A Term Structure Model for Emerging Economy Bonds

Marco Bonomo  
EPGE, Fundação Getulio Vargas and CIREQ.

Alexandre Lowenkron  
PUC-Rio and BBM

May 28, 2006

Abstract

We derive and estimate an affine no-arbitrage model with default risk and macroeconomic state variable to investigate the determinants of the term structure of emerging market external debt for Brazil, Colombia and Mexico. In particular we assess the importance of US macroeconomic factors, country’s solvency ratios and latent variables in determining the risk premium term-structure. Our results indicate: (i) that the single most important variable is the joint latent variable interpreted as international liquidity, being more important in longer yields - it accounts for around 35% of the emerging markets 6 months duration spread movements and around 46% of 10 years duration spreads; (ii) the contribution of US macro variables ranges from 15% of the movements in short spread to 10% in the longer spreads; (iii) solvency variables are relatively more important to explain spreads in Mexico than in Brazil and Colombia; (iv) the idiosyncratic latent factor - interpreted as political risk - is also relevant accounting for approximately 30% of the movements, but its effect is larger on shorter yields.

JEL classification: G12, E44, E43, G15, F3

1 Introduction

What makes the risk-premium of emerging market countries external bonds to increase? Is it the increase in risk aversion from the part of international investors, triggered by US macroeconomic cycle or is it a deterioration of the emerging country solvency indicators? Or it is explained mainly by other factors, which can be attributed to political conditions?

To answer these questions, we propose a term structure approach, based on recent contributions to the finance literature. The general class of non-arbitrage affine models (Duffie and Kan 1996), which allows for time-varying risk premia, are known to produce good pricing fit. However, its original implementation relied on non-observable factors\(^1\), and therefore could not be used to discover the economic driving forces

\(^1\)Following Litterman and Scheinkman (91), the literature interpreted them as level, slope and curvature.
behind price movements. Concurrently, the empirical macro literature\(^2\) was attempting to assess the relative importance of macro factors in determining the yields by using simple regressions and VARs, but ignoring the restrictions implied by the absence of arbitrage or the empirical failure of the expectations hypothesis (usually attributed to time-varying risk premia). Ang and Piazzesi (2003) have recently merged those two branches of the literature by imposing no arbitrage restrictions in a model where the set of state variables included also macro variables following a VAR dynamic\(^3\). Our paper extends this approach by allowing for the possibility of default, therefore allowing us to use such models to explain emerging market spreads.

We present a discrete time model, which is closely related to Duffie and Singleton (1999), who derive an affine term structure for the default spread. In setting the model, we faced the question of what is the necessary hypothesis to get an affine formula for the term structure of default spread in discrete-time, and what it implies to the possible applications. We found out that the crucial condition is that the stochastic discount factor should be predetermined with respect to default. In our specific application it should not depend on emerging country variables that affect default contemporaneously (as for example, solvency ratios). The emerging markets external debt is an environment that fits well this necessary assumption, once one finds plausible the hypothesis that the stochastic discount factor of the relevant investors (US based, for example) is not affected by the emerging markets risk\(^4\).

Our specification strategy relies on the assumption that the pricing of external bonds of emerging countries reflects the preferences of US investors. Then, the price of risk is an affine function of US macro and latent factors, but the dynamics of default risk for each emerging economy depends on both US variables and the country probability of default. The resulting term structure of risk spread for an emerging economy is an affine function of the country’s solvency indicators, one idiosyncratic latent variable, and one extra latent variable common to all emerging economies analyzed. We interpret the joint latent variable as international liquidity for the set of emerging countries and each idiosyncratic latent variable as the country’s idiosyncratic political risk.

We estimate the model with base on US and three emerging market countries: Brazil, Colombia and Mexico, using monthly data\(^5\). The US macro variables are monthly CPI inflation and monthly unemployment. In order to capture the probability of default, we use emerging country’s external and government solvency ratios: external debt over export, and public debt over GDP. Since we work with four latent variables (one for the USA, 1 that affects all three emerging countries, and one idiosyncratic latent variable as the country’s idiosyncratic political risk.

\(^3\)A sequel to this was to model the macro dynamics with a structural model in the place of the VAR (Bekaert, Cho and Moreno 2005, Hördal, Tristani and Vestin 2005, Rudebush and Wu 2004, Dewatcher and Lyrio 2005).
\(^4\)The framework would not be as suitable to the analysis of the term structure of default spread of firms in the same country.
\(^5\)In order to estimate the model for those emerging countries, we also had to estimate a default free term-structure model with US data.
for each emerging country) the estimation is done by Kalman filter.

Even though we are interested in the emerging countries spread, we need also to estimate a default-free affine term structure model for the US. In order to simplify the estimation, we estimated the US model separately first. Then, we used the US parameters and the US latent variable obtained as an input for the joint estimation for the three Latin America emerging countries.

