance since an increase in consumption even when not exploited to its potential level is still better than a drop.

Table 2.6 summarizes the welfare analysis for each of the starting initial values of k_0.

Expected Shock $(\tau_0 = 0.1,\ \tau_{\tilde{t}} = 0.2)$			
Initial value (k_0)	Welfare (no shock)	Welfare	Cost
$0.5k^*$	-124.5585	-133.5139	8.9554
k^*	-117.3117	-126.0254	8.7137
$1.5k^*$	-112.6665	-121.1948	8.5283

Table 2.6: Welfare analysis for three distinct initial values for k_0 upon an expected shock hitting the RCK growth model.

2.4.3 Multiple, sequential shocks

Another very interesting application of the framework is to simulate multiple sequential shocks, something not seen in the literature due to the complexity of determining the new optimal paths from the original necessary optimality conditions. Also, we allow for the optimization to occur from a non-steady state, like in Section 2.4.2. Otherwise, if we allowed enough time for the economy to convert back to a new steady state, these sequential shocks could be simulated using traditional methods by connecting the responses to each shock. The deed is even more complex for anticipated shocks as the system becomes increasingly complex.

Again, such analysis is made possible by the fact that the framework transforms the problem into an equivalent problem that can be solved numerically using a direct method, meaning that optimization is done *a posteriori*.

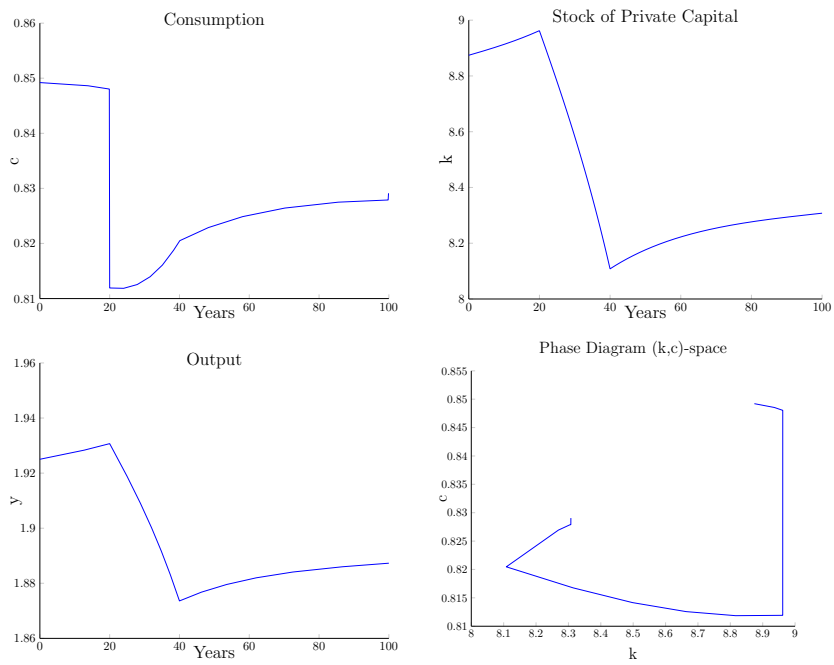

Figure 2.7: Responses to multiple sequential shocks, both expected. The first is an anticipated increase in τ_c at $t = 20$ and the second is a decrease in τ_k at $t = 40$.

Figure 2.7 shows the responses of the RCK model augmented with taxes to an anticipated increase in the consumption tax τ_c at time $\tilde{t} = 20$ from $\tau_{c,0} = 0.1$ to $\tau_{c,\tilde{t}} = 0.2$ followed by a decrease in the capital tax τ_k at time $\tilde{t} = 40$ from $\tau_{k,0} = 0.3$ to $\tau_{k,\tilde{t}} = 0.1$. It is interesting to observe that from both a qualitative and a quantitative point of view households will be worse off, even if the tax decrease on capital could potentially increase the long-term output of the economy, therefore making for the levying of the consumption tax. Also, it is interesting to observe the adjustment trajectories for consumption c and for the stock of private capital k.

2.5 Summary

We have proposed a new framework capable of solving and simulating the transitional dynamics of nonlinear continuous and discrete growth models. This is made possible by the theorem that assures that we can represent an infinite-horizon with a finite-horizon formulation, so that we can solve the underlying optimal control using a direct method. Although such methods are widely used in the control of industrial processes, this approach is new in the economic growth literature, in which most of the relevant numerical procedures making use of indirect methods. The procedure is extremely powerful as it is not limited to problems whose NOCs can be derived and solved analytically. We have already highlighted some of the main advantages when compared to the available procedures, but it is worth emphasizing that this framework allows for the study of the transitional dynamics of models that are not at their steady-state, something that to the best of our knowledge has not been done previously. It also makes it extremely easy to investigate the adjustment trajectories of when multiple shocks hit the economy at different times.

In short, this framework opens a whole new realm of possibilities, being able to cope with extremely complex and nonlinear dynamic systems, continuous or discrete, and making it extremely easy to study expected and unexpected shocks, single or multiple. We believe it will be an important asset in the toolkit of a macro-growth researcher.

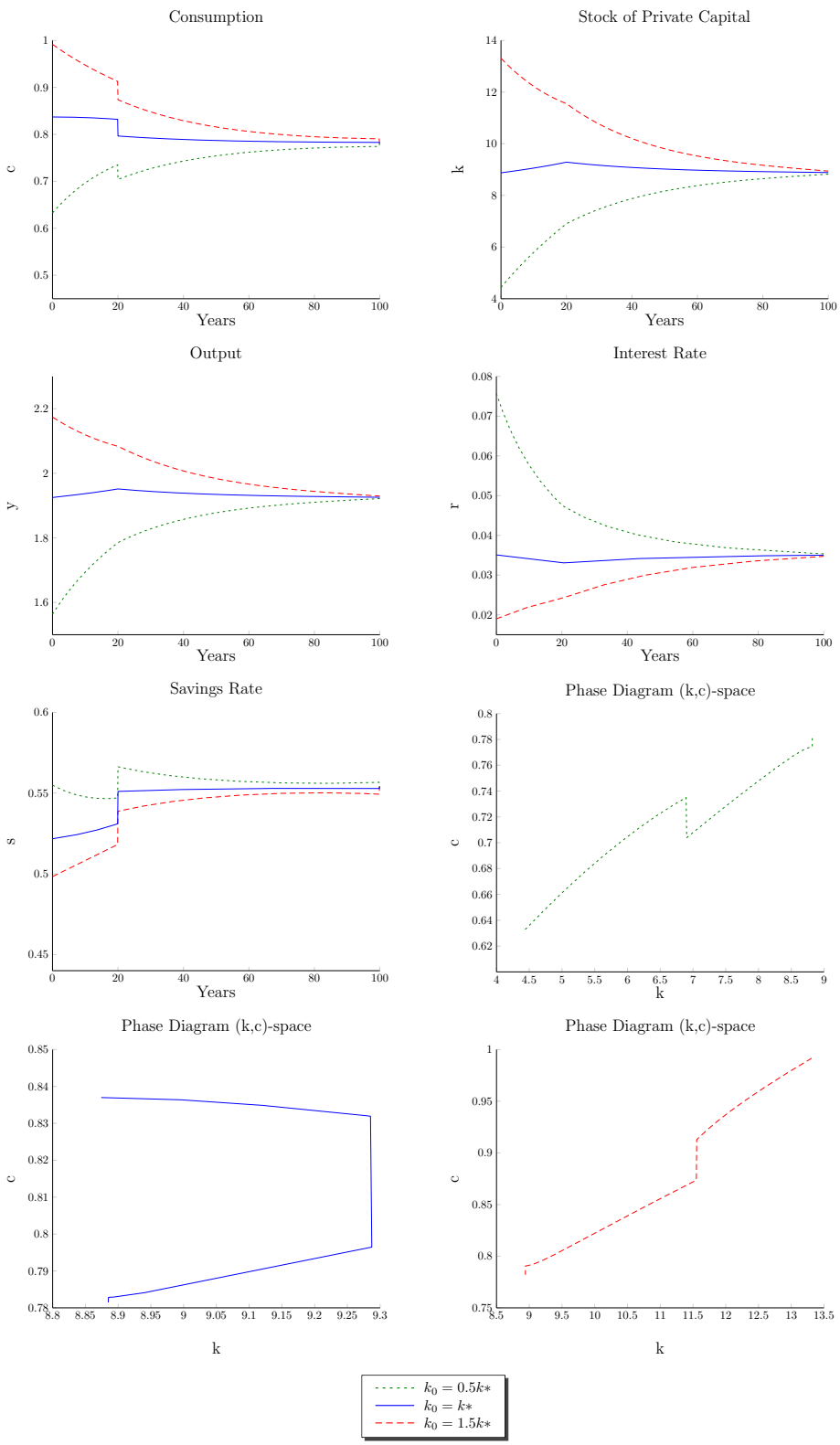

Figure 2.6: Impulse responses to an anticipated increase in τ_c at $t = 20$ for three distinct initial values for k_0. The straight blue line exhibits the adjustment trajectories for when the economy departs from steady state, $k_0 = k^*$. The red dashed line exhibits the very same trajectories for when the economy departs from over its steady state value, $k_0 = 1.5k^*$. Finally, the green dashed line shows the behavior from a starting point of half its steady state value, $k_0 = 0.5k^*$.

Chapter 3

Optimal debt adjustment in a nonlinear endogenous growth model

"If austerity is so terrible, how come Germany and Sweden have done so well?"

– Roberto Barro

3.1 Introduction

In this work we study the optimal debt adjustment strategy in terms of the best policy to conduct a fiscal consolidation, gauging its impact by estimating the welfare costs of the transition back to a sustainable path. To do so, we construct an endogenous growth model with public capital and government debt, time-varying tax rates, a controllable path of public expenditure and a quantitative budget rule. Debt is issued at the domestic interest rate in international capital markets for which the demand for bonds is unlimited, so we ignore the effects of the risk premium on the yield-to-maturity of bonds. The model builds on two related strands of literature: the literature on endogenous economic growth and the literature on dynamic optimal taxation, but it parts ways with the traditional analysis by (i) employing a framework that allows the nonlinear form of the model to be solved, thereby escaping from the recurrent loglinearization inaccuracies; and (ii) introducing time-variant tax rates, an assumption that renders the analytical models intractable and, therefore, irresoluble. We are thus able to overcome this issue.

The work of Romer (1986) marked the revival of the literature on economic growth, further reinforced by the seminal work of Lucas (1988). These models suggested that distortions may affect the long-term rate of growth of income, consumption and accumulation of capital. The recent sovereign debt crisis afflicting developed countries turned this topic ever more significant. In fact, it led to the resurgence in the economic literature of theoretical and empirical research trying to uncover *if, how,* and by *how much* can debt tamper with growth, the optimal level of public debt and strategies to curb explosive debt levels and bring it back to a safe trajectory. It is a response to the negative effects and financial failure caused by the limited and sometimes prohibitive access to the international capital markets, severely constraining how governments finance their expenditure and outstanding debt obligations.

Following this strand of the literature, Maebayashi et al. (2012) examine how to reduce the levels of debt down to a given target level. The results obtained suggest that there exists multiple equilibria and that lower target levels for public debt will increase the growth rate in the high-growth rate equilibrium. Minea and Villieu (2012) solve the model by slightly altering how the log-linearization is done and the model exhibits a single path to equilibrium, reverting some of the results previously obtained, although a negative link between debt and growth is still present. This is not exclusive of the economic growth literature. New-Keynesian dynamic stochastic general equilibrium models perform no better. As soon as log-linearization is introduced, qualitative errors may occur, as a recent discussion at a Federal Reserve Bank conference[1] and the work of Braun et al. (2012)

[1] 37th Annual Federal Reserve Bank of St. Louis Fall Conference. October 2012. http://research.stlouisfed.org/conferences/annual/index.html

have shown.

The empirical literature has also not provided a definite answer on the subject. Results on the best way to reduce debt and on how debt can affect growth are ambivalent. The work of Alesina and Ardagna (2009) imply a steady reduction in the level of public expenditure that is growth-enacting and Reinhart and Rogoff (2010) warn that high levels of debt might impede economic growth. At the opposite side, studies like Blanchard and Leigh (2013) say that strong fiscal consolidations have a detrimental effect on growth due to amplified fiscal multipliers in times of recession. The relationship between debt and growth is key to adjusting fiscal consolidation, but it is still uncertain.

In such a context, we study the optimal fiscal consolidation policy that is able to invert the ascending trajectory of public debt, while minimizing potential welfare costs, under a new perspective. Building upon the numerical framework proposed by Amorim Lopes et al. (2013) for solving infinite-horizon endogenous growth models, we do not an exogenous budget nor a fiscal adjustment rule and thus the system finds the optimal adjustment for the trajectory of the stock of public debt. This approach does not require fixing the functional form of the budget rule, which otherwise would impose a pre-defined path for the fiscal consolidation and subsequent debt adjustment. In lieu, the **quantitative band** allows the system to operate freely in the search for an optimal solution. In this scenario, where both lower and upper bounds are imposed on the budget deficit, the economy is able to make use of the best fiscal policy available for a particular period and then revert it, if convenient. For instance, this strategy would allow for a countercyclical macroeconomic policy to be conducted, as long as it benefits the adjustment. Morever, this approach allows to experiment with **time-variant tax rates**, bringing into play a dynamic fiscal policy, an instrument that can be used to minimize the welfare adjustment costs of the transitional dynamics or as an alternative scenario, to reinforce the investment in public capital and debt service.

Our preliminary results are the following:

1. Like in other endogenous growth models, the optimal tax rate on capital income is zero in the long-run. In the short-run, the optimal policy consists in taxing capital and income, given that the supply of capital is inelastic in the short-term; in the long-run, since capital income taxes discourage capital accumulation, there is a shift in the tax policy and the tax revenues are now fully supported by a tax on consumption, with the tax on capital and income converging to zero;
2. Government chooses to withhold investment in public capital in the first years of the adjustment, suggesting a sharp expenditure cut. Investment in public capital is resumed afterwards, converging again to its equilibrium trend. This suggests that an optimal fiscal consolidation relies heavily on curbing the levels of public spending;

3. The initial level of public debt will condition the adjustment trajectory. For low levels of public debt, it is optimal to issue further public debt prior to proceeding with a fiscal consolidation so as to keep taxes low and promote investment in growth-enacting private capital. For high levels of public debt, adjustment commences right away, at time zero. This suggests that the relationship between debt and growth is non-monotonic with an inverted U-shape function. Our simulations indicate the inflection point to be around 160 and 170 percent of the GDP;
4. For an horizon of thirty years, the larger the level of initial public debt, the lower the short-run and long-run growth rate of the economy. The effects of high levels of public debt are notorious and long-lasting. For a longer time window, the economies converge to the same growth rate, but since the growth rates exhibited during the transition are different, they converge necessarily to different equilibriums;
5. Although debt can promote a short-run surge in growth, unsustainable levels are detrimental to the economy and take a long period to accommodate for. This suggests that fiscal policy should be very careful of unsustainable debt levels and act accordingly.

The remaining of this work is as follows. In Section 3.2 we review the existing literature, provide an empirical background on public debt, explain fundamental concepts like those of fiscal and debt sustainability and standard approaches to fiscal consolidation and debt adjustment. In Section 3.3 we introduce the framework and the model under which we will analyze the optimal adjustment of debt, along with the numerical results. Section 3.4 leaves some remarks to the policy maker, along with policy implications of the suggested adjustment plan. Finally, Section 3.5 concludes with suggestions for future work.

3.2 Background on public debt

In an economy where a government exists and provides public goods and services, mainly in the form of investment in public capital, spurious government consumption and social transfers through tax credits or outright subsidies, then any budget needs can be financed in two distinct ways. Government either levies a tax mix on capital, labour and consumption, or it resorts to debt financing instruments, issuing public debt in the form of treasury bills and government bonds, therefore deferring immediate payment over a defined period of time. In some countries, the holders of government debt owe a privileged drawing right over the fiat reserves of the domestic currency held by the country or central bank. Initially, the reserves used to be gold. With the advent of the fractional reserve banking system, the amount of money in circulation has long surpassed the market value of the available reserves of gold, and so it is a practice that no longer applies.

When government issues debt, it is committing to a payment of interest, along with a redemption of the face value of the coupon at the date of maturity. Servicing debt adds an extra burden to the government budget. If it can no longer abide by its debt obligations, it is on a path of fiscal unsustainability that could eventually trigger a default notice, either partial or in full, if a fiscal consolidation is not immediately undertaken. Such financial stance can result in massive losses to the bondholders, prompting subsequent withdrawals from the debt market, prices to plummet, yields to rise and, in the worst-case scenario, making it impossible to issue further debt bonds in international capital markets due to a falling demand. Facing the impossibility of issuing debt, the government would be unable to run a fiscal deficit. In this scenario, where the government is then forced to run a balanced budget, taxes would have to equal expenditure, *pari passu*. Fiscal policy would have to immediately compel to this restriction, either by raising taxes or by reducing public spending in a very sharp and sudden adjustment.

The financial advantage of issuing debt is to avoid the upfront payment of the amount borrowed, which otherwise could take years of accumulated surpluses before a large and potentially beneficial investment could take place. This way, payments can be split over a period of several years, benefiting from the potential increase in economic growth brought forward by a strategic investment in productive public investment. In the particular case of public debt, it may also serve a morality principle. Since investment in public capital will be used by future generations, it may be argued that they also must share the burden. The contrary may also apply: present generations may use debt as a way to postpone their financial obligations and charge them to future generations. The struggle for a balanced sharing of the costs is of extreme interest, although outside the scope of this work. If debt has no productive use, it may have no effect on growth. In fact, since interest has to be paid on debt and taxes may have to be raised in order to do so, it could be adverse to economic

growth. These cases expose a potential link between public debt and growth, and such relationship is the core of the discussion on how much debt should the government be allowed to pile up.