Our specification provides sensible interpretation of variables, and to relate them with recent events that affected the term structure of risk spread for emerging economies. We are able to interpret the joint latent factor as the international liquidity to Latin-America (or the risk propensity for Latin-America countries) and each idiosyncratic factor as a "political risk". The model captures well some recent stylized facts: (i) the recent increase in international liquidity; (ii) some contagion events on emerging markets as the Russian crisis, the Brazilian real devaluation and the Brazilian electoral crisis; and also (iii) the decrease in political risk in Brazil after the election of Lula for presidency with a later increase in this variable (which can be associated with the recent political scandals).

The methodology also allow us to asses the relative importance of each one of the variables analyzed in explaining movements in the term structure of emerging market spread. The single most important variable is the joint latent variable, interpreted as international liquidity. Its movements accounts for about 42% of the variability in emerging markets spread. Its importance varies from 56% in Brazilian spread to 24% of Mexican spread but it is much more important for bonds with longer duration. Contribution of US macro variables to spread movements ranges from 15% in short spreads to 10% in the longer spreads.

The solvency variables seems to be more important in Mexico than in Brazil and Colombia. Another related result is that the idiosyncratic latent variable (political risk) is much more important in Brazil and Colombia than in Mexico. We conjecture that the cause of these two phenomena is the fact that Mexico is an investment grade country among the three emerging economies.

The remaining part of the paper is divided in four sections. In the next section we propose a no-arbitrage framework for pricing defaultable bonds of emerging market economies. In section 3 we present the data and the estimation method and estimation results. Term structure results are presented and analyzed in the fourth section. The last section concludes.

2 A Term Structure Model of Defaultable Bonds with Macroeconomic State Variables

In this section we present a no-arbitrage framework for pricing defaultable bonds of emerging market economies based on US and emerging market state variables. We follow Duffie and Singleton (1999) by specifying a model for the term structure for the default spread, and Ang and Piazzesi (2003) by using
macroeconomic and latent variables as driving factors for the US term structure. We further introduce emerging markets solvency ratios as factors that jointly with US macro variables and latent factors determine the default spread term-structure.

We proceed as follows. First, we derive a discrete-time recursive equation for the price of the defaultable bond, which will depend on dynamics of two processes: the stochastic discount factor for the US economy, and the recovery intensity. The latter process is defined to be one minus the product between the probability of default and the percentage loss in case of default. Then, we specify a VAR dynamics for the state variables, which include US latent and macro variables, and emerging market latent and solvency variables. The price of risk is assumed to be a linear function of US latent and macro state variables, and the recovery intensity is modeled as a linear function of all state-variables and their innovation. With those assumptions we are able to derive an affine term-structure for the default spread as a function of US macro variables, US and emerging markets latent variables, and emerging markets solvency variables.

2.1 A No-Arbitrage Framework for the Term Structure of Defaultable Bonds

Define \( P_{t}^{us(N)} \) as the time \( t \) price of a default-free zero coupon bond in US dollars that makes a $1 payment at time \( t+N \). By this definition, \( P_{t}^{us(0)} = 1 \), \( \forall t \) since there is no default. The yield to maturity \( Y_{t}^{us(N)} \), is defined as \( P_{t}^{us(N)} = \frac{1}{(Y_{t}^{us(N)})^{N}} \). The absence of arbitrage implies the existence of a positive stochastic discount factor \( m_{t} \). Therefore:

\[
P_{t}^{us(N)} = E_{t} \left( m_{t+1} P_{t+1}^{us(N-1)} \right).
\]

(1)

Given this formula and the absence of default at any given time \( t \) bonds of different maturities can be priced recursively by \( P_{t}^{us(N)} = E_{t} \left( m_{t+1} m_{t+2} \cdots m_{t+N} \right) \).

Assuming that the variables are lognormally distributed, we have:

\[
p_{t}^{us(N)} = E_{t} \left( \ln m_{t+1} + p_{t+1}^{us(N-1)} \right) + \frac{1}{2} \text{var}_{t} \left( \ln m_{t+1} + p_{t+1}^{us(N-1)} \right),
\]

(2)

where \( p_{t}^{us(N)} \equiv \ln(P_{t}^{us(N)}) \) and \( y_{t}^{us(N)} = \ln(Y_{t}^{us(N)}) \).

When a bond is subject to default risk, the pricing equation (1) is no longer valid. Let \( P_{t}^{(N)} \) be the price of a dollar denominated \( N \)-period emerging market defaultable bond, and \( V_{t+1} \) its value next period. In this case, \( V_{t+1} \) can be much lower than \( P_{t}^{(N-1)} \), the price in case of no default up to \( t+1 \), due to the possibility of default in \( t+1 \). Then, the basic pricing equation gives:

\[
P_{t}^{(N)} = E_{t} \left[ m_{t+1} V_{t+1}^{(N-1)} \right]
\]

Let \( R_{t}^{N} \) denote the recovery value of a bond with maturity \( N \) in the event of default at time \( t \). We will assume that once the default occurs the proportional loss is independent of the maturity of the
Let $R_N^t$ denote the recovery value of a bond with maturity $N$ in the event of default at time $t$. We will assume that the proportional loss, in case of default, is independent of the maturity of the bond: 

$$L_{t+1} = \frac{P_{t+1}^N - R_{t+1}^N}{P_{t+1}^N}$$ for every $N$

The other crucial assumption is that the stochastic discount factor $m_{t+1}$ is conditionally independent of the occurrence of default in the emerging market bond. Notice, however, that this still allow that the probability of occurrence of default $\theta(j)$ and the loss conditional on default $L(j)$ are both affected by state variables affecting the stochastic discount variable. It is as if the realization of the stochastic discount factor preceded that of the realization of default or not. This idea is depicted in the tree (figure 1) below.

Does this crucial assumption rule out our application to emerging markets bonds? We think it does not. For emerging markets external debt denominated in US dollars, it is reasonable to assume that the relevant stochastic discount factor is the American one. Therefore, it should not be affected by the incidence of default in emerging market bonds. On the other hand, US macroeconomic conditions may trigger the occurrence of emerging bonds default.$^6$

---

$^6$This hypothesis could be non-realistic if we wanted to apply the model to bonds of US companies.
In order to proceed with our derivation, let \( \mathcal{F}_{t+1} \) be the partition of states which represent all the possible events in \( t+1 \) and let \( \mathcal{F}^m_{t+1} \) be a coarser partition generated by the realizations of the stochastic discount factor. Each element \( a^m_{d,t+1} \in \mathcal{F}^m_{t+1} \) contains two other separate events \( \left\{ a^m_{d,t+1}, a^m_{n,t+1} \right\} \); the only difference between them being that in \( a^m_{d,t+1} \) the default occurs. Thus, we have

\[
P^N_t = \int_{\mathcal{F}_{t+1}} m_{t+1}(s) V^N_{t+1}(s) \pi_t(ds)
\]

We can break this integral in two sets of events: one where default occurs \( (a^m_{d,t+1}) \) and the one where it does not \( (a^m_{n,t+1}) \). Notice that under condition (2) the distribution of \( m_{t+1}, \theta_{a^m_{n,t+1}}, P^N_{t+1} \) and \( L_{t+1} \) are the same under both partitions. In other words, the only differences between the set of events is that \( (a^m_{t+1}) \) occurs with probability \( \theta_{a^m_{n,t+1}} \) leaving the price equals to \( P^N_{t+1}(a^m_{t+1}) \), while \( (a^m_{n,t+1}) \) occurs with probability \( 1 - \theta_{a^m_{n,t+1}} \) and leaves the price equals to \( R^N_{t+1}(a^m_{n,t+1}) = (1 - L_{t+1}(a^m_{t+1})) P^N_{t+1}(a^m_{n,t+1}) \). This last equality follows from condition (1). So we can write:

\[
P^N_t = \int_{a^m_{n,t+1} \in \mathcal{F}^m_{t+1}} m_{t+1}(a^m_{n,t+1}) \left( 1 - \theta_{a^m_{n,t+1}} \right) P^N_{t+1}(a^m_{n,t+1}) \pi_t(da^m_{t+1})
+ \int_{a^m_{d,t+1} \in \mathcal{F}^m_{t+1}} m_{t+1}(a^m_{d,t+1}) \theta_{a^m_{n,t+1}} \left( 1 - L_{t+1}(a^m_{t+1}) \right) P^N_{t+1}(a^m_{d,t+1}) \pi_t(da^m_{t+1}).
\]

Putting terms in evidence and summing them up, we have:

\[
P^N_t = \int_{a^m_{n,t+1} \in \mathcal{F}^m_{t+1}} m_{t+1}(a^m_{n,t+1}) \left( 1 - \theta_{a^m_{n,t+1}} \right) \left( P^N_{t+1}(a^m_{n,t+1}) \right) \pi_t(da^m_{t+1})
+ \int_{a^m_{d,t+1} \in \mathcal{F}^m_{t+1}} m_{t+1}(a^m_{d,t+1}) \theta_{a^m_{n,t+1}} \left( 1 - L_{t+1}(a^m_{t+1}) \right) \pi_t(da^m_{t+1})
= E_t \left( m_{t+1} \Theta_{a^m_{n,t+1}} P^N_{t+1} \right),
\]

where \( \Theta_{a^m_{n,t+1}} \equiv \left( 1 - \theta_{a^m_{n,t+1}} L_{t+1}(a^m_{t+1}) \right) \) measures the recovery intensity. Notice that it depends on the state of the nature since, differently form Duffie and Singleton (1999), we do not restrict \( L_{t+1} \) to be known at \( t \).\(^7\) This allows the recovery intensity \( \Theta_{a^m_{n,t+1}} \) to depend on the state of the nature and will enrich our analysis. As before, we have a recursive formula. Imposing lognormality we have:

\(^7\)Duffie and Singleton (1999) propose the following pricing equation for a defaultable bond that has not defaulted up to time \( t \):

\[
P^N_t = h_t e^{-r_t} E_t^Q \left( R^N_{t+1} \right) + (1 - h_t) e^{-r_t} E_t^Q \left( P^N_{t+1} \right)
\]

where it is clear that the probability of default in \( t+1 \) is predetermined in \( t \).