In the next sections we will look at some empirical data showing how governments have been issuing debt, rooting some of the causes of the sovereign debt crisis; in addition, we will provide a brief overview of the components of the government budget, the inter-temporal constraint on the budget, sustainable and unsustainable fiscal policies; furthermore, the relationship between debt and growth is also studied and an overview of the last developments in the literature is given.

3.2.1 Empirical data

Historically, massive debt issuance programs have been associated with wars. In order to finance a surge in military spending, governments would issue bonds and sell them at face or discount value redeemable at a specified maturity. Looking at the long-term data series of public debt issued in the USA, depicted in Figure 3.1, it can be seen that the largest bond programs took place prior to the World War I and II, and have been declining ever since, at least up until the 80s. The Great Depression, although not a war, was also a time of great financing needs in order to support the New Deal Act — a program of public works of unprecedented scale — put forward by President Franklin D. Roosevelt[2].

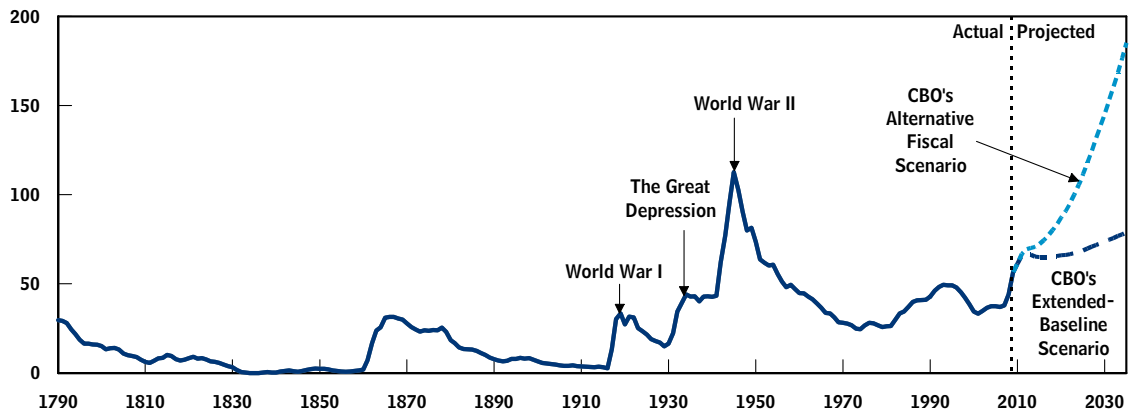

Figure 3.1: Public debt-to-GDP ratio of the United States of America, 1790-2010. Simulation for the path of the public debt-to-GDP until 2030. Source: CBO - Congressional Budget Office, Figure 1 of "Federal Debt and the Risk of a Fiscal Crisis", July 27 2010.

Over the years that followed the end of wartime, the debt-to-GDP ratio started to

[2]For an unorthodox perspective on how the New Deal might have delayed by 7 years the economic recovery of the USA, see Cole and Ohanian (2004).

decline. Aizenman and Marion (2011) point to a putative effect of both inflation and an expanding economy driving down the debt-to-GDP ratio. Growth, on the one hand, increases tax receipts and reduces the budget deficit. Inflation, on the other hand, depreciates the real value of debt, acting as a transfer from lenders to borrowers. In the case of government debt, inflation delivers a transfer from the bondholders to the state.

In the last couple of decades public debt has been accumulating again, despite no particular upsurge in government military spending. The situation has had a major impact in Europe, leading to a large sovereign debt crisis afflicting several countries like Greece, Portugal, Ireland and Cyprus, and probably causing a slow down of the growth rate of the remaining European Monetary Union (EMU) constituents. The response of the European Central Bank, the European Commission and the IMF comprised of *bail out* packages to Portugal, Ireland and Greece, a debt haircut to bondholders of the greek public debt and a *bail in* in the case of Cyprus, imposing an overnight tax on all savings accounts above a certain threshold.

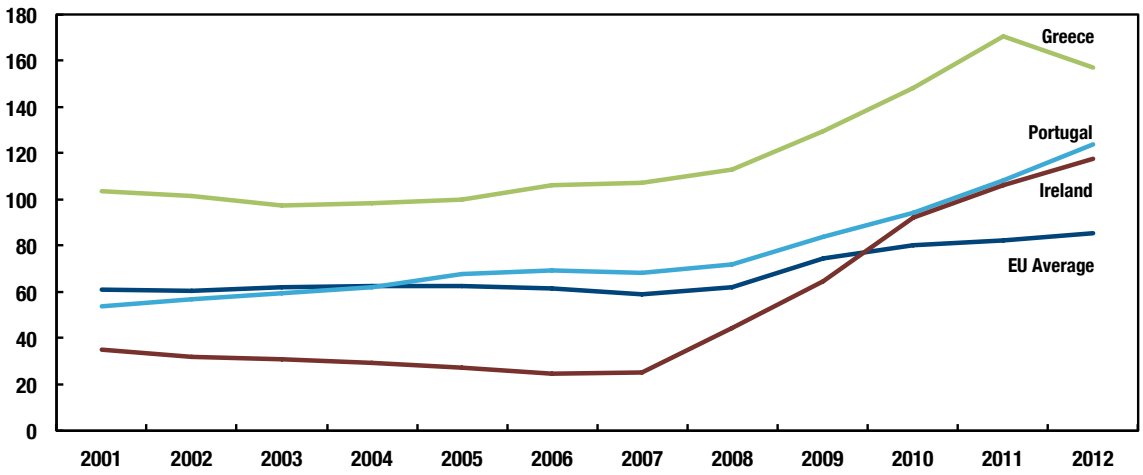

Figure 3.2: Public debt-to-GDP ratio of some of the countries recently afflicted by the sovereign debt crisis, along with the EU average. 2001-2012. Source: Eurostat, General government gross debt - annual data, 26 July 2013.

Figure 3.2 highlights the recent trajectories of the public debt-to-GDP ratios of some of the countries hit by the sovereign debt crisis, along with an average for the European Union. As the picture shows, the debt-to-GDP ratio aggravated after the latest financial crisis, exhibiting a sharp upward trend in the middle of 2010, after two years of extremely large fiscal deficits. For the year of 2010, Ireland recorded an all-time high budget deficit of 30.8% of the GDP. This happened as a result of the governmental injection of capital into its domestic banks.

The Maastricht Treaty and the Stability and Growth Pact establish a ceiling for the level of public debt, fixed at 60% of the GDP. However, countries like Portugal and Greece had long surpassed that limit before the crisis hit, with Greece exhibiting levels of debt

over 100% of the GDP when it joined the Euro.

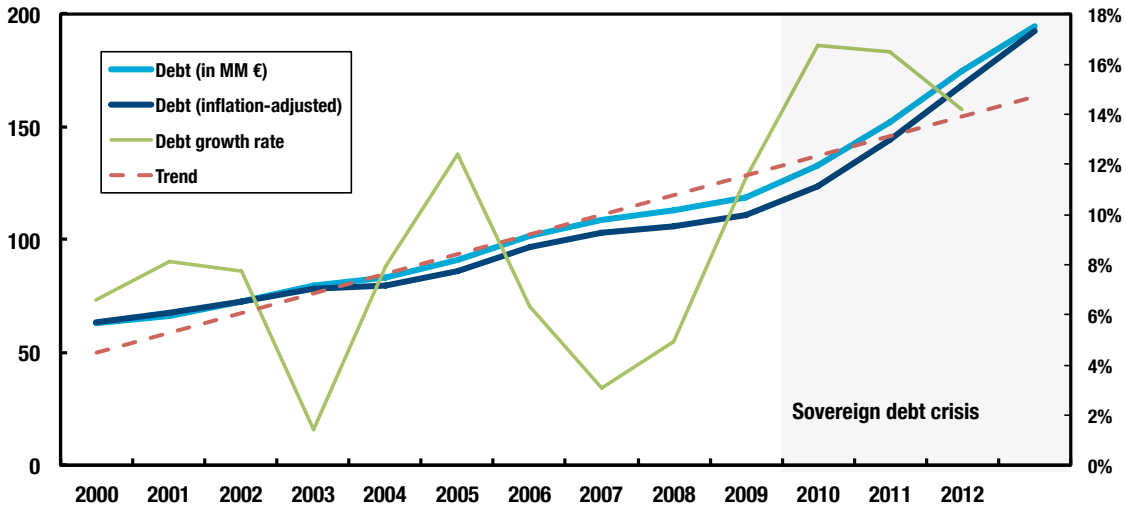

Figure 3.3: Level of the Portuguese public debt between the period of 1999-2012, measured in thousands of million Euros. Source: Agência da Gestão e da Tesouraria da Dívida Pública, annual series; Banco de Portugal, GDP deflator, annual series. Author calculations for the inflation-adjusted level of public debt.

Nonetheless, as Figure 3.3 shows, the financial crisis that started in 2007 was not the only culprit. For instance, in absolute levels and adjusting for inflation, the level of the Portuguese sovereign debt increased twofold between 2004 and 2011, jumping from 85 959 million Euros to 168 341 million Euros (constant prices). Also, public debt has been growing at positive yearly rates between 2% and 16% for the last decade.

Rising debt levels and soaring interest rates on bonds have again raised the red flag. The financial burden generated by the high levels of public debt might severely harm the growth rate of the economy, as it significantly raises the amount of money required to service it. Furthermore, and as it can be seen in Section 3.2.4.1, the unsustainability of the public finances of a country automatically increases the implicit and perceived risk of default, thereby increasing the risk premium demanded by investors. Such effects add further financial strain to an already precarious situation.

Figure 3.4 is revelatory of the upcoming financial effort that is required to service the outstanding public debt of Portugal, one of the countries currently under a bail out program. Assuming an inflation of $\pi = 1.5\%$ and a growth rate of $\gamma = 1.5\%$ per year and using the GDP in 2012 as the base year, the costs of servicing debt in the years of 2016 and 2021 will consume a considerable fraction of the GDP, up to 10%. Considering that a significant part of this debt is held by foreign investors and institutional creditors, as can be seen by the decomposition of the redemptions, a significant and critical amount of the output produced will be towards paying debt obligations, with money leaving the domestic economy.

Three signs indicate that the aforementioned combination of inflation and growth

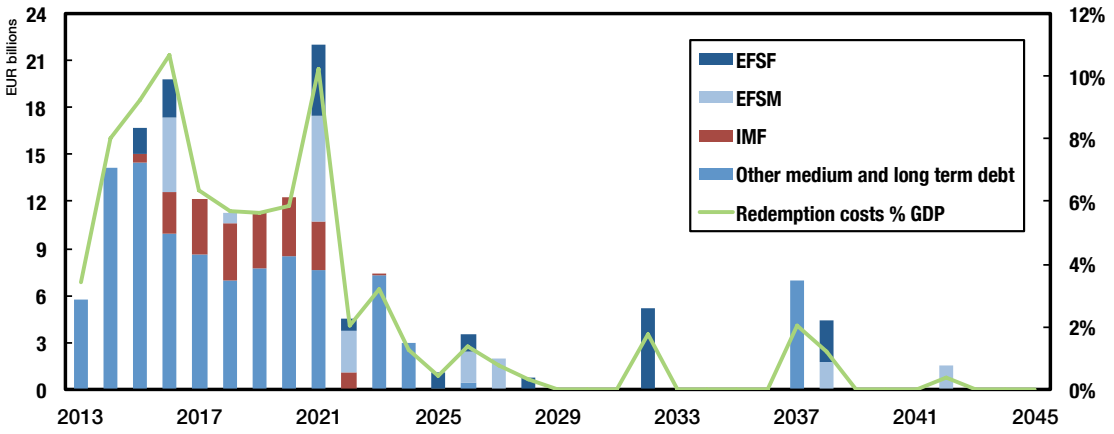

Figure 3.4: Outstanding public debt issued by the Portuguese government to be paid between the period of 2013 and 2045. Note: the redemption costs as a percentage of the GDP were calculated assuming an yearly inflation $\pi = 1.5\%$ and a growth rate of $\gamma = 1.5\%$ and using the GDP for 2012 (165 409 200 Euros) as the base year. Source: Agência da Gestão e da Tesouraria da Dívida Pública, annual series; PORDATA and author calculations.

observed in the past may not again be an option to tackle the current public debt-to-GDP ratios observed in these European countries. Firstly, independent central banks, like the ECB, follow a strict inflation-targeting policy, ensuring that inflation, measured by the Consumer Price Index (CPI) or any other price level index, is kept at low values, below the 2% threshold. In addition, in order to attract more investors, governments have been issuing bonds that are indexed to inflation and Figure 3.5 highlights the upward trend in the demand for U.S. Treasury Inflation-Protected Securities (TIPS). Secondly, the growth trend has been declining in most of the developed countries, exhibiting positive yet decreasing rates over the last decades. For the particular case of Portugal, it is interesting to observe a substantial declining of the growth rate of the economy, especially during the last decade. Figure 3.6 shows the percentage change in growth, year over year, from 1961 to 2011; a simple linear statistical regression shows a clear downward trend. Thirdly, nominal interest rates are close to their *zero lower bound* (ZLB), implying that expansive monetary policies to boost investment and reduce unemployment are spurious. This counter-cyclical monetary policy is an instrument used by central banks to boost aggregate demand in the short-run and generate a temporary surge in the output of the economy. The recovery of the economy would automatically help with the adjustment of the public debt, directly by raising the tax revenues and reducing deficits and, indirectly, by increasing the denominator of the debt-to-GDP ratio, the nominal GDP. Furthermore, the European countries that are members of the EMU gave up the monetary instruments able to foster aggregate demand and the GDP and, thus, reduce the debt-to-GDP ratio.

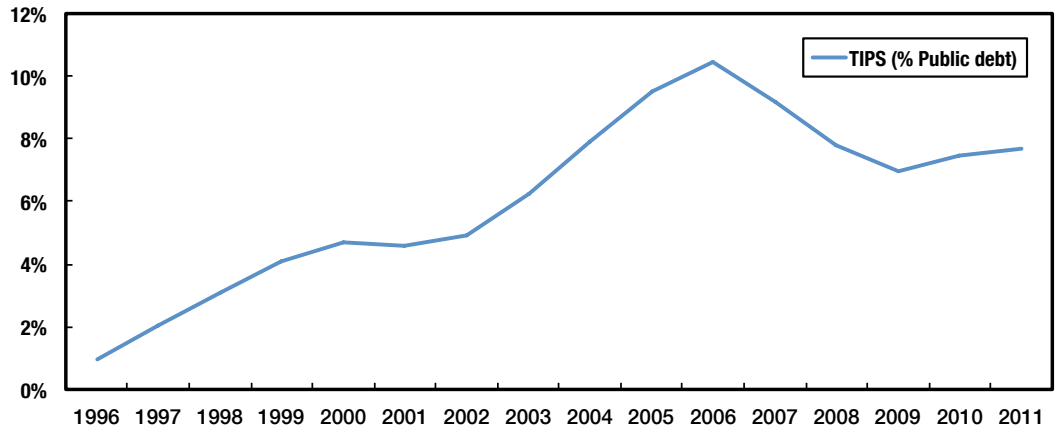

Figure 3.5: Share of the public debt of the USA held in Treasury Inflation-Protected Securities (TIPS). Source: U.S. Treasury, U.S. Securities Outstanding.

Considering the low growth rate and stable inflation trends registered for the Portuguese economy and also observable in other developed countries, reducing the level of public debt will require resorting to a fiscal consolidation by raising taxes, decreasing public expenditure or a mix of both. Section 3.2.4 briefly discusses other available approaches that fall under the domain of political economy to curb down debt. The next sections expose the theoretical underpinnings of the dynamics of the stock of public debt.

3.2.2 Debt and growth

Academics, and non-academics alike, have recently witnessed an upsurge in the debate over the effects of public debt on economic growth. Such intensified discussion has also unleashed a considerable amount of theoretical and empirical literature trying to find a causal relationship between these two facets of a dynamic economy. In fact, such question is of the utmost importance due to its indirect effects on the fiscal sustainability of public finances and, consequently, on the general welfare of society.