They also define

\[
L_t = \frac{P^N_{t+1} - R^N_{t+1}}{P^N_{t+1}}
\]

that is, that the proportional of loss in case of default is predetermined in \( t \). Although they do not comment on the restrictions necessary to be able to write such an equation, their formulation is clearly more restrictive than ours.
\[ p_t^{(N)} = E_t \left( \ln m_{t+1} + \ln \Theta_{t+1} + p_{t+1}^{(N-1)} \right) + \frac{1}{2} \text{Var}_t(\ln m_{t+1} + \ln \Theta_{t+1} + p_{t+1}^{(N-1)}). \]  

\[ (6) \]

### 2.2 State variable dynamics

We specify a VAR dynamics for the state variables, which include US latent and macro variables, and emerging market latent and solvency variables.

There are \( K \) state variables. Their dynamics are given by:

\[ X_t = \mu + \Phi' X_{t-1} + \Sigma \varepsilon_t. \]  

Without loss of generality, we can split the vectors \( X_t \) and \( \varepsilon_t \):

\[ X_t = \begin{bmatrix} X_{t us}^u & X_{t em}^u \end{bmatrix}'. \]

where \( X_{t us}^u \) refers to US variables that will affect both the US and the emerging spread term structure and \( X_{t em}^u \) refers to variables that influence only the emerging spread the term structure. We also split \( \varepsilon_t \) accordingly:

\[ \varepsilon_t = \begin{bmatrix} \varepsilon_{t us}^u & \varepsilon_{t em}^u \end{bmatrix}'. \]

The vector of US variables \( X_{t us}^u \) will be composed of the US unemployment rate, \( Y_{t us}^u \), inflation rate, \( \pi_{t us}^u \), and one latent variable, \( U_{t us}^u \):

\[ X_{t us}^u = \begin{bmatrix} U_{t us}^u & \pi_{t us}^u & Y_{t us}^u \end{bmatrix}'. \]

The vector of variables for the emerging market economy \( X_{t em}^e \) will be composed by two latent variables \( J_{t em}^e \) and \( U_{t em}^e(i) \), and two solvency ratios - net external debt over exports \( D^*X_t \) and debt GDP ratio \( DY_t \):

\[ X_{t em}^e = \begin{bmatrix} J_{t em}^e & U_{t em}^e & D^*X_t & DY_t \end{bmatrix}'. \]

The latent variable \( J_{t em}^e \) will be common to all emerging countries, while \( U_{t em}^e(i) \) will be specific to each emerging country.\(^8\)

\(^8\)On our first estimate, we didn’t have this joint latent variable \( J_{t em}^e \), only one idiosyncratic latent variable for each emerging country, \( U_{t em}^e(i) \). However, the results showed us the importance of introducing \( J_{t em}^e \): when we plotted the series of \( U_{t em}^e \) for Brazil, Mexico and Colombia we noticed that even though they were estimated totally independently, the series were almost coincidental with a correlation of more than 0.9. We decided then to include this extra latent variable with a systematic effect on every emerging country.
It is assumed that $ Cov(\varepsilon_t) = I$ and that $\Sigma$ is a lower triangular Cholesky matrix such that

$Cov(\varepsilon_t) = \Sigma \Sigma'$. We impose that most factors have non-correlated contemporaneous shocks, including the unobservable factors,\(^9\) with exception made to US inflation and output shocks:

$$
\sum = \begin{bmatrix}
\sigma_{uu}^\varepsilon \\
0 & \sigma_{xu}^\varepsilon \\
0 & 0 & \sigma_{xx}^\varepsilon \\
0 & 0 & 0 & \sigma_{xx}^c \\
0 & 0 & 0 & 0 & \sigma_{xx}^d \\
0 & 0 & 0 & 0 & 0 & \sigma_{xx}^d \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{xx}^d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{xx}^d
\end{bmatrix}.
$$

We also impose that the US economy is not affected by the emerging economy although the emerging economy is affected by the US economy:

$$
\Phi = \begin{bmatrix}
\delta_{uu}^\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_{uu} & \delta_{uy} & 0 & 0 & 0 \\
0 & \delta_{uy} & \delta_{yy} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_{xx}^c & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_{xx}^c & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_{xx}^c \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{xx}^c
\end{bmatrix}.
$$

The log short rate $r_t$ for the US is assumed to be an affine function of the US state variables, so it encompasses any form of linear Taylor Rule followed by the Federal Reserve:

$$
r_t = \delta_0 + \delta_1 X_t^us.
$$

### 2.3 The price of risk

In modeling the US term structure we follow Ang and Piazzesi (2003), the only differences being that they use 2 latent variables, while we use only one, for parsimony, since we our main interest is the emerging spread term structure.

We follow the tradition in the literature by specifying the price of risk as an affine function of state variables. However, given our assumption that the bond pricing is determined by US investors, we assume that the price of risk depends only on US state variables:

$$
\lambda_t = \lambda_0 + \lambda_1 X_t^us.
$$

\(^9\)This is standard practice in the term structure model (see Dai and Singleton (2000) and Ang and Piazzesi (2003) among others. Since these factors are unobservables, we can do normalizations that give us observationally equivalent systems.
where $\lambda_0$ is a $K \times 1$ vector and $\lambda_1$ is a $K \times K$ matrix that we assume to be block diagonal:

$$
\lambda_1 = \begin{bmatrix}
\lambda_{UU} & 0 & 0 \\
0 & \lambda_{\pi\pi} & \lambda_{\pi y} \\
0 & \lambda_{y\pi} & \lambda_{yy}
\end{bmatrix}.
$$

Then, we can recover the stochastic discount factor through the following relation:

$$
m_{t+1} = \frac{\xi_{t+1} e^{-r_t}}{\xi_t},
$$

and

$$
\xi_{t+1} = \xi_t \exp \left( -\frac{1}{2} \lambda_t' \lambda_t - \delta_t' \varepsilon_{t+1}^{us} \right),
$$

where $\xi_{t+1}$ is the Radon-Nikodym $\xi_{t+1}$ derivative that converts the US dollars state variables from probability measure $Q$ into $P$, and $\varepsilon_{t+1}^{us}$ is the vector of unpredictable shocks to the US state variables $X_{us}^{us}$.\footnote{Note that, consistent with our hypothesis that emerging market variables do not affect US investors risk aversion, we assumed that the innovation in $\xi_{t+1}$ is given only by shocks that affect US state variables.}

Substituting 8 and 10 into 9 we find that the stochastic discount factor is driven by:

$$
\ln m_{t+1} = -\frac{1}{2} \lambda'_t \lambda_t - \delta_0 - \delta'_1 X_{t}^{us} - \lambda'_t \varepsilon_{t+1}^{us}.
$$

As in Dai and Singleton (2002), and differently from a multi-factor Vasicek (1977), our model presents a time varying risk-premium. Recent empirical work, as Almeida (2003), shows that this is an important feature to reproduce the term structure dynamics.

### 2.4 The Recovery Intensity

The log of the recovery intensity $\Theta_{t+1}$ is a reduced variable measuring the riskiness of the debt by the interaction of the default probability and the loss of market value processes. We assume that it depends on the full vector of state variables, from both the US and the emerging economy:

$$
\ln \Theta_{t+1} = -\theta_0 - \theta_1 X_t - \beta' \varepsilon_{t+1}.
$$

We leave this process totally unrestricted in order to let the data to tell us which variables drive its dynamics.

### 2.5 The default free US term structure as a function of the US state variables

The derivation of US term structure is straightforward and familiar from the affine term structure models literature. Using pricing equation 1 and the fact that $P_{t+1}^{us(0)} = 1$ we find that:
The price of the one-period bond is a linear function
\[ p_t^{us(1)} = \ln E_t [m_{t+1}] = -r_t \]
\[ = -\delta_0 - \delta'_1 X^{us}_t. \]

The price of the one-period bond is a linear function \( p_t^{us(1)} = A_1^{us} + B_1^{us} X^{us}_t \) where \( A_1^{us} = -\delta_0 \) and \( B_1^{us} = -\delta'_1 \). Is it possible to show by induction that this affine form for the logarithm of the price of the bond also holds for any maturity, that is \( p_t^N = A^{us}_N + B^{us}_N X_t \).

**Theorem 1** Under the state variable dynamics (7) the price of an \( N \) period default-free bond is given by
\[ p_t^N = A^{us}_N + B^{us}_N X_t, \]

where
\[ B^{us}_N = \left( B^{us}_{N-1} (\Phi^{us} - \Sigma^{us} \lambda_1) - \delta'_1 \right) \]
and
\[ A^{us}_N = A^{us}_{N-1} + B^{us}_{N-1} (\mu - \Sigma \lambda_0) + \frac{1}{2} B^{us}_{N-1} \Sigma \Sigma B^{us}_{N-1} - \delta_0. \]

**Proof.** See appendix.

Defining \( A_N = -\frac{A^{us}}{N} \) and \( B_N' = -\frac{B^{us}}{N} \), we can write the equation for the yields:
\[ y_t^N = A_N + B'_N X_t. \]