Establishing an implicit relationship between debt and growth is quintessential for ascribing a proper strategy to curb the trajectory of debt. In the neoclassical growth model, the functional form of the endogenous variables that are responsible for economic growth is neutral towards debt. This results from the fact that in a closed economy all assets are held by families, comprised either of private capital or public debt, and so servicing debt in fact reverts back to the economy in the form of new investment in productive capital or welfare-maximizing consumption. Consider, for instance, Maebayashi et al. (2012). Taking out the fiscal policy rule, debt can grow indefinitely without having a direct effect

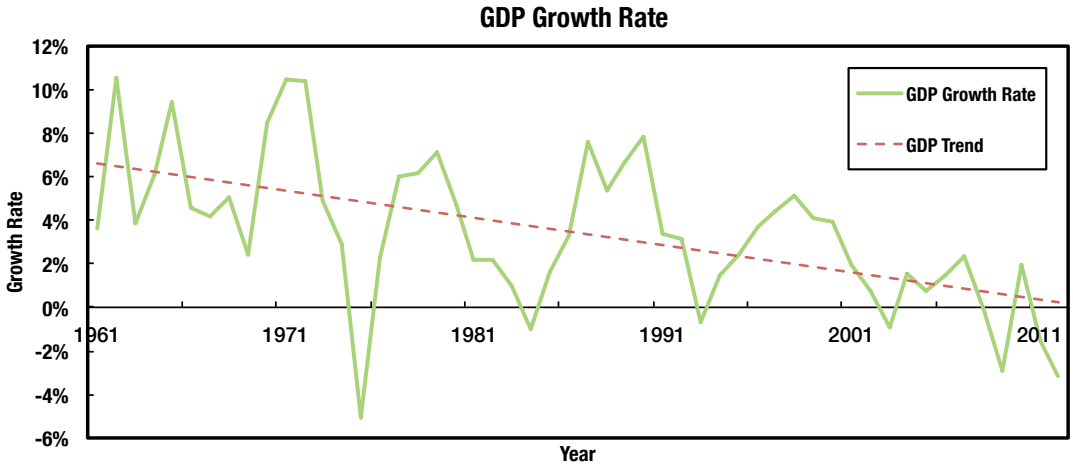

Figure 3.6: Growth rate and statistical trendline of the GDP of Portugal for the years between 1961 and 2012. Source: PORDATA.

on the accumulation of capital. Only when a policy rule binding the budget surplus/deficit and public debt is chosen, does debt start affecting growth by the taxes channel. In a world characterized by open economies, international capital markets, free flows of capital and risk premia on sovereign default, such models fall short on fully depicting how the capital markets react to ever-mounting increases in debt, and so a fiscal feedback rule is necessary to define how fiscal policy should react to a variation in the stock of public debt and to enforce that the government remains solvent.

Four possible causal relationships from debt to growth can be put forth *a priori:*

1. **Debt is neutral**. It has no effect on the growth rate of the economy, neither increasing or decreasing it;
2. **Debt has a positive effect on growth**. It allows public investment in infrastructure to take place, leading, in turn, to positive externalities in the production sector. The financial burden caused by borrowing is offset by the ever increasing tax revenues made possible by the effect;
3. **Debt has a negative effect on growth**. This one is probably the most worrisome possibility as public debt may lead to lower growth rates and an unsustainable fiscal stance due to rampant interest rates, a situation usually seized with fiscal austerity, inflation or default[3]. In this scenario, public debt is probably not being put to serve productive investment but rather spurious consumption;
4. **Debt can feature all of the above at different points in time**. This is a feature arising from a likely nonlinear, U-shaped relationship between public debt and growth,

[3]Cochrane (2011) provides an interesting discussion of the short and long-term consequences of inflation and how it could lead to a run on the currency.

expressed as a non-monotonic connection between variables. Such scenario would considerably encumber the task of figuring out the connection, if any, between public debt and economic growth.

The question remains. What is the effect of public debt on growth?

3.2.2.1 Mathematical analysis

At this time, it might be useful to proceed with a strictly aprioristic analysis of the problem, laying down the core concepts before conjecturing a causal link. From a purely mathematical perspective, this is a standard case of a relative change. If a given variable decreases at rate α, it will have to grow at a rate equal to $\gamma = 1/(1-\alpha)$ to go back to its original level. It is immediate to see that for $\alpha \in [0,1)$, then $\gamma > \alpha$ for the case of linear growth, which can be assumed to be the case over the cycle. This effect is present in the mathematical relationship between public debt and output. Public debt can be approximated by α and the growth rate of the output by γ.

To see the impact of this property, consider an economy producing 100 units of goods at time t, that is, $Y_t = 100$. Suppose that servicing debt costs the equivalent to 7.5% of the aggregate output, such that $rB = 7.5$. If the markets equilibrate in a closed economy, assets held by families are either capital or bonds, so the servicing of debt for a given tax rate does not affect the level of aggregate demand as bonds can easily be converted into new investment in private capital. Let us neglect that for a moment and assume that debt is only held by international investors, implying that its payment will in fact reduce the size of the national output by rB units.

In order to ensure that the output level remains constant, i.e. $Y_{t+2} = Y_t$, the economy at time $t+1$ needs to grow enough to offset the debt service. Output growth is assumed to be exponential but since we are dealing with a short period of time we can approximate it to a linear form without much loss of generality. So, the growth rate γ will have to be equivalent to $Y_t = Y_{t+2} = \gamma Y_{t+1}$. Solving with respect to γ we obtain $\gamma = Y_t/Y_{t+1} = 100/92.5 = 1.081$ that implies a growth rate of 8.1%, higher than the percentage of resources that was taken out of the economy (7.5%). Holding everything else constant and ignoring all the other interactions within the economy, we can generalize this result by saying that if rB/Y is reduced from the economy, the growth rate γ that puts it back at the original level is larger than the debt service to GDP ratio rB/Y by the amount $\gamma' - \gamma$, with $\gamma' = 1/(1-\gamma)$.

The above line of reasoning is necessarily true if, as previously mentioned, the problem is reduced to the exploitation of the mathematical properties of the variables. Incorporating economic evidence like the positive and significant effect of public capital inputs in the private sector output or the possibility of bonds being held by domestic

investors would draw the quantitative conclusion less meaningful. Servicing debt to domestic bondholders eases the budget constraint of families, allowing them to reinvest in capital or consume, which could boost growth by itself. Despite the fact that incorporating such economic properties might invalidate the conclusion, it does not render the exercise just conducted useless. Notice that servicing debt is at odds with growth since it can potentially drain resources from the domestic economy, and if that is not the case, it has to be offset by a mechanism capable of compensating for the differential. Thus, if a relation exists, it has to be stronger than the *a priori* assumption that $\gamma > rB/Y$. Economically, this means that the marginal productivity of debt has to be high enough to compensate for the yield necessary to service its costs.

3.2.2.2 Theoretical and empirical evidence

The unfolding of a paper by Thomas Herndon (2013), rebutting the evidence found by Reinhart and Rogoff (2010) that pointed to a negative effect of public debt on growth when a 90% threshold of the public debt to GDP ratio is surpassed, generated a huge turmoil in the academic community. According to Reinhart and Rogoff (2010), data shows a steady negative correlation between debt and growth and a sharp drop of the mean growth rate from 3% to -0.1% as the 90% threshold is attained. The critique does not go against the negative connection found between debt and growth but rather against the conclusion that a sudden jolt to growth takes place as the 90% line is crossed. They find a much smoother decrease from 3% to 2.2% in the mean growth rates, as depicted in Figure 3.7.

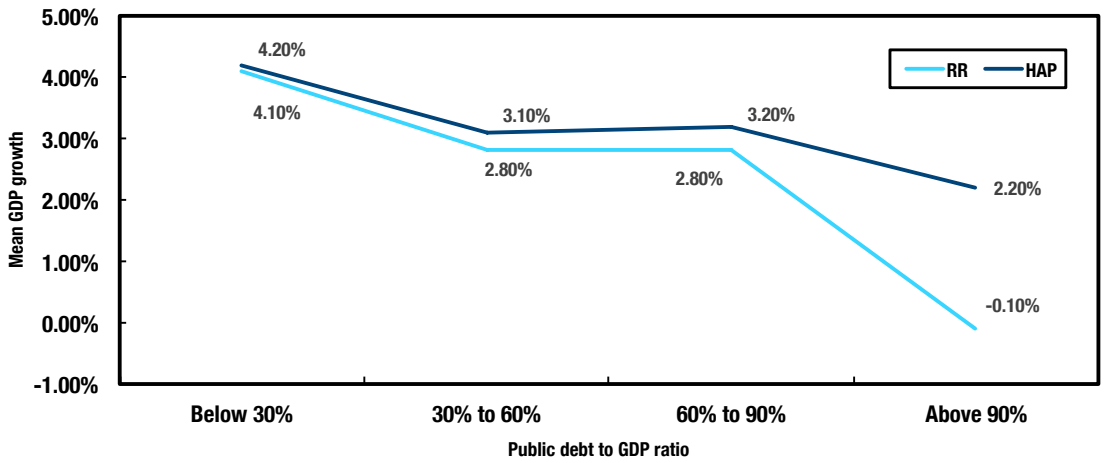

Figure 3.7: Empirical evidence on growth and the debt-to-GDP ratio obtained by Reinhart and Rogoff (2010) (RR) and the revised calculations by Thomas Herndon (2013) (HH).

It is outside the scope of this work to cover the full extent of the discussion, but it is certainly insightful to shed a light on the topic. Literature is mixed. From an empirical standpoint, the work of the IMF economists Kumar and Woo (2010) points to a stance

similar to that of Reinhart and Rogoff (2010). They observe a negative relationship between public debt and growth and quantitatively they find a decrease in 0.2% in the rate of growth for each increase of 10% in the pre-existing level of public debt. Furthermore, they find an alarming nonlinearity for high levels of public debt, which could be responsible for a disproportionate effect on the subsequent growth.

The economists Cecchetti et al. (2011) at the Bank for International Settlements reached a similar conclusion. Their results suggest that public debt plays a very negative effect on growth for levels over 85% of the debt-to-GDP ratio. For such cases, the recommended policy would be to act rapidly and decisively, which can be translated from the policy jargon as performing an abrupt (*"cold-turkey"*) fiscal consolidation. This very same conclusion is also shared by Baum et al. (2012) at the European Central Bank. Their empirical results find a pattern very similar to that widely reported in the literature. The short-run impact of small levels of public debt of up to 67% of the debt-to-GDP ratio is positive and statistically significant, exhibiting a detrimental effect on growth for levels over 95% of the debt-to-GDP ratio. Moreover, they also find a link between debt-to-GDP ratios above 70% and the surge in the risk premium demanded by potential bondholders.

Panizza and Presbitero (2013) take a more holistic approach and review the whole body of literature covering the connection between public debt and growth. They find ambiguous answers in the theoretical literature and so they resort mainly to the empirical evidence. Their conclusion is that although several studies suggest a negative correlation between public debt and growth, none of them makes a clear case for a causal relationship *from* debt *to* economic growth. They also find out that the existence of thresholds and a non-monotonic relationship between debt and growth is not robust to small changes, and therefore conditions the results obtained.

The property of being monotonic implies that a positive link between debt and growth cannot suddenly turn into a negative relationship, and *vice-versa*. The problem can be restated as follows: does an inverse U-shaped functional form exist between public debt and growth? Greiner (2012) puts the hypothesis to the test under a framework of an endogenous growth model with public capital and public debt. According to the theoretical model, the non-monotonic relation only holds for the particular case of exogenously fixed public deficits to equate the public investment at each point in time. The corollary of such result is that under a more general debt policy, a negative monotonic relation between debt and growth exists, implying that increasing levels of debt-to-GDP cause a decrease in the long-term growth rate.

Some other recent studies like IMF (2012) are in line with the conclusions reached by Panizza and Presbitero (2013). Their results seem to indicate that no clear threshold for the debt-to-GDP ratio exists, although their policy implications do not refrain from the

need of a fiscal adjustment to preclude further debt overhang. The juxtaposition of that suggestion comes in the form of a recent review by Reinhart et al. (2012) of their previous study (cf. Reinhart and Rogoff, 2010), incorporating updated evidence and focusing on the above 90% threshold debt-to-GDP ratio, again reiterating a clear negative connection between high ratios and a lower or negative economic growth rate.

Notwithstanding, the conundrum still remains. Reaching a conclusion would help in establishing the optimal set of policies for adjusting the levels of public debt. Since no final answer has been attained yet, this work focuses on the reduction of the levels of public debt and in finding optimal adjustment policies to accomplish it. We follow this approach given the recent sovereign debt crisis triggered by ever-increasing levels of public debt. The next sections cover the theoretical and empirical aspects of fiscal consolidations and, then, concentrate on how debt adjustment usually takes place before introducing an optimal debt adjustment policy under a framework of endogenous growth.

3.2.3 Debt dynamics

The overview of how public debt evolved over time provided previously exposes the dynamic nature of public debt. Likewise with other stock variables, public debt increases with positive flows, decreases with negative flows, and depreciates with inflation.

Although the law of motion for the stock of public debt may be easy to understand in terms of positive and negative flows, the functional relationship between variables, if any, — i.e., how and to which extent the variation of one variable affects the others — is not, and it is still subject to great dispute amongst economists. Therefore, an extensive introduction to debt dynamics is required. We will provide a mathematical introduction to the time-dependent equation of public debt and consult the literature for theories extrapolating links between variables.

Governments usually levy taxes capital, labor, and consumption, although it can also obtain a stipend from some other form of indirect taxation, like on alcohol, tobacco, oil, gas emissions, customs, financial transactions, property taxes, etc. Taxation is raised in order to finance public expenditure, G, which can be decomposed in public investment, I_g, public consumption, C_g, and social transfers, C_s. For simplicity, we assume the tax revenue to be a share of the output of the economy, $T = \tau Y$. We can then extract the **primary balance** accounting identity, PB. It is the difference between tax revenues and public expenditure.

$$PB = T - G = \tau Y - I_g - C_g - C_s. \tag{3.1}$$

When $PB > 0$, the government exhibits a primary surplus, meaning that tax revenues are in excess of public spending. For $PB < 0$, the opposite occurs and the government is

facing a fiscal deficit. The government is said to run a balanced budget when revenues equal public expenditure ($PB = 0$).

When the government faces a fiscal deficit ($PB < 0$), the differential has to be financed by offering government bonds, B, in an amount of no less than PB. In order for investors to be interested in holding assets not as liquid as paper money, bonds pay a fixed interest rate, rB, on its face value, i.e., the initial selling price. Since capital is usually taxed, earnings from bonds are no exemption, the government earns an additional τrB at each period. The following accounting identity describes the law of motion of the stock of public debt:

$$\dot{B} = rB - \tau rB - \tau Y + I_g + C_g + C_s + SFA, \tag{3.2}$$

where SFA stands for the stock-flow adjustment variable, i.e., the statistical increase in debt that is not explained by direct budget deficits, which in theory should be zero. The identity given by Equation (3.2) can be rearranged in order to obtain the dynamic budget constraint on the government:

$$\dot{B} = rB(1-\tau) - PB. \tag{3.3}$$

Note that we are assuming a closed economy and therefore all interest paid remains in the economy. No spill outs occur. Under such circumstances, when debt is solely held by domestic investors, no resources drain out of the economy, but this may not always be the case. In fact, this is not the only factor that may constrain how public financing is conducted. Driscoll (2003) covers some of the issues a government may face. They include i) the need to remain solvent; ii) the distortionary effect of taxes on the supply-side and on tax evasion; iii) time lags arising from delays in implementing policies; iv) credibility problems from time-inconsistency of announced policies and v) limitations imposed by political or legal boundaries.

Equation (3.3) is the book-keeping identity that governs public finances. If $\dot{B} > 0$, the government is issuing bonds, adding to the stock of public debt. On the other hand, if $\dot{B} < 0$, the government has a budget surplus and is thus able to reduce the amount of public debt in circulation.

In order to fully assess the dynamics of public debt another fundamental variable, the gross domestic product (GDP), must be included. Indeed, how the output evolves affects all variables of the book-keeping relation identified in Equation (3.3). Tax revenues are raised by the growth of output (recall that $T = \tau Y$), contributing to an automatic decrease of the public deficit, $-PB$, when everything else is held constant. To do so, we derivate

b $(\equiv B/Y)$ with respect to time and replace accordingly in Equation (3.3) to obtain:

$$\dot{b} = d + (r - \gamma)b, \tag{3.4}$$

where d is the ratio of the primary balance to the GDP and γ is the GDP growth rate.

Looking at Equation (3.4) and assuming the real interest rate, two particular cases stand out when evaluating the path of the public debt:

$r > \gamma$: when the real interest rate exceeds the real growth rate of the economy then $b \to \infty$ when $t \to \infty$. Debt is labelled as explosive once, even with a balanced primary budget, the debt-to-GDP ratio rises endogenously due to debt servicing ("*snowball*" effect). In practice, investors realize that the government is running insolvent and so no longer buy treasury bonds, eventually reselling them in the secondary market, causing prices to plunge and the market interest rates to rise, which only further aggravates the problem. Ignoring the scenario of seignorage (i.e., paying for bonds with high powered money creation), the government may default, raises taxes T and/or decreases public expenditure G in order to bring debt back to a sustainable position;

$r \leq \gamma$: in this case, accumulated debt pays itself and, given G, fiscal deficits converge to balanced budgets to the ever-increasing tax revenues. If so, $b \to 0$ when $t \to \infty$. For the particular case that the interest rate equals the growth rate, $r = \gamma$, then $b = b_0$ for all $t \geq 0$. Under such circumstances, the government continues solvent for the times to come and debt is on a sustainable path, i.e., obeying the intertemporal budget constraint.

By considering the Fisher equation $(r = i - \pi)$ and substitute in Equation (3.4) we obtain the differential equation that also takes into account the effect of nominal inflation

$$\dot{b} = d + (i - \pi - \gamma)b. \tag{3.5}$$

How the interaction between inflation and the stock of public debt occurs will depend on several factors. If public debt bonds are indexed to inflation as with TIPS, when nominal interest rates rise either through inflation or by increases in the real interest rate, debt servicing costs expand, which are positive flows in the equation of public debt, thereby adding to it. If, on the contrary, bonds are not indexed to inflation, this monetary effect will depreciate the real value of money, making it easier to pay for the outstanding debt obligations. When government or the central bank purportedly increase the growth rate of the money supply in order to cause a rise in inflation, it is attempting at the monetization

of debt, reducing the real value of its financial commitments. This process is only possible when bonds are issued in the domestic currency, which sometimes is not the case, especially for countries with fragile domestic currencies or an history of currency devaluations. Seignorage (inflation through high-powered money creation) can be, as a matter of fact, a strategy for debt adjustment, although at the expense of outstanding creditors (an "inflation tax" on bondholders).