### 2.6 The defaultable emerging markets term structure as a function of the US and emerging market state variables

Using pricing equation 6 and the fact that \( p_t^{(0)} = 0 \) we find that:

\[
\begin{align*}
p^{(1)}_t &= E_t (m_{t+1} + \theta_{t+1}) + \frac{1}{2} \text{var}_t (m_{t+1} + \theta_{t+1}) \\
&= E_t \left( -\frac{1}{2} \lambda'_t \lambda_t - \delta_0 - \delta'_1 X^{us}_t - \lambda'_t \varepsilon^{us}_{t+1} - \theta'_1 X_t - \beta' \varepsilon_{t+1} \right) + \\
&\quad \frac{1}{2} \text{var}_t \left[ -\lambda'_t \varepsilon^{us}_{t+1} - \beta' \varepsilon_{t+1} \right].
\end{align*}
\]

The partition of \( X_{t+1} \) above induce similar partitions for the vector \( \theta_1 \):
\[ \theta_1 = \begin{bmatrix} \vartheta^{us} & \vartheta^{em} \end{bmatrix}, \]

By assuming \( \sigma(\varepsilon) = I \), we imposed that \( \varepsilon^{us}_{t+1} \) and \( \varepsilon^{em}_{t+1} \) are uncorrelated. Therefore:
\[ p_t^{(1)} = \frac{1}{2} \lambda'_t \lambda_t - \delta_0 - \delta'_t X_t^{us} - \theta_0 - \theta'_t X_t + \]
\[ \frac{1}{2} \text{var}_t \left[ -\lambda'_t e_{t+1}^{us} - \beta^{ust} e_{t+1}^{us} - \beta^{emt} e_{t+1}^{em} \right] \]
\[ = -\delta_0 - \delta'_t X_t^{us} - \theta_0 - \theta'_t X_t + \frac{1}{2} \beta' \beta + \beta^{ust} \lambda_t \]

Noticing that \( p_t^{us(1)} = -\delta_0 - \delta'_t X_t^{us} \), we have:

\[ p_t^{(1)} = p_t^{us(1)} + \left( \frac{1}{2} \beta' \beta + \beta^{ust} \lambda_0 - \theta_0 \right) + (\beta^{ust} \lambda_1 X_t^{us} - \theta'_t X_t) \quad (11) \]

Or equivalently,

\[ p_t^{(1)} = p_t^{us(1)} + \overline{A} + \overline{B} X_t \quad (12) \]

where

\[ \overline{A} = (\theta_0 + \frac{1}{2} \beta' \beta + \beta^{ust} \lambda_0) \]

and

\[ \overline{B} = \left[ \begin{array}{c} (\theta^{ust}_1 + \beta^{ust} \lambda_1) - \theta^{emt}_1 \\ -\theta^{emt}_1 + \beta^{ust} \lambda_1 \end{array} \right] \]

In fact, the property that the difference between emerging market and risk-free bond prices is linear holds for all maturities, as can be shown by induction.

**Theorem 2** Under the state variable dynamics (7) the price of an \( N \) period defaultable bond is given by

\[ p_t^N = p_t^{us(N)} + \overline{A} + \overline{B} X_t \]

where

\[ \overline{B}_N = \overline{B} + \overline{B}_N \Phi - \left[ \begin{array}{c} \beta^{ust} \lambda_1 \\ 0 \end{array} \right] \]

and

\[ \overline{A}_N = \overline{A} + \overline{A}_{N-1} + \overline{B}_N \Phi - \left[ \begin{array}{c} \beta^{ust} \lambda_1 \\ 0 \end{array} \right] \]

\[ -\beta^{ust}\lambda_1 \overline{B}_N - \beta^{emt} \overline{B}_N + \overline{B}_N \Sigma^{ust} \overline{B} \overline{B}^{ust} \]

**Proof.** See appendix.

Defining \( s_t^N \equiv y_t^{(N)} - y_t^{us(N)}, A_N^s \equiv -\overline{A}_N \) and \( B_N^s \equiv -\overline{B}_N \), and we can write an affine equation for the spread between the emerging country and the risk-free bond yields:

\[ s_t^N = A_N^s + B_N^s X_t. \quad (13) \]
3 Estimation

Notice that we have specified the US state variables independently of the emerging market state variables, and that stochastic discount factor depends only on US state variables. Therefore, we are able to estimate the model for the US economy independently of the emerging economy information. We use the output of the US model - the parameter estimates for the dynamics for US state variables, and the series for the US latent variable - in a joint estimation for the parameters of the three emerging market economies of our sample.

3.1 Data

We use monthly data for US, Brazil Colombia and Mexico. The yields analyzed are 1, 2, 3, 5, 7 and 10 years. For the US we use data on treasuries provided by the Federal Reserve. As Rudebusch and Wu (2004), we analyze only the Greenspan period, from Jan/1988 to Oct/2005. For the emerging economies, we use data on the yield curve for sovereign bonds denominated in US dollars and which is calculated by Bloomberg. They interpolate the yields of available sovereign bonds at each date, providing their time series, from Mar/1998 to Oct/2005. We present below the data on yields, where we can notice similar patterns for those countries US dollar denominated yields, specially among the emerging countries:

For the default free US curve we follow most studies in this literature by choosing two variables in order to account for inflation pressure and economic activity. More specifically, we use are inflation and unemployment series from BLS\textsuperscript{11}.

\textsuperscript{11}Ang and Pizzesi (2003), Rudebusch and Wu (2004) and Hordal, Tristani and Vestin (2004) respectively uses, for
For Brazil, Mexico and Colombia, we use monthly data on \((\text{total public debt})/\text{GDP}\) and on \((\text{net external public debt})/\text{exports}\) provided by each country’s Central Bank\(^{12}\). The \((\text{total public debt})/\text{GDP}\) ratio is included to account for the solvency condition of the sovereign government, while the \((\text{net external public debt})/\text{exports}\) is included to account for the capacity of a country to generate dollars to repay its debts.

### 3.2 The space-state form, the Kalman filter and the point estimates

We estimate the model by maximum likelihood, following the recent literature. However, depart from most term structure papers, which follow Chen and Scott (1993) by inverting the unobservable variables from some of the yields, and imposing that they are observed without error. In our paper, we use a Kalman filter in order to avoid this exogenous asymmetric imposition on pricing errors, treating all yields equally. The state-space form of the joint estimation is presented in the Appendix.

The estimation results are presented in table (??) below. Most of the coefficients are significant. The discussion of its signals will be left for the next section, since our main interest is not the coefficients itself, but the \(A\)’s and \(B\)’s that are the factor loading of the yields, and these are basically a interaction of these parameters, given by theorems 1 and 2.

\(^{12}\)All those data are in monthly frequency, except for Colombia’s solvency indicators which are only available quarterly. We then interpolate them to monthly frequency in order for us to introduce Colombia in the joint estimation.
The pricing errors are reasonably low. The average in-sample pricing errors are 19 basis points for the USA, 26 for Mexico, 49 for Brazil and 53 for Colombia.
First, we briefly analyze the results for the US default-free term structure. Then, we investigate the determinants of the term structure of emerging market external debt for Brazil, Colombia and Mexico, which is our main goal. In particular, we assess the importance of US macroeconomic factors, country’s solvency ratios and latent variables in determining the risk premium term-structure. After a brief analysis of the results for the US, we undertake a more detailed scrutiny of the results for the emerging market risk premiums.

4.1 Results for the USA

We present below the model for the USA. Usually latent variables are interpreted as slope, level or curvature. The reason is that the factor loading of these variables on the yield have this shape. More recently, Rudebusch and Wu (2004) interpreted the level variable as a proxy for inflation trend in the USA. In our model we only use one latent variable, since the macro variables capture important part of the dynamics. There are two additional reasons: (i) parsimony, since our may concern is the analysis of the emerging spreads; and (ii) because the first principal component of the US yields analyzed accounts for 96.4% of the variance of all yields\textsuperscript{13}. It is possible to note in figure (4.1) a negative trend in this latent variable for the US from 1988 to 2005.

In figure (4.1), in order to better represent the relative intensity of the effect of each state variable we chose to show the product between the factor loading and the standard deviation $\frac{B_{us}}{N_{x}}\Sigma^{us}$, instead of presenting just the factor loading $\frac{B_{us}}{3xN}$.

\textsuperscript{13}This is due to the fact that we use monthly data. The famous 3 latent variables is usually used for models of daily data.
What drives US Treasury Term Structure?

**St. Dev * Factor Loading**

As usual, we can label the US latent variable as level, since its effect is almost the same in all treasury yields. Notice that CPI and the latent variable affect the yields positively and almost with the same pattern, i.e. with a slight negative slope. This corroborates Rudebusch and Wu’s interpretation of this latent variable as trend inflation. Furthermore, the effect of unemployment is negative, with a much more important impact on short rates. In other words, an increase in US inflation or in the latent (expected US inflation) provokes an increase in all US interest rates, while an increase in unemployment provokes a decrease in all US term structure, affecting also its slope.

Confirming the finds of Ang and Piazzesi (2003) and Rudebusch and Wu (2004), the inclusion of observable macroeconomic variables shows its importance. More than 40% of the movements in the US treasury yields are explained by the movements in these unemployment and in CPI, as can be seen in table (4.1).

### 4.2 Results for the Emerging Economies

First, we present the latent variables realizations and their interpretation. Then, we show how important each of the state variables is for explaining the movements of the spread term structure. Finally, we present impulse response functions of yield spreads for shocks in each state variable.