3.2.3.1 Intertemporal budget constraint

Since all borrowed debt needs to be paid, a government intertemporal budget constraint must be imposed. It requires the present value of government expenditure to equal the present value of taxation. Simplifying to a discrete, 2-period framework, it means that the following must held and thus,

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r}. \tag{3.6}$$

The 2-period budget constraint from Equation (3.6) can be translated into a continuous infinite-horizon version:

$$B(0) = \int_0^\infty e^{-\int_0^\mu (1-\tau)r(\nu)d\nu} S(\mu)d\mu \leftrightarrow \lim_{t\to\infty} e^{-\int_0^t (1-\tau)r(\mu)d\mu} B(t) = 0 \tag{3.7}$$

In practice, one way of ensuring that the intertemporal budget constraint is verified is by defining a fiscal policy rule, i.e., a public finance rule of thumb that sets a particular feedback function on the government budget as a response to changes in the public debt.

3.2.4 Debt adjustment

Equation (3.5) contains all the flows and variables that can change the stock of public debt and so it is a good starting point for drawing strategies to surmount alarming levels of public debt. As previously seen, the stock of public debt increases with flows of primary budget deficits, d, and with nominal debt costs, rb. The real interest rate, r, increases the financial burden of servicing debt, and inflation, π, can either help or hinder how these flows vary, playing a dual role. Growth exhibits a less ambiguous effect as it is always beneficial insofar as it increases tax revenues, therefore helping with balancing the government budget. At the same time, it increases the denominator of the debt-to-GDP ratio. Less clear is, as we have seen, how debt may affect growth.

Improving the fiscal budget so as to reduce deficits or promote surpluses is termed fiscal consolidation. That deals with the primary budget, d, and is one instrument of public

finance for reducing the level of public debt. Whether to increase taxes or cut government spending is a choice concerning economic policy. We can divide the available strategies for adjusting public debt into two separate strands of the literature, economic theory and political economy. Under the umbrella of economic theory fall the monetary and fiscal policies aimed at putting the economy back on a sustainable path. They work by either proceeding with a fiscal consolidation, monetizing debt through inflation or promoting growth-enacting policies. Other strategies like negotiating a partial or a full default of the outstanding debt, trying to reach a debt relief or coming to terms on longer maturities along with reduced interest rates concerns the domain of political economy. They are not exclusive. In certain cases where it is no longer financially possible to recover from a debt overhang, reaching an agreement on a debt haircut or debt pardon may be the only option available, although it may come at considerable economic and reputation costs (Borensztein and Panizza, 2008).

Although monetizing debt, pushing for a higher growth rate or negotiating reduced interest rates may be valid ways of reducing debt, they are instruments not under the direct control of governments. Countries that joined a monetary union under a fixed exchange-rate between members, like the EMU, no longer control monetary policies capable of generating inflation, and they are severely limited in using expansionary fiscal policies due to financial constraints. Moreover, modern central banks conduct inflation-targeting policies, keeping tight supervision of variations in the price-level indexes. This severely restricts the use of policies that rely on inflation to reduce debt-to-GDP ratios. As for economic growth, structural reforms usually take years to bear fruit and depend on so many political factors that the degree of control a government exerts over growth rates is also severely limited. Moreover, it does not tackle the most immediate needs of an out of control budget. Likewise, negotiating changes in interest rates or the maturity of the contracts with the bondholders is also a complex and not fully controllable task and so it cannot be used as a long-lasting and reliable instrument for adjusting debt.

Another valid strategy consists in promoting economic growth by performing structural reforms and other growth-enacting policies. But, as we have Seen in section 3.2.2, the theory of how debt affects growth is far from settled. In fact, the relationship between debt and growth can be even more puzzling than the relationship between debt and inflation, so far making it very difficult to sketch a debt adjustment program solely focused in promoting economic growth. Evidence is clear in showing that the growth rate of the economies is not constant (see Figure 3.6). Fluctuations have been consistently observed in the growth rate, an empirical fact that by itself invalidates the conjecture of a monotonic and always positive constant growth rate, although the secular trendline appears to be so.

Reducing the costs of servicing debt by either defaulting partly or in full, or by rene-

gotiating the interest rates and maturities, is also possible. Defaulting on debt and renegotiating is under the domain of political economy. As for the interest rate demanded by the bondholders, despite being usually set by the government in the initial offering, it is benchmarked against the yield-to-maturity obtained in the secondary markets, which is subject to the law of supply and demand. This implies that besides not being constant, the interest rate is also not solely under the direct influence of governments. Some of the factors that might influence the yield required by rational investors when buying assets include variations in wealth, expectations about the future interest rate or future inflation and a preference for liquidity[4], all of them applying to some extent to the particular case of public debt bonds. The one factor that is paramount to our analysis is that of the risk premium. The riskier the prospect, the higher the expected return demanded by investors will be in order to compensate for a potential loss. Uneven and increasing levels of debt may lead to an unsustainable position, where servicing debt would be unattainable and could cause a depletion of the whole of the economy if carried on.

Considering this, the objective of adjusting public debt in the short-run can only consistently rely on controllable instruments like that of fiscal policy. Following this rationale, adjustment and assistance programmes like those promoted by the IMF usually depend on structural reforms to promote a higher long-run growth rate, but require short-run fiscal consolidation to reduce budget deficits by tapering public consumption or public investment and/or by increasing tax rates.

The present work abstains from considerations and strategies pertaining to political economy, and so the scope is restricted to economic policies. Moreover, by focusing on countries belonging to a monetary union, like the EMU countries which no longer possess the ability to decide on matters of monetary policy, the study is further limited to the optimal fiscal consolidation policy, which we will consider for the remainder of this work.

3.2.4.1 Fiscal sustainability

Let us consider the following definition of fiscal sustainability.

Definition 1. *For an initial level of public debt $B(0)$, considering the expected growth rate $E_t(\gamma)$ and the expected interest rate $E_t(r)$ and a given model of the economy M, the fiscal policy $M(G(\cdot),\ T(\cdot))$ is sustainable if the government remains solvent.*

This definition is the requisite to ensure that debt is sustainable and can be evaluated in a simple question: is the debt-to-GDP ratio compatible with the government's intertem-

[4]For a more extensive reference of factors influencing interest rates, check Mishkin (2007).

poral budget constraint? If sustainable fiscal policies are to be pursued then debt remains under control and will not reach exploding levels.

How fiscal deficits may affect fiscal sustainability is better grasped with the following example. If it is assumed that the interest rate r, the growth rate of the economy γ and the budget deficit d are constant, the the debt trajectory can be obtained by solving the linear differential equation (3.4):

$$b(t) = \frac{d}{\gamma - r} + \frac{b_0(\gamma - r) - d}{\gamma - r} \cdot e^{(r-\gamma)t}. \tag{3.8}$$

From Equation (3.8) it can be seen that perpetual budget deficits do not necessarily lead to fiscal problems, as long as $\gamma - r$ is large enough to offset deficit d. Unfortunately, neither the growth rate nor the interest rate are constant, so such budget rule cannot be binding. On top of that, the government has no direct control over these two variables, which can suddenly vary. Hence, if for a given period $r > \gamma$ is observed then the only way of bringing debt back to a sustainable trajectory is by conducting a fiscal consolidation.

By using Definition 1 along with Equation (3.8) it can inferred that the yearly government budget has to be adjusted in order to compensate for deviations in the interest rate r and in the growth rate γ. Note that, the adjustment does not have to occur in every period if in the long run the dynamics remain sustainable, but the longer it takes to adjust, the steeper the fiscal consolidation has to be.

Any given optimal debt adjustment strategy has to attend to Definition 1 to ensure that in the terminal time the economy is in a fiscal sustainable state.

3.2.4.2 Debt sustainability

Building on the Definition 3.2.4.1, we can define sustainable debt as a situation where the debt-to-GDP ratio stays constant or decreases and, thus, debt grows at a rate lower or equal to the output of the economy.

Definition 2. *Public debt is sustainable if the growth rate of level of public debt, $\frac{\dot{B}}{B} = \gamma_B$, is less than or equal to the growth rate of the economy $\frac{\dot{Y}}{Y} = \gamma$, such that $\gamma = \gamma_B$ or $\gamma > \gamma_B$, which implies that the debt-to-GDP ratio either stays constant or decreases.*

Without surprise, the definition of public debt depends on the previous definition of fiscal sustainability (see Definition 1). Since fiscal policies command the growth rate of the level of public debt. Following the dynamics of the public debt in Equation (3.4), in order for $\dot{b} < 0$ to hold, it implies that

$$b \leq \frac{-d}{r - \gamma} \Leftrightarrow \frac{B}{Y} \leq \frac{S/Y}{r - \gamma}. \tag{3.9}$$

Equation (3.9) states that in order for debt to be sustainable, the debt-to-GDP ratio, $b \equiv B/Y$, must be less than or equal to the ratio of the primary budget surplus, S/Y, divided by the interest rate minus the growth rate of the economy, $r - \gamma$. This means that the debt-to-GDP ratio has to grow at a rate inferior to the absolute value of the budget when detrended of the interest rate of the growth of the economy.

3.2.4.3 Fiscal consolidation

Fiscal consolidation consists in reducing the underlying fiscal deficit of a government, thus decreasing or inverting the positive flow effect that the budget deficit plays on the evolution of public debt. Like with all other economic problems, the way variables are interconnected may completely alter *how*, *when* or *if* fiscal consolidation should be conducted.

Equation (3.4), that describes the law of motion of the stock of public debt, exposes one particular scenario where no fiscal consolidation would be required, from a strictly financial perspective: that of the exponential and continuous growth of the economy, capable of mitigating the costs of servicing debt and paying off primary deficits. That is, when the growth rate is higher than the real interest rate and enough to offset fiscal deficits.

This is seldom the case, so an optimal strategy for conducting fiscal consolidation is required in order to adjust the levels of public debt. Mauro et al. (2009) also find strong evidence pointing in that direction: since higher public debt raises solvency risks and increases the borrowing costs, fiscal consolidation is necessary to curtail the negative effect of budget deficits and government debt levels on the interest rates. It is then quintessential to analyze not only how to keep the public finances in a sustainable path but also how to bring them back into track when adverse negative shocks or misadjusted policies force them out of a sane trajectory. Failure to do so can eventually lead the country to extreme austerity, inflation, or the possibility of a sovereign default.

Like the effect of debt on growth (see Section 3.2.2.2), the theory of optimal fiscal policy, either when conducting fiscal stimuli or doing fiscal adjustment, is also surrounded by much controversy. The study by Alesina and Ardagna (2009) probably is one of the most cited studies when referring to the dispute of whether fiscal consolidation can be expansionary, or not. When implementing fiscal stimuli, they are bold in advising for tax cuts instead of spending increases. As for the subject at hand, fiscal consolidation, they conclude that adjustments based upon spending cuts instead of tax raises are more effective at reducing the levels of public debt and curbing deficits. Moreover, they also state that an adjustment based on spending cuts might not have recessionary effects due to *crowding-in* effects.

The work of Baldacci et al. (2010) seemed to point in a similar direction. They found that successful fiscal consolidations obtained mainly by reductions in the public expenditure, along with structural reforms aimed at promoting growth and a supportive monetary policy, when available. Notwithstanding, they also find benefit from increasing tax rates for those countries in need of large fiscal adjustments. According to the European Comission (2007), expenditure cuts are more effective, as they are usually followed by reforms that improve the public services' efficiency, reducing squander of taxpayers money. In contrast, Leigh et al. (2007) warn that increasing taxes may postpone structural reforms, signaling a weak commitment to proceed with such endeavor. As a matter of fact, consistent resemblances can be found between this research paper and the practical implementation of the Memorandum of Understanding (MoU) undersigned by the Portuguese government and the members of the Troika (ECB, EC and the IMF). The first two years consisted fundamentally of raising taxes, with no notorious and significant structural reforms undertaken, other than a change to the rigidity of the labour market legislation. More recently, IMF economists have retreated from their initial stance. Blanchard and Leigh (2013) present the case of larger fiscal multipliers that increase the negative effect of the fiscal adjustment on the output growth. This view is also shared by Auerbach and Gorodnichenko (2010), confirming that fiscal multipliers tend to be larger when in times of recession. If that is effectively true, fiscal consolidation is not expansionary and could potentially be self-defeating, contributing to an increase of the debt-to-GDP ratio through a reduction of the GDP denominator. Another cause for such effect is the phenomenon of *hysteresis*, an hypothesis put forward by DeLong and Summers (2012). The analysis is conducted under a scenario of nominal interest rates at their zero lower bound, rendering the monetary policy ineffective. Under such scenario, and because short term cycles might influence long term trends, namely causing short-termed unemployment to persist, the authors argue that consolidation should be postponed to better yet distant times. Alternatively, fiscal stimulus might finance itself since the unemployed would rejoin the active labour force and make a positive contribution through taxes instead of requiring social transfers.

Owyang et al. (2013) elaborate a contrasting view on the subject. Using new quarterly historical data, they find no evidence whatsoever that fiscal multipliers in the US tend to be higher during times of recession and high unemployment, and are inclusively below unity. They do find some evidence of higher multipliers during times of slack in Canada. If such hypothesis holds true then conducting fiscal consolidation during a recession is no costlier than doing so afterwards. Consider, for instance, the work of Almeida et al. (2011) for a small euro area economy. Under a New-Keynesian framework, they find that, in the long run, fiscal consolidation increases both consumption and output. On

top of that, Barrios et al. (2010) find evidence pointing to a vigorous and short-termed fiscal consolidation, also dubbed *"cold-turkey"* or *"cold-shower",* as a means to rapidly shift the economy towards a higher growth path.

Again, no conclusive position has been attained. The strongest case against a fiscal adjustment when nominal interest rates are at their zero lower bound comes from the already referenced study by DeLong and Summers (2012). This study does not take into account how bond investors perceive increasing fiscal deficits. Mauro et al. (2009) shed a light on the subject, claiming that fiscal deficits have a negative influence on interest rates. If that is so, investors either have to be fully convinced that fiscal deficits will not add to the risk of holding bonds or the strategy is simply unfeasible.

This work relies on the explanatory properties of the proposed model when solving for an optimal adjustment trajectory of the public debt in order to find out which policy leads to better results, if a primary fiscal surplus or fiscal deficit, or perhaps a combination of both. No single 'one-size-fits-all' strategy will ever exist, since a lot of factors condition the economy. For instance, EMU countries are constrained by an independent central bank maintaining a no bailout policy. Furthermore, some particular countries have restricted access to the bond markets or are under intervention, meaning that they have no self means of pursuing fiscal stimuli even if desirable. The study here reported is best appropriated to such cases, where fiscal consolidation is not optional but rather compulsory.

3.2.4.4 Fiscal policy rules

For the purpose of this work, we define *fiscal policy* or *budget rule* as a permanent constraint on fiscal policy, following the definition of Groneck (2009). Such rule exerts a direct influence on the adjustment path of the stock of public debt. Technically, the budget rule conditions the shape and smoothness of the adjustment path, limiting the set of feasible solutions. For instance, if the rule allows for running budget deficits, the adjustment might be conducted over a longer timespan. Alternatively, a balanced budget rule tends to lead to a more rapid adjustment in order to take the economy back to its long-run trend as fast as possible, even if that comes at the expense of an extreme and harsh fiscal consolidation in the short-run, with a sudden increase in tax rates, a rapid decrease in public spending, or both.

Usually, an indirect relationship between the stock of public debt and growth arises from these models the moment a fiscal budget rule is defined. Such rule is set as an exogenous and imperious impediment on the government to continuously run fiscal deficits and accumulate debt indefinitely. The rules found in the literature are most commonly *qualitative*, in the sense that they define *how* fiscal policy should react to a change in the level of public debt, even if an additional quantitative boundary applies. The most com-

mon rules either (i) set a target level for the government bonds (Futagami et al., 1993); (ii) force the government to run a balanced budget fiscal policy (Greiner, 2008); (iii) define a fixed deficit regime or (iv) a golden rule of public finance (Groneck, 2009). For a given constant tax rate, these rules influence the composition of the government expenditures and its variation over time[5], in fact imposing a response function of the government budget to the variation of the debt-to-GDP ratio. *Quantitative* rules, on the other hand, do not impose a particular functional form but only an upper or lower bound under which the budget may operate.

The choice of the budget rule to use has growth and welfare implications, as noted by Greiner (2008), since such rules affect the transitional dynamics of the economy, i.e., the adjustment trajectories along the Balanced Growth Pact (BGP). Eventually, they may even give origin to multiple equilibria characterized by distinct growth rates, as in the case of Maebayashi et al. (2012). Moreover, the choice of the budget rule sets *apriori* constraints on the adjustment and, therefore, reduces the admissible candidates to an optimal solution. Although an optimal debt adjustment path might be found within the problem domain defined by a particular budget rule, a better solution might lie near if only the dynamic system obeyed a different rule.

Alternatively, a *quantitative* budget rule may be used instead. In fact, such a rule is the method of choice for the prevailing fiscal budget guidelines adopted by the EMU, underwritten in the Stability and Growth Pact (SGP) and in the Maastricht Treaty, and discussed in Rubianes (2010). The SGP imposes a maximum fiscal deficit of 3%, not excluding debt service, that may only be violated under very specific conditions. The noncompliance may lead to an excessive deficit procedure. Moreover, the debt-to-GDP ratio should not exceed the 60% threshold. The rationale behind this is that a deficit, d, of 3% on an economy growing at a nominal rate γ of 5% (3% real growth and 2% inflation) will drive the debt-to-GDP ratio to $d/(\gamma+\pi)$, which for this particular case is 3/5 (60%)[6].

Given the potential impact of the budget rule on determining the fiscal policy leading to an optimal debt adjustment, an overview of the most commonly used fiscal budget rules along with the underlying assumptions is required in order to fully entail all the feasible strategies for adjusting debt. We will go through the rules most commonly found in the literature, providing a brief overview of the underlying mechanism of each fiscal feedback function in the context of economic growth models.

[5]See Blanchard and Giavazzi (2003) for an interesting discussion on the impact of some of these rules.

[6]Recall that the debt-to-GDP ratio b is given by B/Y. These variables are assumed to grow at exponential rates, but for the cycle we can approximate the functions to exhibit linear growth. So, if we assume that B grows at a constant budget deficit of d, given by $\dot{B} = d$, and that Y grows at rate γ, given by $\dot{Y} = \gamma$, and we consider the l'Hôpital rule that says that $\lim_{x\to c} B/Y = \lim_{x\to c} \dot{B}/\dot{Y}$, it results that the debt-to-GDP ratio converges to $\lim_{x\to c} \dot{B}/\dot{Y} = d/\gamma$, holding everything else constant. The same result can be obtained by solving the linear differential equations and finding the limit, i.e., $\lim_{t\to\infty}(dt + B_0)/(\gamma t + Y_0) = d/\gamma$.

3.2.4.4.1 Bond-level targeting

In the original formulation of Futagami et al. (1993), followed by Maebayashi et al. (2012), it is assumed that the government adjusts its bonds gradually to a target level according to the following rule

$$\dot{b} = -\phi(b - \bar{b}), \tag{3.10}$$

with $b \equiv B/K$ being gauged by the level of private capital K and $\bar{b}$ being the target level at the end time. The adjustment coefficient $\phi(> 0)$ defines the smoothness of such adjustment. A large value for ϕ implies a *"cold-turkey"* approach to fiscal consolidation, while a small ϕ allows for a slower, progressive adjustment. What this rule implies is that the government reduces debt by cutting government spending or raise taxes at each period according to the difference between the current and target level for the public debt, respectively. A positive difference, that is, $b > \bar{b}$, it implies a reduction in the budget deficit in the proportion given by ϕ. Conversely, if $b < \bar{b}$, there is room for expanding investment in public capital. Since under this model the tax rate is constant and exogenous, the government exerts no further influence other than setting the public expenditure level in each period.

Under a loglinearized representation of the original dynamic system, the adjustment rule given by Equation (3.10) leads to a multiple equilibria of two steady states, one of low-growth and another of high-growth. Furthermore, there is also a possible indeterminacy of the transition path to the high-growth BGP. To overcome this, Minea and Villieu (2012) suggest for the level of public bonds, B, to be gauged by the level of output Y, with $b \equiv B/Y$. This is not an economic issue, since the change of base of a variable should exert no influence on the outcome of the system. It is a mathematical hindrance arising from the linearization of complex dynamic systems.

According to their suggestion, the adjustment rule would then be

$$\dot{\theta} = -\phi(\theta - \overline{\theta}), \tag{3.11}$$

with $\theta \equiv B/Y$ and the target level for the public debt $\overline{\theta}$ defined as a ratio to the size of the economy. Budget rule (3.11), despite not being substantially different from the rule (3.10) as the endogenous variables grow at the same rate in the BGP ($\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \gamma$), gives origin to a substantially different dynamic system. As found by Minea and Villieu (2012), using a debt-to-GDP ratio for the target level of the public debt causes the previous ambiguities to vanish. The system exhibits then a single unique BGP, thus the adjustment path to equilibrium is determinate.

3.2.4.4.2 Balanced budget

Another way of setting an adjustment path involves defining a function between the primary budget surplus and the debt-to-GDP ratio, a strategy followed by Greiner (2008). Assume, for instance, that the primary budget surplus to GDP ratio is a positive linear form of the debt-to-GDP ratio plus a constant, as in

$$\frac{S}{Y} = \phi + \beta \frac{B}{Y}, \tag{3.12}$$

where $\beta \in \mathbb{R}^+$ reflects how fast the primary budget surplus varies upon a change in the public debt and $\phi \in \mathbb{R}$ determines whether the primary surplus should rise (fall) following an increase (decrease) in the GDP.

As shown by Greiner (2008), Equation (3.12) is also a necessary condition for the commitment to the intertemporal budget constraint, imposing that changes in the debt-to-GDP ratio have to be accommodated by a proportional variation of the government budget. This avoids any attempt at exploiting a Ponzi game on public debt, where the government would be constantly rolling over its debt obligations.

This budget rule gives origin to a scenario where government permanently runs a balanced budget, not implying that the initial level of public is nil, but that it will converge to zero, with $\dot{B} = 0$ in the long run. Greiner (2008) shows that when the government starts with a balanced budget and shifts to a fiscal deficit in order to finance public investment, it raises the growth rates of private and public capital and, consequently, of the output growth rate as well, since $Y = f(K, K_g)$ over the transitional path. However, it leads to a lower growth rate in the long run, unless the government takes fiscal action in order to rebalance its budget again. This thesis confirms the Keynesian postulates on counter-cyclical fiscal expansion but raises a warning on perduring detrimental effects in the long-term.

A balanced budget imposes fiscal discipline but it restricts the path if adjustment. In an hypothetical scenario of public capital exhibiting a high marginal productivity, a surge in investment in productive public capital could potentially increase the growth rate of the economy. When subject to a permanent balanced budget, such investment would have to be financed upfront with taxes or budget cuts, draining resources from the productive sector and potentially leading to a state of dynamic inefficiency, this is the case when a long-term and beneficial investment may not be put forth due to a short-term decrease in the overall welfare.

3.2.4.4.3 Golden rule

According to Groneck (2009), the golden rule decrees that the debt issued is to be used strictly to finance investment in public capital, thereby excluding spurious flows of public consumption arising from social transfers or other payouts. The rule states that the new issued bonds $\dot{B}$ will have to equal the net investment in public capital, $\dot{K}_g$, less the amount financed by the tax turnover reserved to public investment, $(1-\psi)\tau Y$. The fraction $\psi \in (0,1)$ represents the share of expenditure assigned to unproductive government consumption.

$$\dot{B} = \dot{K}_g - (1-\psi)\tau Y \tag{3.13}$$

By enforcing the rule stated by Equation (3.13), it then results that public consumption and other financial obligations will have to be paid for by tax receipts, as in

$$\psi\tau Y = C_g + C_s + rB, \tag{3.14}$$

Groneck (2009) shows that employing this rule of public finance increases the long run growth rate comparatively to adhering to a fixed deficit regime.

3.2.4.4.4 Fixed deficit rule

Under a fixed deficit regime, borrowing is restricted to a certain fixed amount, but the turnover from the government bonds is not specifically set aside for a particular objective of public policy, be it either investment in public capital or spurious public spending, as happens with the golden rule of public finance (see Section 3.2.4.4.3). Following Groneck (2009), a fixed deficit rule can be represented as

$$\frac{\dot{B}}{Y} = d, \tag{3.15}$$

where d is a constant representing the allowed deficit (as a ratio to the GDP). Since government always runs a fiscal deficit, the growth rate of public debt is always positive in the long run. Nevertheless, the debt-to-GDP ratio might decrease as a result of the output growth rate exceeding that of the public debt, $\gamma > \gamma_B$. It is expected that an adjustment following a fixed deficit rule results in less sharp transitional trajectories, relying on the long-run growth of the economy to rebalance public finances. Unfortunately, this scenario might be sometimes unfeasible in practice due to temporary restrictions in accessing the foreign capital markets, as the recent European sovereign debt crisis has shown.

Furthermore, the outright benefits of running a fiscal deficit depend on the type and duration of the public expenditure. Shortsightedness might lead to investment in projects that temporarily rise employment and have a positive effect on the GDP, but that are hardly

justifiable when brought to the light of the net present value (NPV), therefore dragging on the long term growth rate. Also, excessive investment in public capital could potentially lead to the *crowding-out* of the private sector.

In the light of political economy, and not ignoring some of the potential social benefits arising from redistribution, the case against running a permanent deficit can be put forth when we consider social phenomena like corruption. This is something to be taken into account by the policy maker, as it drains resources out of the productive sector of the economy. This subject is outside of the scope of this paper, despite being extremely relevant when refining the type and amount of public expenditure. For an extended review of the effects of corruption on growth, investment and public expenditure, refer to Paolo (1999).

3.2.4.4.5 No-rule rule

By not defining an active rule no relationship is established between the public debt and growth. Since most economic growth models operate under a closed economy, where assets W are composed of private capital K or public bonds B, servicing debt does not cause any leakage of resources to the international capital markets, therefore not affecting growth. Hence, bonds can be immediately reconverted into productive private capital, K, adding to the output of the economy.

In this case, and considering that in the BGP all the endogenous variables — consumption, private and public capital and debt — grow at the same constant rate, with $\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{B}}{B} = \gamma$, the government would be running fiscal deficits and adding up to the infinitely growing stock of public debt.

This scenario is consistent from a theoretical point of view, and has also been observed empirically. The amount of debt is usually close to the GDP level. But, contrary to what is observed in real economies, the model would allow for debt to exceed the GDP by a severalfold, especially when the production function exhibits decreasing returns to scale, while the stock of public debt is unbounded. This is hardly desirable in a world characterized by open economies where investors demand a risk premium to compensate for a default and where free capital movements hold. To avoid the unlimited growth of debt while not applying an active rule, an intertemporal budget constraint like Equation (3.27) can be defined. This way, it is guaranteed that the public debt is sustainable in the long-run.

3.3 Optimal debt adjustment

In this section, we follow Maebayashi et al. (2012) by studying the reduction of public debt under an endogenous growth model with productive government spending. The realized expenditure, in the form of public investment, is then converted into public capital either by adding to the stock of public capital or by replacing depreciated capital units. This model is based on the work of Futagami et al. (1993) changing only the *type* of the input provided by the government. Its contribution to the production function is in the form of productive public capital, whereas the original formulation considers it to be a flow of public expenditure on services, in the same fashion as the pioneering work of Barro (1990).

Literature on debt adjustment under an endogenous growth framework adopts one of the several possible fiscal rules covered in Section 3.2.4.4. This work parts ways with the current literature on the subject since no fiscal adjustment rule is assumed. Incorporating a bond issuance rule limits the role of fiscal policy and reduces the scope of application of the model in studying the optimal adjustment trajectories following a reduction in public debt, since a given path for the stock of public debt is implicitly imposed.

More importantly, we deviate from the original formulation and from most of the endogenous growth literature in the following aspects. First, like Greiner (2005), we decompose taxes into an income and capital tax rate, τ, and a consumption tax rate, τ_c, adding to the roster of fiscal policy instruments; also, we allow the tax rates to be time-variant, adding further degrees of freedom to the conduction of fiscal policy. Typically, as noted by Futagami et al. (1993), in order to maintain the tractability of the dynamic system the tax rate is exogenously set at a constant, given level. This does not pose a problem in the proposed approach since it employes recent numerical methods to solve the model, as in Amorim Lopes, A C C Fontes, and B M M Fontes (2013).

Secondly, the model does not require a *qualitative* budget rule, forcing a relationship between the level of public debt and how the government budget should react. Instead, we define a quantitative boundary for the government budget, under which it must operate using whatever welfare-maximizing fiscal policy it finds adequate. Furthermore, under a *qualitative* budget rule, the level of public investment is defined endogenously (see, for instance, Maebayashi et al., 2012). By not specifying such rule, public investment can be handled as an exogenous control variable and thus adjusted at each point in time.

We also decouple the international capital markets and the domestic productive and household sector. Within most endogenous growth frameworks with public debt, households own assets composed of private capital or government debt bonds, and they earn interest on the assets. In equilibrium, the market clears and the households are the sole lenders and creditors of the government. In the proposed approach, domestic households

show a preference for *not* holding public debt and so the government has to finance its needs in the international capital markets at the domestic interest rate. It is also assumed that the price demand for bonds is perfectly elastic for all interest rates and so the government has unlimited access to foreign markets and all supply of bonds at a given face or discount value find a corresponding demand. Furthermore, we neglect risk premium on public debt. Nonetheless, the government still faces the challenge of finding the best allocation for its expenditure since under this setup, payment of interest on debt is a resource that fully exits the economy.

The rationale behind this premise is the following: in a closed setup, interest charged on the resources borrowed reverts back to the economy and never affects the accumulation of capital directly, unless an exogenous budget rule exists and taxes have to be raised to pay for debt. There is no restriction to the accumulation of debt, and it happens independently of the available resources. If that is the case, there is no implicit requirement that the marginal productivity of debt has to equal the interest rate. In fact, interest paid on debt adds to the economy in the form of an additional endowment to families. By decoupling the international capital markets, we internalize the requirement that the loans need to be productive and in excess of the interest rate charged on them. This setup does not imply that levying taxes is always a preferred alternative to raising debt, since no interest rate is paid on taxes. In fact, simulations suggest that the relationship is non-monotonic, since more debt is borrowed when the initial level of debt is relatively low.

Finally, we follow the same procedure as Turnovsky and Fisher (1995) and most of the strand of literature on economic growth models and we exclude all nominal effects by abstracting from money, since we are only interested in the real effects on the economy.

3.3.1 The model

We consider a decentralized economy populated by an infinitely long-lived household with an infinite planning horizon and perfect foresight. We assume no population growth and its size is normalized to unity, $L = 1$, without any loss of generality. Households maximize the time-discounted utility of consumption, firms produce goods using a combination of both private and public capital, and government finances investment in public capital by either levying taxes on income, capital and consumption or by issuing debt in the foreign capital markets. [7]

[7] Unless noted otherwise, all variables are time-variant so we omit the time subscript t.

3.3.1.1 Firms

A single homogeneous good Y is produced by firms with private capital K, public capital K_g, and labour L. This is a standard Cobb-Douglas production function that incorporates public capital which is assumed to be a labour augmenting public good[8]. In addition, we consider a Hicks-neutral technical progress $A > 0$ affecting none of the inputs directly. In this particular case, A is the total factor productivity. The output is then produced as follows:

$$Y = AK^{\alpha}(K_g L)^{1-\alpha}. \tag{3.16}$$

The parameter α is the output elasticity of capital, a constant determined by the available technology. We assume constant returns to scale by imposing α to be positive but less than one, which results in a linear homogeneous function. Any other range would imply that public capital has no productive use (case $\alpha = 1$) or that it has a negative effect on the production function (case $\alpha > 1$).

The first-order conditions for profit maximization are given by

$$f_K = \alpha A K^{\alpha-1} K_g^{1-\alpha} \tag{3.17}$$

$$f_L = (1-\alpha) A K^{\alpha} K_g^{1-\alpha} \tag{3.18}$$

where f_K and f_L denote the marginal productivity of private capital (MPK) and labour (MPL), respectively. Assuming perfectly competitive markets of goods and inputs, these first-order conditions can also be interpreted as the interest rate of the private capital, r, and the wage rate of labour, w. For each given K and ignoring depreciation, each profit-maximizing firm equates the marginal product of capital to the rental price, r. Hence, $f_K = r$. Furthermore, it can be shown that Y exhibits positive but diminishing marginal physical productivity on all factors. Hence, $f_K > 0$, $f_{KK} < 0$, $f_{K_g} > 0$, $f_{K_g K_g} < 0$ and $f_L > 0$, $f_{LL} < 0$.

Contrary to Aschauer (1988), where the marginal productivity of the government expenditure can vary freely and be negative insofar as government intervention has a negative effect on the production function (e.g., government regulations or inefficient red tape), we consider productive public investment that can only, in effect, augment the productivity of the labour factor. Consider, for instance, the installment of a new port or any public institutions that strive to enforce the rule of law, contribute to a reduction of

[8]The discussion whether public capital can be a substitute or a complement to private capital in the production function is still subject to great dispute. Lynde and Richmond (1992) argue that they are complements rather than substitutes, reinforcing our stance.

risk and uncertainty or create positive externalities in the productive sector[9]. Therefore, we are only interested in the case where $f_{K_g} > 0$, which requires that $\alpha > 0$, as already noted.

3.3.1.2 Households

Consider a representative household that benefits from consumption, who wishes to maximize the overall utility over an infinite-time horizon, which can be written as

$$\max_{C} \int_{0}^{\infty} U(C)e^{-\rho t}dt \tag{3.19}$$

where C is private consumption and $\rho > 0$ is the constant rate of time preference or subjective discount rate. We consider an isoelastic or constant relative risk aversion (CRRA) utility function $U(C)$

$$U(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta}, & \theta \neq 1 \\ \ln C, & \theta = 1 \end{cases} \tag{3.20}$$

where $\theta > 0$ is the coefficient of risk aversion. The household's budget constraint is given by

$$(1-\tau)(rK+w) = \dot{K} + (1+\tau_c)C, \tag{3.21}$$

where $\tau \in (0,1)$ is the income/capital tax rate and $\tau_c \in (0,1)$ is the consumption tax rate. As is common in the literature, the dot gives the derivative with respect to time and all variables are in per-capita quantities since we normalize labour to unity. Equation (3.21) states that families earn gains from capital rK and wage w and use their after-tax income $(1-\tau)(rK+w)$ to consume $(1+\tau_c)C$ or to invest in new capital, $\dot{K}$. Households can only save by investing private capital since they do not want to hold government bonds. Families maximize (3.19) subject to (3.21).

Furthermore, the transversality condition expressed by Equation (3.22) is a necessary bar to Ponzi schemes and must be verified.

$$\lim_{t\to\infty} C^{-\theta}K \cdot e^{-\rho t} = 0. \tag{3.22}$$

[9]Evaluating whether the marginal efficiency of the public investment is greater than zero or whether it might cause the crowding out of the private sector is a relevant issue but outside the scope of this work. For a review of the evidence, check Lansing (1995).

3.3.1.3 Government

Government finances public expenditure by levying a time-variant tax on income and capital at rate τ, on consumption at rate τ_c, or by issuing new bonds $\dot{B}$ in the international capital markets, which provide access to unlimited funding. This allows for the government to either run budget deficits ($S < 0$) or budget surpluses ($S > 0$) in order to finance investment in public capital $\dot{K}_g$, that will then revert into the stock of public capital K_g. Similar to private capital, we are ignoring the depreciation of public capital. The evolution of public capital is then given by an exogenous path of public expenditure, G

$$\dot{K}_g = G. \tag{3.23}$$

Assuming no tax is charged on public debt bonds, the law of motion that governs the accumulation of public debt is given by

$$\dot{B} = rB - S, \tag{3.24}$$

where B are government bonds, S is the primary budget surplus and rB is the interest paid on the debt bonds, which we assume to be tax exempt. We consider the interest rate paid on bonds to be equal to the domestic rate of interest. The primary budget consists of the tax receipts less the government expenditure, given by Equation (3.25)

$$S = (\tau_c C + \tau Y) - G \tag{3.25}$$

By inserting Equation (3.25) into Equation (3.24) we obtain the government budget constraint:

$$\dot{B} = rB + G - (\tau_c C + \tau Y). \tag{3.26}$$

Equations (3.25) and (3.26) imply that whenever the mix of taxes levied $(\tau_c C + \tau Y)$ is not enough to pay for the public expenditure G and for the costs of servicing debt rB, the government is forced to issue new debt $\dot{B}$.

The intertemporal budget constraint of the government is verified if

$$B(0) = \int_0^\infty e^{-\int_0^\mu r(\nu)d\nu} S(\mu)d\mu \leftrightarrow \lim_{t\to\infty} e^{-\int_0^t r(\mu)d\mu} B(t) = 0 \tag{3.27}$$

holds for every t. Equation (3.27) is the net present-value borrowing constraint that imposes public debt at time zero to equal the future net present-value surpluses. This no-Ponzi game condition prevents the government from consistently servicing its debt (principal and interest) by issuing new debt, barring the rollover. This work focuses in the fiscal

adjustment from high levels of debt, and so the initial level of debt $B(0)$ does not obey Equation (3.27). Notwithstanding, the transversality condition is *a posteriori* condition, having to be observed only at the final time.

Instead of imposing a *qualitative* budget regime, we will define a *quantitative* band under which the government can run the budget. The budget as a percentage of the output of the economy has to stay between a lower bound μ and an upper bound η, operating according to the following rule:

$$\mu \leq S/Y \leq \eta \tag{3.28}$$

The rule given by Equation (3.28) enforces no particular fiscal policy or automatic feedback system of the total budget $S + rB$ to an increase or decrease in the level of public bonds, B. Also, no relationship between debt and growth is exogenously inserted into the model. Setting a maximum deficit ratio of μ is the current practice in the EMU to ensure that debt stays in a sustainable path (see European Comission, 2007 for further details) and is required to ensure that the government does not accumulate debt indefinitely.

The upper bound η for the budget surplus serves a similar purpose as the adjustment coefficient ϕ in the bond level targeting rule (Section 3.2.4.4.1) or the parameter β in the balanced budget rule (Section 3.2.4.4.2). It indirectly affects the smoothness of the adjustment, i.e., whether fiscal consolidation should be more abrupt and sharp or slow and gradual. In fact, the upper limit set by η does not define what kind of adjustment should be conducted, therefore being less imposing than the parameters ϕ or β. What it does is to prevent a case of postponing the adjustment up until the end time and then proceeding with an abrupt fiscal consolidation.

3.3.1.4 Equilibrium conditions and the balanced growth path

In order to derive the equilibrium of the economy, it is assumed that the goods and the labour market clear and that the government has unlimited access to a foreign capital market willing to buy public debt bonds at the domestic interest rate. The labor market equilibrium condition is $L = 1$ since population size is unity and each household supplies one unit of labour inelastically. When the goods market clears, the rate of interest on capital equals the marginal productivity of private capital and the wage equals the marginal productivity on labour and so expectably $rK + w \equiv Y$, since the demand-side of the economy must equal the supply-side. The overall constraint of the economy is then (for derivation, see Appendix A):

$$\dot{K} = (1-\tau)Y - (1+\tau_c)C. \tag{3.29}$$

The decentralized optimal control problem can be then be stated in the following way:

$$\max_{C} J = \int_{0}^{\infty} U(C)e^{-\rho t}dt, \quad s.t. \tag{3.30}$$

$$\begin{aligned}
\dot{K} &= (1-\tau)Y-(1+\tau_c)C && K(0)=K_0,\ K\geq 0\\
\dot{K}_g &= G && K_g(0)=K_{g,0},\ K_g\geq 0\\
\dot{B} &= rB+G-(\tau Y+\tau_c C) && B(0)=B_0,\ B\in\mathbb{R}\\
Y &= AK^{\alpha}K_g^{1-\alpha}\\
S/Y &\in [\mu,\eta]\\
C &> 0\\
\tau &\in (0,1)\\
\tau_c &\in (0,1)
\end{aligned} \tag{3.31}$$

and the transversality conditions of the household

$$\lim_{t\to\infty} C^{-\theta}K = 0, \tag{3.32}$$

and of the government

$$\lim_{t\to\infty} e^{-\int_0^t r(\mu)d\mu}B = 0. \tag{3.33}$$

The household chooses an allocation $\{C(t)\}_{t=0}^{\infty}$ that maximizes J and the government choosing an allocation $\{G(t),\tau(t),\tau_C(t)\}_{t=0}^{\infty}$ that obeys the intertemporal budget constraint and the quantitative budget rule given by Equation (3.28). The control variables are C, G, τ and τ_C and the state variables are K, K_g and B, subject to the constraints in (3.31).

Before solving the model, we derive the conditions for a decentralized equilibrium and for a balanced growth path. An equilibrium allocation is defined as follows:

Definition 3. An equilibrium is a sequence of variables $\{C(t),K(t),K_g(t),B(t)\}_{t=0}^{\infty}$, a sequence of prices $\{w(t),r(t)\}_{t=0}^{\infty}$, and a sequence of fiscal policies $\{G(t),\tau(t),\tau_C(t)\}_{t=0}^{\infty}$ where the firm maximizes profits, the household solves Equation (3.19) subject to constraints (3.21) and the government stays solvent by abiding to the budget constraint set according to (3.26).

We now characterize a balanced growth path, using the same definition as Greiner (2008).

Definition 4. A balanced growth path (BGP) is a path such that the endogenous variables of the economy, consumption, private and public capital grow at the same strictly positive constant rate such that $\dot{C}/C=\dot{K}/K=\dot{K}_g/K_g=\gamma,\ \gamma>0$ and $\gamma=\bar{\gamma}$.

In Greiner (2008), one of the following three additional conditions is also defined, given that the balanced budget rule forces a feedback mechanism of the primary budget on the debt level and households are also the creditors of the public debt: (i) $\dot{B} = 0$, under which the government always runs a balanced budget; (ii) $\dot{B}/B = \gamma_B,\ 0 < \gamma_B < \gamma$, under which the government always runs budget deficits, although at a rate inferior to that of the growth of the economy; (iii) debt grows at the same level as the economy, with $\dot{B}/B = \dot{C}/C = \dot{K}/K = \dot{K}_g/K_g = \gamma$.

This imposition on the growth level of the public debt stems directly from the budget rule adopted. The quantitative budget band set here imposes no particular trajectory for the growth of debt, letting it vary freely as long as condition (3.28) is verified. In fact, it is very likely that during the transitional dynamics of the debt adjustment the level of public debt might grow at a positive rate, i.e., $\dot{B}/B > 0$, in the first years of the adjustment.

From Definition 4 it is known that, in the long-run, the economy grows at rate

$$\gamma = \frac{\dot{K}}{K} = (1-\tau^*)AK^{\alpha-1}K_g^{1-\alpha} - (1+\tau_c^*)\frac{C}{K}. \tag{3.34}$$

Without solving the model, Equation (3.34) tells us that the long-run growth rate of the economy depends negatively on both tax rates, the income/capital tax rate τ, and the consumption tax rate τ_C. On the other hand, it depends positively on the level of public capital K_g, which is financed by raising taxes or by issuing bonds. Moreover, it shows that consumption, insofar as it funnels resources out of the productive sector, can be detrimental to growth. Its effect is minored, though, when supported by an increase in private capital from previous investment, causing the ratio C/K to decrease. This is, in fact, a verification of Say's Law, purporting the view that (aggregate) supply creates (aggregate) demand, as in, supply creates the resources to allow for the demand for consumption and increase investment in the future.

3.3.2 Numerical solution

As can be seen from the optimization problem stated in Equation (3.31), the system gives origin to a strongly nonlinear set of equations. Similar problems are solved analytically by first removing the nonlinear dependence by dividing all the endogenous variables by the stock of private capital K or by the output of the economy Y. The result is an approximated dynamic system of linear differential equations where an exact optimal control indirect method, like Pontryagrin's Maximum Principle, can then be used to obtain the necessary optimality conditions.

In our particular case, it is not possible to analytically solve the system. The introduction of time-variant tax rates makes it impossible to remove the nonlinearity of the model.

Moreover, the use of a quantitative budget band affecting a combination of endogenous variables inserts several hard constraints for each variable, adding extra complexity to an already non-trivial system.

Therefore, to study the macroeconomic effects of the debt adjustment we resort to the numerical framework laid by Amorim Lopes et al. (2013) to solve infinite-horizon non-linear economic growth models. The procedure consists of first transcribing the infinite-horizon problem (P_∞) into an equivalent finite dimensional representation (P_T). Afterwards, we use the Imperial College London Optimal Control Software[10] (ICLOCS) to discretize and transform the problem into a Nonlinear Programming Problem (NLP). A state-of-the-art NLP solver, Interior Point Optimizer[11] (IPOPT) can then be used to find optimal trajectories.

3.3.2.1 Finite-horizon problem

The initial step involves the definition of the equivalent finite-horizon problem. To do so, we need first to calculate the boundary cost W and define the boundary conditions S of the optimal control problem. For further details, an extended mathematical explanation and examples of application of the framework in endogenous growth models, refer to Amorim Lopes, A C C Fontes, and B M M Fontes (2013).

The boundary cost is calculated as follows. First, we normalize consumption so that it is constant at the end time $\bar{t}$. From Definition 4 of the BGP we know that consumption will grow at a constant rate γ. We can express this in the following equation.

$$C_t = C_{\bar{t}} \cdot e^{\gamma(t-\bar{t})},\ t \in [\bar{t}, +\infty) \tag{3.35}$$

The boundary cost is then obtained by integrating Equation (3.19) along with the definition of $C(t)$ given by Equation (3.35). We then obtain (assume $\theta \neq 1$ for the risk aversion coefficient of the utility function)

$$W = \int_{\bar{t}}^{\infty} \frac{(C_{\bar{t}} e^{\gamma(t-\bar{t})})^{1-\theta}}{1-\theta} e^{-\rho t} dt, \tag{3.36}$$

which we can integrate to obtain:

$$W = \frac{e^{\gamma(1-\theta)} - \rho}{\rho - \gamma(1-\theta)} \cdot \frac{(C_{\bar{t}} e^{-\gamma \bar{t}})^{1-\theta}}{1-\theta}. \tag{3.37}$$

The boundary cost can then be calculated by using Equation (3.37) and inserting into the

[10] http://www.ee.ic.ac.uk/ICLOCS/
[11] https://projects.coin-or.org/Ipopt

stage cost function J defined in Equation (3.30).

The boundary condition S is, by definition, an invariable set for which the problem has a finite solution. Again, from Definition 4 we know that under a balanced growth path the endogenous variables grow at a constant rate. From this we can extract the definition of the set S

$$S = \{(K,\ K_g) \in \mathbb{R} :\ \frac{\dot{K}}{K} - \frac{\dot{K}_g}{K_g} = 0\}, \quad K_{\bar{t}},\ K_{g,\bar{t}} \in S. \tag{3.38}$$

The definition of the boundary cost W in Equation (3.37) and the boundary condition S in Equation (3.38) guarantee that P_T is an equivalent representation of P_∞.

3.3.2.2 Base case and calibration

We calibrate the model in order to mimic as close as possible the underlying structural parameters of an EMU country. The economy does not necessarily start from a steady-state as that is not required by the framework we use to solve the model.

The intertemporal discount rate ρ is set to 0.05, a value frequently used in the literature. The risk aversion coefficient θ is set to 1.1, a value such that $\theta > 1$, which results in risk-averse agents and a strong incentive to smooth consumption. The elasticity of production with respect to private capital, α, is set to 0.75, a standard value found in studies like Barro (1990), Greiner (2007), and Maebayashi et al. (2012). The total factor productivity is set to $A = 0.1313$. When considering an average inflation of 3% per year, this allows for a nominal growth rate of up to 5%, well within the values registered in the last two decades (see Figure 3.6). Without loss of generality, we are abstracting from the depreciation of both private and public capital with $\delta_K = \delta_{K_g} = 0$. For the quantitative boundaries of the government budget as a share of the GDP, we are defining a lower bound of $\mu = -0.05$, which corresponds to a maximum budget deficit of 5% of the GDP, and an upper bound of $\eta = 0.05$, meaning that the government cannot run budget surpluses in excess of 5% of the GDP. The simulations indicate that increasing the upper bound limit and allowing for larger budget surpluses postpones the fiscal adjustment and then executes the consolidation in a very sharp way. Therefore, this limit is maintained within real and frequently observed boundaries. The maximum time-horizon for the debt adjustment to take place is 30 years.

The initial values for the endogenous state variables are set as follows. We set the initial stock of private capital is set to $K_0 = 1$. The initial stock of public capital as a ratio to the stock of private capital is set to $K_{g,0}/K_0 = 0.34$. This value is taken from Checherita-Westphal et al. (2012). The initial level of public debt as a ratio to the GDP is 127% i.e., $B_0/Y_0 = 1.27$. This is the debt-to-GDP ratio currently observed in Portugal. All the parameters and initial values for the variables are summarized in Table 3.1.

Parameters		Initial values	
ρ	0.050	K_0	1
θ	1.100	$K_{g,0}/K_0$	0.340
α	0.250	B_0/Y_0	1.270
A	0.131		
δ	0.000		
μ	-0.05		
η	0.05		

Table 3.1: Structural parameter values and the initial levels for the endogenous state variables.

3.3.3 Transitional dynamics

The main purpose of a debt adjustment program is to bring the debt-to-GDP ratio back to a sustainable trajectory in the least costly way that fiscal policy and the underlying structure of the economy allow, measured in terms of minimizing the loss in welfare created by the adjustment. Actually, the matter of concern in debt sustainability is the level of debt relative to expected future budget surpluses, with the debt-to-GDP ratio being only a point of reference. Figure 3.8 shows an optimal trajectory for the adjustment of the stock of public debt in absolute levels (top left panel) and when gauged as a share of the GDP (top right panel). The pre-existing level of the public debt-to-GDP ratio was defined as $B_0/Y_0 = 1.27$ or 127% of the GDP, a value commonly found in the European countries hit by the sovereign debt crisis.

When the adjustment starts, public debt continues to accumulate both in absolute and in relative terms. As can be grasped from Figure 3.8, it takes up to seven years to reach the inflection point where the trajectory of the debt-to-GDP ratio inverts its ascending trend and starts a long-lasting descent. This year, the debt-to-GDP ratio reaches a maximum of $B/Y = 1.45$ or 145% of the GDP. The variation in time of the growth rate of the level of public debt, $\dot{B}/B$, is portrayed in the bottom left panel of Figure 3.8. It follows that during the first seven years of the adjustment, debt continues to rise at a positive, almost constant rate of $\sim 3\%$, with $\dot{B}/B > 0$. At the inflection point, debt starts to decline at a rate of over 3% per year, with $\dot{B}/B < 0$.

Why does not the adjustment start right away, during the first year? We know that the initial level of public capital as a percentage of the level of private capital, $k_g(0) \equiv K_{g,0}/K_0$, was set to equate the long-term equilibrium ratio, $k_g(0) = k_g^*$. Therefore, the marginal productivity of public capital was at its optimal level and so no initial upfront public investment is required. In fact, the investment in public capital is zero for the first

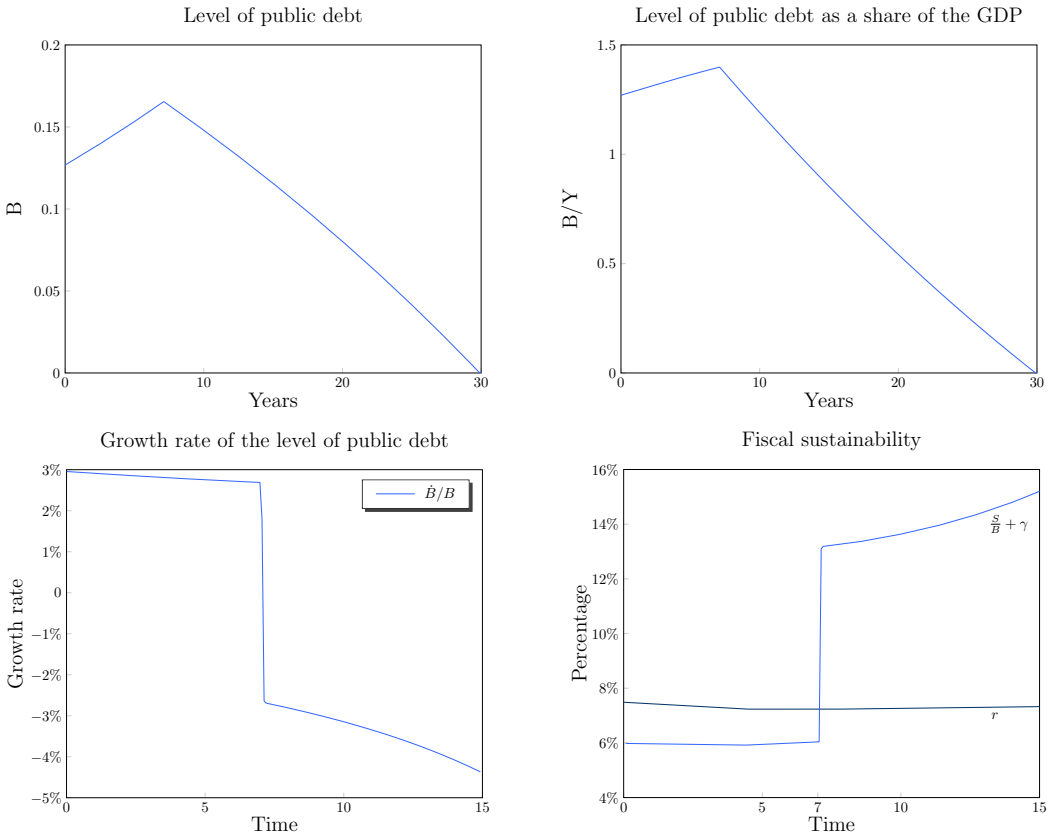

Figure 3.8: At the top, the optimal trajectory for the reduction of the absolute level of public debt (left panel) and as a share of the GDP (right panel). At the bottom, the growth rate of the stock of public debt ($\dot{B}/B$) during the first fifteen years of the time-horizon (left panel) and the evolution of the fiscal position of the economy (right panel).

four years of the adjustment (see Figure 3.11). The reason may lie with the positive effect of public debt in the private sector, postponing tax revenues and hence promoting a surge in private investment in the short-run. Figure 3.9 highlights this point. In the short-run, debt provides a tax relief on the private sector since expenditure does not have to be offset with tax revenues, with the government allowed to run fiscal deficits. Lower distortionary tax rates promote economic growth through increased investment in private capital, $\dot{K}$. In the long-run, the positive effect exhausts and the negative effects come in. Tax rates have to be increased in order to ensure that the intertemporal budget constraint is observed, and debt has to be serviced which is another line of the budget. From that moment on, the long-run growth rate decreases.

The right panel of Figure 3.8 provides an insight into the evolution of the financial position of the economy. Rearranging the debt sustainability condition in Equation (3.9) with respect to the interest rate charged on public debt bonds, we obtain the relationship

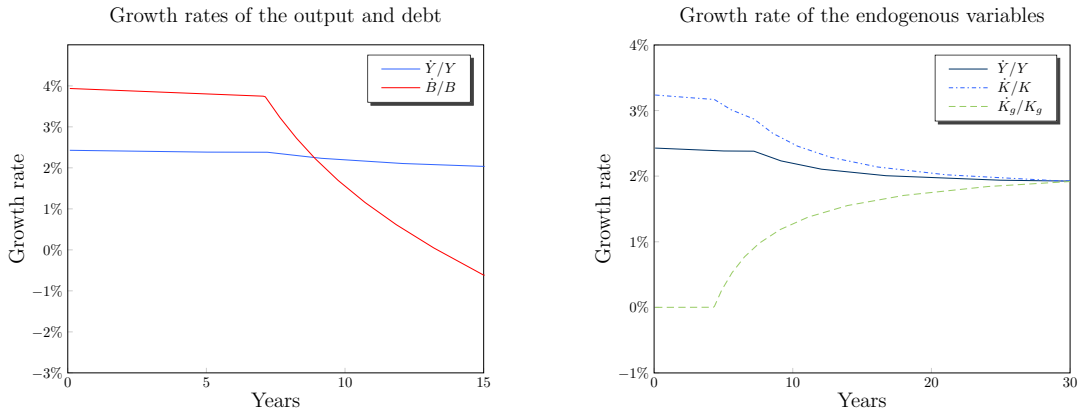

Figure 3.9: Growth rate of the level of public debt $(\dot{B}/B)$ and of the output of the economy $(\dot{Y}/Y)$ (left panel) and growth rates of the stock of private capital $(\dot{K}/K)$, of the stock of public capital $(\dot{K}_g/K_g)$ and of the output of the economy $(\dot{Y}/Y)$. .

that guarantees a fiscal sustainable position is obtained.

$$r \leq S/B + \gamma. \tag{3.39}$$

Plotting both the left-hand and right-hand side of Equation (3.39) for the baseline scenario it can be shown that a fiscal sustainable position is attained only after the seventh year of the adjustment (Figure 3.8).

Figure 3.9 provides another insight into the evolution of the rates of growth of public debt and of the output of the economy. Depicted in the left panel of the picture we have a decreasing, yet positive, evolution of the level of public debt up until the seventh year, when inflection takes place and the fiscal adjustment starts. When debt inverts its trajectory we observe a negative effect on growth, with the growth rate of the output decreasing slightly. The effect will be thoroughly explained in the next sections, but it consists mostly of the upsurge in tax revenues required to service debt. On the right panel we have the growth rates of the remaining endogenous variables, namely of private and public capital, which reinforce the previous stance on the effects of taxation on the accumulation of capital. Despite the recovery of the investment in public capital after the fourth year, its positive effect is offset by the decreasing growth rate of private capital. In the end, the long-term growth rate of the economy is penalized by the decreasing rate in the accumulation of private capital.

3.3.3.1 Fiscal policy

The next sections provide an extensive overview of the results obtained in matters of taxation and the tax mix that better supports recovery. We analyze the variations required in the investment in public capital, and we also study the government budget, whether

fiscal consolidation is rapid or slow, and whether the government should run fiscal or budget surpluses.

3.3.3.1.1 Taxes

The results obtained for the optimal tax policy under adjustment can be found in Figure 3.10. The results suggest starting with the taxation of income and capital at around $\tau = 5\%$ of the GDP, followed by a twofold increase in the fifth year of the adjustment. At that moment, government also starts levying a tax on consumption of no more than $\tau_c = 4\%$. At the seventh year, an inflection point takes place. The income/capital tax rate τ drops to zero and the consumption tax rate jumps to around $\tau_c = 34\%$, starting a smooth descent over the years to $\tau_c = 20\%$.

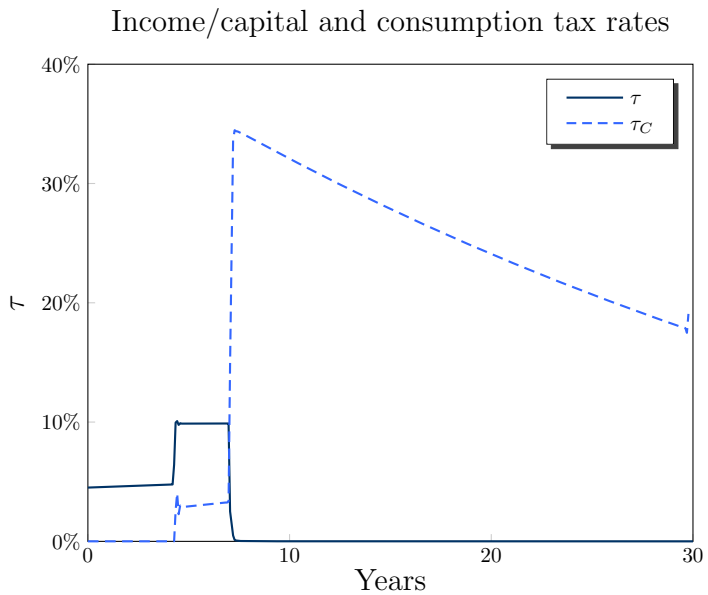

Figure 3.10: Optimal income/capital and consumption tax rates during the adjustment.

This tax mix is consistent with optimal taxation plans in dynamic growth models. As Roubini and Milesi-Ferretti (2002) point out, under a Ramsey consumption-maximizing framework and in the short run when the supply of production factors is relatively inelastic, it is optimal to heavily tax private capital; in the long-run, taxing capital has a degenerate effect on private investment, discouraging capital accumulation. Chamley (1986) and Judd (1985) show that under an endogenous growth model with human capital in the long run the optimal tax rate on capital converges to zero, which in fact happens at the inflection point observed in the seventh year. Note that these results may not hold when public expenditure is endogenous and productive, like in Barro (1990). In that case, the tax rate on capital income may be different from zero. As seen in Section 3.3.3, the

seventh year coincides with the inversion of the debt trajectory, bringing it back into a sustainable and decreasing path.

When tax on income and capital drops to zero, the equation describing the growth rate of the economy in the BGP simplifies to:

$$\gamma = \frac{\dot{K}}{K} = AK^{\alpha-1}K_g^{1-\alpha} - (1+\tau_c)\frac{C}{K}. \tag{3.40}$$

Equation (3.40) implies that, as expected, both consumption C and the tax rate on consumption τ_c may have a detrimental effect on the growth rate, as they reduce the capacity to accumulate the productive factor capital. Nevertheless, it can be seen that it is optimal to tax consumption instead of capital while the supply factors are no longer inelastic, i.e., in the long run.

Furthermore, the budget constraint of the government no longer depends on taxes on capital and income, as the following equation shows

$$\dot{B} = rB + G - \tau_c C, \tag{3.41}$$

which implies that the financing of the public expenditure G will depend exclusively on taxation over consumption, $\tau_c C$. Crossing this equation with the reduction in the debt service costs (right panel of Figure 3.12), it becomes clear that the tax rate on consumption starts decreasing the moment that the debt service costs start reducing. In fact, whenever they converge to zero, the taxes levied should equal the amount necessary to invest in public capital so as to keep the marginal productivity of public capital at its optimal level.

3.3.3.1.2 Public expenditure

Most of the studies concerning the evolution of public capital and public debt assume the amount of government spending to be exogenously provided (see, for instance, Barro, 1990) or determined endogenously, resulting from the application of a given budget rule (see Futagami et al., 1993, 2008; Maebayashi et al., 2012). In the case considered here, government expenditure is a control variable and so it can be subject to sudden upheavals or reversals.

Figure 3.11 illustrates the trajectory of public expenditure G during the adjustment (left panel). Recall that G is direct investment in public capital K_g and so it affects the evolution of the stock of public capital (right panel). As it can be seen, until the fourth year, investment in public capital is nil, with $G = 0$. The effect in K_g is direct and linear, causing a stagnation in the evolution of the stock of public capital. Investment in public capital resumes after the forth year, with a jump to approximately 7% of the GDP devoted

to public investment.

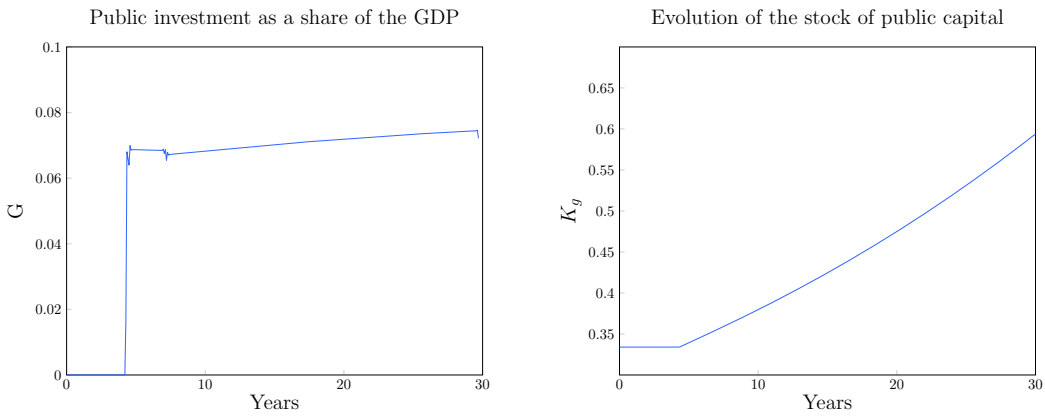

Figure 3.11: Optimal public expenditure in public capital during the adjustment.

3.3.3.1.3 Government budget

The government budget reflects the fiscal policy adopted, uniting both the tax policy and the schedule of the public expenditure. The total budget also includes the costs of servicing debt, rB, while the primary budget excludes the financial costs. The left panel of Figure 3.12 depicts the sequence of budget deficits and surpluses during the adjustment. As it can be seen, the total budget as a share to the GDP, S/Y, stays between the lower and upper bounds defined, such that $-0.05 \leq S/Y \leq 0.05$. Until the seventh year, government runs a sequence of budget deficits of 5% of the GDP. Afterwards, the policy shifts to a sequence of budget surpluses of 5% of the GDP. If we discount the costs of servicing debt, the schedule of the primary budget is always negative, pointing to fiscal surpluses. The differential between the two lines reflects the amount of the government budget devoted to servicing debt.

In fact, when observing the right panel of Figure 3.12 along with the stall in public expenditure suggested in Figure 3.11 we verify that the first seven years are strictly for paying off debt in order to bring debt back into a sustainable trajectory.

3.3.3.2 Private sector

The adjustment in the debt-to-GDP ratio affects the private sector mainly through two channels: (1) by increasing taxes τ and τ_C, either the investment in productive private capital reduces, possibly decreasing the long-run growth rate of the economy, or (2) the levels of consumption have to decrease due to the diminishing net wealth of the households. The other channel is the investment in public capital, or lack thereof, and how it can influence the production function and, thus, the output of the economy. Equation

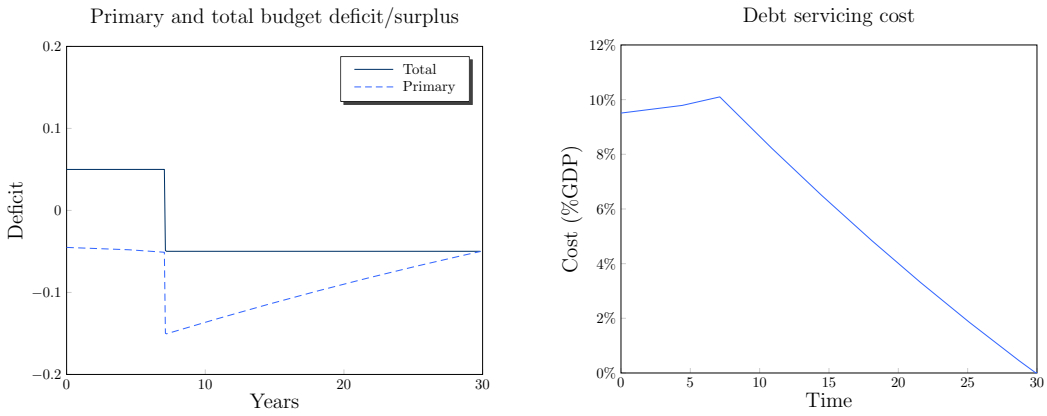

Figure 3.12: Primary budget, discounting the payment of interest on debt plotted along with the total budget during the adjustment (left panel); the interest payments on public debt as a share of the GDP (right panel).

(3.21) shows the effect of taxes levied by the government on the evolution of the stock of public capital and on consumption.

Expectably, the private sector is affected by the adjustment. Investment-wise, as Figure 3.13 shows, the evolution of private capital suffers a decrease after the seventh year of adjustment (left panel). As a share of the output of the economy (right panel), private capital increases considerably to about $K/Y = 10.37$, decreasing back to its initial level at the beginning of the adjustment, $K/Y = 10.02$.

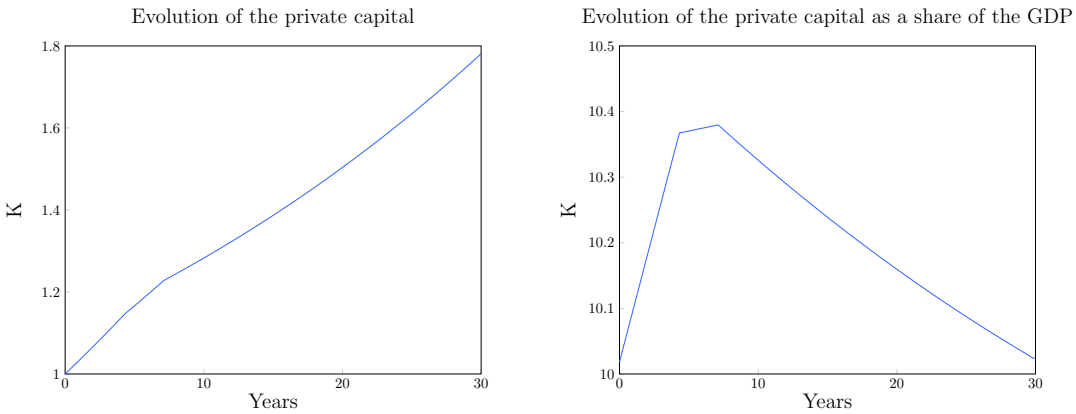

Figure 3.13: The evolution of the private capital in absolute levels (left panel) and as a share of the GDP (right panel).

Consumption-wise, the left panel of Figure 3.14 highlights that in absolute levels consumption continues to grow on a steady trend from its initial level C_0, following the growth of the economy. As a share of the GDP, it decreases slightly in the first seven years of the adjustment from an initial level of $C/Y = 0.63$ to $C/Y = 0.627$. Nevertheless, it grows considerably in the following years, reaching a maximum of 70% of the share of the GDP (right panel). Given the decrease in the growth rate of private capital, it can

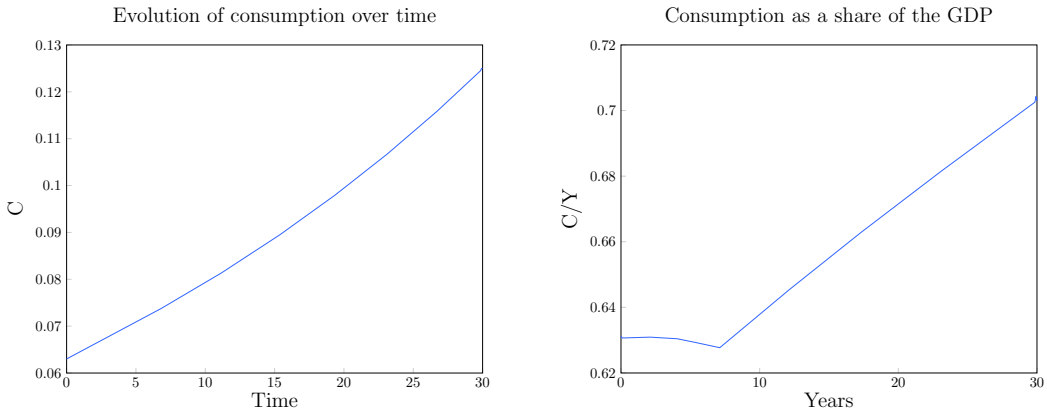

Figure 3.14: The evolution of consumption in absolute levels (left panel) and as a share of the GDP (right panel).

be seen that consumption takes a larger share of the GDP, but it still verifies the Barro-Ricardo equivalence. Despite changes in the rates, families keep their consumption levels as a share of private capital relatively proportional.

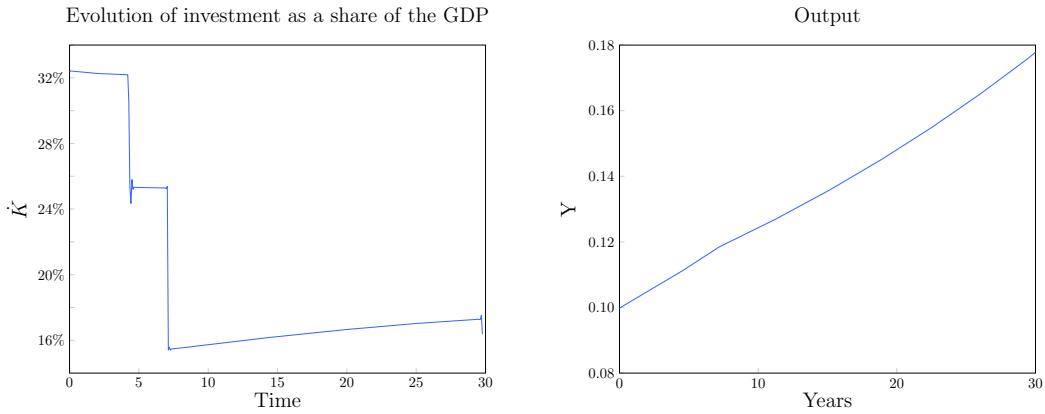

Figure 3.15: The evolution of the formation in fixed capital or net investment as a share of the GDP (left panel) and the evolution of output (right panel).

Two other macroeconomic indicators may be relevant. The first is the evolution in the formation of fixed capital or net investment as a share of the GDP, observable in the left panel of Figure 3.15. The decrease in investment observed right before the fifth year, going down from about 32% to 25% of the GDP ratio can be explained by observing the variation in the capital/income tax rate τ in Figure 3.10. In the seventh year, investment plummets again to below 16% of the GDP. Although the capital/income tax rate is now set to $\tau = 0$, tax on consumption increases dramatically, affecting the accumulation of private capital, as households try to maintain their level of consumption. Afterwards, following a decreasing trend in the consumption tax rate τ_c, investment starts recovering again, albeit at a very slow pace, converging to around 17% of the GDP in the long run.

The other indicator is the output of the economy depicted in the right panel of Figure

B_0/Y_0	$(B/Y)_{MAX}$	γ_0	γ^*
0.6	1.01	0.0252	0.02099
0.8	1.10	0.0249	0.02048
1.0	1.22	0.0246	0.01996
1.27	**1.45**	**0.0242**	**0.01926**
1.4	1.49	0.0241	0.01892
1.6	1.64	0.0238	0.01840
2.0	2.0	0.0160	0.01761

Table 3.2: Long-run growth rates for different values of the initial debt-to-GDP ratios.

3.15. It can be seen that after the seventh year of the adjustment, the slope of the output function decreases slightly. In fact, looking at the downfall in investment in both public and private production factors it is expectable to observe a decrease in the output. What this shows is that the short-run adjustment required to curb down debt has a long-term impact on the output of the economy. This point is better reinforced in the next section.

3.3.4 Debt and the growth

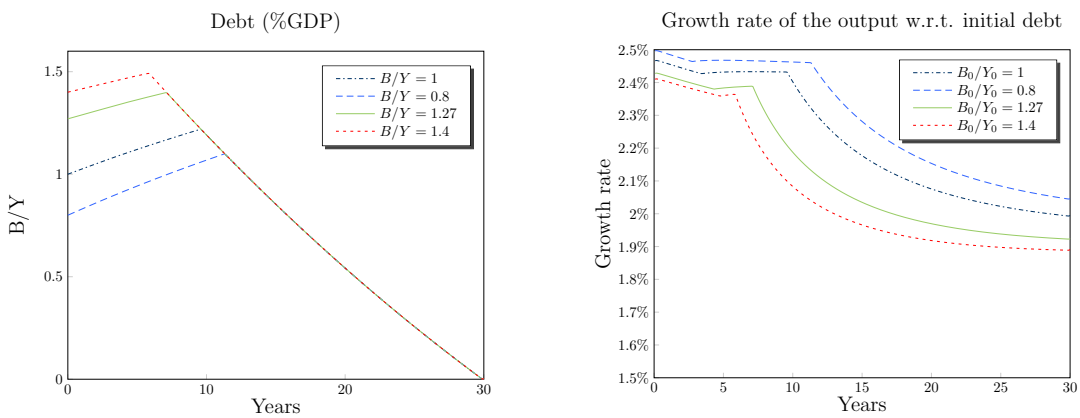

Figure 3.16: The number of years that debt is allowed to grow prior to the fiscal adjustment (left panel) and the growth rates of the economy for varying initial levels of public debt (right panel) $B(0)/Y(0) = (0.80,\ 1.00\ 1.27\ 1.40)$.

The key to understand the relationship between debt and growth and how government-backed bonds can maximize the short-run growth rate is better reflected when running multiple simulations holding everything constant except for the initial level of public debt, $B(0)$. Figure 3.16 and Table 3.2 expose this point. There is a clear negative relationship between the pre-existing level of public debt and the long-run growth of the economy

at the final time. When running the simulation with an initial level of $B_0/Y_0 = 0.6$ a final growth rate of $\gamma = 0.02099$ is obtained, while for an initial level of $B_0/Y_0 = 1.6$ the economy grows at a rate of $\gamma = 0.01840$, a significant decrease of $\sim 13\%$ in the long-run growth rate. Furthermore, as the right panel of Figure 3.16 attests, the short-run growth rate of the economy is also hit by the initial level of public debt in two distinct ways: when debt is allowed to grow, the growth rate of the economy is higher due to the fact that no taxes have to be levied; low levels of initial debt allow higher short-term growth rates to be sustained for longer.

The left panel of Figure 3.16 puts in evidence two additional interesting remarks. First, regardless of the initial value of public debt, debt follows exactly the same path the moment it inverts its ascending trajectory. Secondly, when adjustment starts will depend on the initial amount of public debt. Low levels of public debt require no immediate fiscal consolidation and debt is allowed to grow for a period. High levels of public debt require immediate attention and the adjustment starts much sooner, although not at the initial time.

These observations are the basis for the following conjectures.

Conjecture 1. *The higher the initial level of public debt $B(0)$, the lower the short-run and the long-run growth rates of the economy, assuming all the other initial parameters and conditions are held constant.*

Resulting from Conjecture 1, it is immediate to see that the longer it takes for a government to decide to curb its level of public debt, the greater the penalty on the growth rates of the economy and, thus, as we will demonstrate, on the general welfare of the society. To see why, recall that in equilibrium, the endogenous variables grow at the same rate γ and so we have that $\frac{\dot{C}}{C} = \gamma$. Solving w.r.t. C we obtain $C_t = C_0 e^{\int_0^t \gamma_\upsilon d\upsilon}$ for the short-run, when the growth rate γ is time-dependent, and $C_t = C_0 e^{\gamma t}$ for the long-run when the growth rate is constant. Inserting the equation for C_t into Equation (3.19) we obtain the present value of the welfare of the economy

$$J_0 = \frac{{C_0}^{1-\theta}}{(\theta-1)\rho} + \int_0^\infty \left(\int_0^t \gamma_\upsilon dv \right) e^{-\rho t} dt. \tag{3.42}$$

Therefore, the higher the long-run growth rate γ, the higher the welfare of the economy. ■

The short-term increase in the levels of public debt and its contribution to a higher short-run growth rate stems from the fact that issuing debt and running fiscal deficits postpones levying taxes on capital/income or consumption, which have a distortionary effect on the accumulation of capital. Therefore, resorting to debt instead of raising tax rates

provides a boost in the level of private capital, an input factor in the production function and thus, in the output of the economy. This gives origin to the following conjecture.

Conjecture 2. *Up to a certain point, increasing levels of public debt may lead to economic growth by postponing distortionary tax rates and promoting investment in private or public capital, inputs in the production function of the economy that can be accumulated.*

To see why, consider the long-run growth rate of the economy provided by Equation (3.34). The lower the tax rates on income/capital and consumption, τ and τ_c, the higher the growth rate, since:

$$\frac{\partial \frac{\dot{K}}{K}}{\partial \tau} < 0, \ \frac{\partial \frac{\dot{K}}{K}}{\partial \tau_c} < 0. \tag{3.43}$$

This reduction in the tax rates leads to a surge in the accumulation of private capital, which boosts the growth rate of the economy.

The results also suggest that the function of the evolution of the debt-to-GDP ratio may have a non-monotonic, U-shape form, given that in some adjustments the debt-to-GDP increases prior to being curtailed, but in other scenarios (e.g., when the initial level of debt is 200% of the GDP), the adjustment starts right away and the debt-to-GDP ratio decreases monotonically.

3.4 Remarks and policy implications

No model or scientific achievement is expected to give an absolute sentence on a given topic. A good theory is nothing more than a set of assumptions and deduced conclusions that has survived falsification, at least until the day. As Karl Popper once put it, no theory can ever be proved, only falsified. This is epistemologically valid for social sciences, for economics and for natural sciences alike. Newton's laws still hold true because no evidence has been found on the contrary. At least for large objects.

Provided this, a cautious approach is always recommended when evaluating new developments in economic theory. This work is no exception. We consider that this work serves two different purposes. The first is to present a new numeric framework for solving infinite-horizon endogenous growth model. We have exposed the benefits of such framework by pointing out that no linearization is required, that taxes can now be time-variant and that a quantitative budget rule can be used. We consider such task was duly achieved and that it can be an important contribution to the toolkit of the macroeconomist. The second is to study the optimal debt adjustment in an endogenous growth model by employing the developed framework and derive policy recommendations.

The results are extremely interesting, which we present next:

1. ***The initial phase of the fiscal adjustment consists in a radical cut in government spending and an increase in the capital/income tax rates***. In the short-run, the optimal policy consists in taxing capital and income, given that the supply of capital is inelastic in the short-term; at the same time, a significant reduction in government expenditure, which in this case consists of public investment, should also be undertaken. Government chooses to withhold investment in public capital in the first years of the adjustment.
2. ***As soon as debt inverts its trajectory, the capital income tax drops to zero, consumption tax is the major source of financing and public investment resumes***. In the long-run, since capital income taxes discourage capital accumulation, there is a shift in the tax policy and the tax revenues are now fully supported by a tax on consumption, with the tax on capital and income converging to zero. Investment in public capital is resumed afterwards, converging again to its equilibrium trend. This suggests that an optimal fiscal consolidation relies heavily on curbing the levels of public spending.
3. ***When the fiscal adjustment starts will depend on the initial level of public debt***. The initial level of public debt will condition the adjustment trajectory. For low levels of public debt, it is optimal to issue further public debt prior to proceeding

with a fiscal consolidation so as to keep distorting taxes low and promote investment in growth-enacting private and public capital. For high levels of public debt, adjustment commences right away, at time zero.

4. ***The initial level of public debt conditions the short-run and long-run growth rate of the economy***. The larger the level of initial public debt, the lower the short-run and long-run growth rate of the economy. The effects of high levels of public debt are notorious and long-lasting. For a larger time window, the economies converge to the same growth rate, but since the growth rates exhibited during the transition are different, they converge necessarily to different equilibriums.
5. ***Simulations seem to suggest that the relationship between debt and growth is of an inverted U-shape form***. Although debt can promote a short-run surge in growth as it allows distortionary taxes to stay low, unsustainable levels are detrimental to the economy and take a long period to accommodate for. This suggests that fiscal policy should be very careful of unsustainable debt levels and act accordingly.

We hope this work will open the way to further research in optimal debt adjustment and to help to finally untangle the relationship between debt and growth.

3.5 Conclusion and future work

In this work we studied the fiscal policy that best suits an optimal adjustment of the levels of public debt, taking into consideration the general welfare of the society. We did so following a new approach, that is, by using a nonlinear endogenous growth model with time-variant tax rates and a quantitative budget rule. By doing so, we abstain from the errors introduced by the log-linearization of the model, the strong assumption of tax constancy and the functional form imposed by a qualitative budget rule.

Our results confirm part of the literature with respect to optimal tax policy. As predicted, capital can be subject to strong taxation given the inelastic property in the supply of private capital, shifting afterwards to a consumption-based taxation. Also, they suggest that an initial cut in public investment may be appropriate, at least under our assumption that the country is already well-equipped with public infrastructure. Moreover, we show that the initial level of public debt severely conditions the short-run and long-run growth rate of the economy and, therefore, cannot be ignored. We also find a ceiling in the debt-to-GDP ratio that depends on the initial level of public ratio. Furthermore, the timing of the adjustment, i.e., when it begins, depends on the initial level of public debt. Low initial levels of public debt postpone the adjustment. On the contrary, for levels higher than 160 percent of the GDP, the adjustment starts right away. Finally, we infer that there is a non-monotonic, inverted U-shape relationship between debt and growth.

Nevertheless, more remains to be done. Due to the fact that the model can only be solved numerically, conjectures lack the assertiveness of proved propositions. Therefore, a more detailed simulation of multiple scenarios is required. Moreover, given the novelty of both the framework, some of the properties of the model like time-variant tax rates and a quantitative budget rule, there is a whole new realm of possibilities that have to be experimented with. Future research should study the relationship between debt and growth now that the analytical intractability of the endogenous growth models no longer poses a problem. Hopefully, we have supplied the tools that prosperity demands for.

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Appendix

Appendix A

Equilibrium in decentralized markets

Most of the endogenous growth models with public debt documented in the literature, like Futagami et al. (1993), Greiner (2007) or Maebayashi et al. (2012), consider a closed economy where households own assets, W, consisting of a mix between public debt bonds B and capital K. This implies that since domestic households own the public debt, the debt service costs remain within the economy and can, in principle, be immediately allocated to investment in new capital, to consumption, or to acquire more public debt.

In equilibrium, it is assumed that markets clear and hence $W \equiv B + K$. The household's constraint then becomes

$$\dot{B} + \dot{K} = (1-\tau)(rK + rB + w) - (1+\tau_c)C, \tag{A.1}$$

which coupled with the budget constraint of the government

$$\dot{B} = (1-\tau)rB - (\tau Y + \tau_c C - G) \tag{A.2}$$

gives origin to global constraint of the economy

$$\dot{K} = Y - C - G. \tag{A.3}$$

In the model proposed here, the international capital market for issuing and buying public debt is decoupled from the domestic market. Furthermore, we assume that households have a preference for holding no domestic public debt. Their assets, then, consist only of private capital with $W \equiv K$. In this case, the global constraint of the economy becomes

$$\dot{K} = (1-\tau)(rK + w) - (1+\tau_c)C. \tag{A.4}$$

Since $rK + w \equiv Y$, inserting Equation (3.17) and Equation (3.18) into Equation (A.4) the

global constraint of the economy can be rewritten as

$$\dot{K} = (1-\tau)Y - (1+\tau_c)C. \tag{A.5}$$