#### 4.2.1 Interpreting the latent variables

The estimation for the emerging countries was done jointly in order to recover the joint latent variable, \( J_t \), common to all emerging economies. Besides \( J_t \), each one of the three emerging countries in our sample
has its idiosyncratic factor \( U_t(i) \) for \( i = Bra, Mex, Col \). As will be shown on the next section, the effect of \( J_t \) on all the yields is negative and the effect of each \( U_t(i) \) is positive.

In order to interpret the latent variables, recall that we are controlling for the effect of a set of observable economic variables: US macroeconomic conditions (US CPI inflation, US unemployment and the US latent factor "level") and solvency variables from each emerging country (total debt/GDP and external debt/exports). So the latent variables must be measuring everything else that is not being captured by the effect of those variables. Since one of them is common to all emerging countries, we interpret it as international liquidity or global risk propensity. On the other hand, assuming that our solvency variables adequately capture the relevance of each country economic environment, the country specific latent variable should be reflecting predominantly the country political environment.\(^{14}\)

\[ J_t = \frac{1}{7} \sum_{\text{country}=1}^{7} \sum_{\text{duration}=1}^{3} y_{\text{country,duration}(t)} / 21 \]

The above figure depicts the resulting dynamic of \( J_t \). Since the effect of \( J_t \) on all country spreads is negative, as we show below, we contrast it with the negative of the total average of the sovereign spreads.\(^{15}\) Their pattern is extremely similar, which corroborates our interpretation of this variable as international liquidity to Latin-American countries or the risk propensity for Latin-American countries.\(^{16}\)

We can identify the cause of some important movements in \( J_t \). First, the recent increasing trend in international liquidity, which is pushing down emerging market spreads, can be clearly identified in

\(^{14}\)On an earlier version of this paper, we didn’t include the joint latent factor for the emerging economies, only one idiosyncratic for each emerging. In this formulation, the estimation was done independently for each emerging country. However, after the output produced a dynamic of the resulting latent variables almost coincidental, their correlation being higher that 0.9. For this reason we decided to include this joint latent factor in the model.

\(^{15}\)At each point in time we averaged all the 7 yields and all countries: \( y_{\text{country,duration}(t)} / 21 \)

\(^{16}\)In this sense, \( J_t \) is similar to the negative of an EMBI for Latin America.
the graph. We can also notice crisis events that drove all the spreads down: (i) the Russian crisis of 1998; (ii) the Brazilian Real devaluation of Jan/1999 and (iii) The recent Brazilian electoral crisis in 2002, known as Lula effect\textsuperscript{17}. Even though all of those events were triggered at a specific country, there were undoubtedly contagion to all emerging countries in our sample, since $J_t$ was defined to affect those countries altogether.

![Graph of Idiosyncratic Latent Variables: Political Risk](image)

We plot the dynamics of the idiosyncratic latent factors. As it will be shown later, an increase in these variables leads to an increase in the yields. Since we control for local and US economic conditions, and for international liquidity ($J_t$), we interpret them as specific political risk.

This "political risk factor" is relatively stable in Mexico and has a random pattern for Colombia. In Brazil, however, we can clearly identify trends, and relate it to known events. Until 2002, when Cardoso was in office, we can see a slow downward trend in "Brazil's political risk", until the 2002 election. On July, there is an increase in this variable - not as much as we know the risk actually increased, because much of it is being captured by $J_t$ once there was some contagion to other countries. After it became clear that the approach of the new government would be market friendly, the pace of the downward trend increased. In the end of 2003, after one year of Lula in charge, the "Brazilian political risk" reached its lowest levels. However, in 2004, a series of corruption scandals appeared and, as a consequence, the trend in the "political risk" variable reversed. In 2005, as the deterioration of the political environment continued, the latent factor continue to increase, reaching a level of 41.1 on October, which is higher than the Cardoso’s term average (34.2).

\textsuperscript{17}The market feared the victory of the left wing candidate Lula because of the past non-market friendly commitments of him and his party, including the mention of default on public debt. As the perception that he would have a fiscal responsible and market friendly approach, the crisis reverted.
4.2.2 Accounting for term structure movements

The methodology we use allow us to assess the relative importance of each state variable in explaining movements in the term structure of emerging market spread. The spreads are given by the formula presented in theorem 1: $s^N_t = A^*_N + B^*_N X_t$, where $N$ refers to the duration of the yield, where the factor loadings depend on the parameters that determine the dynamics for the state variables. As above, instead of presenting only the factor loadings, we multiply them by the covariance matrix so that their values are directly comparable. We plot below the graph of $B^*_N X_t$ for Brazil, Mexico and Colombia as a function of term structure durations ($N = 1$ month to 120 months).

Another analysis of interest is variance decomposition. For each duration of the term structure, we are able to measure the percentage of its variance explained by each of the state variables. We present below the graph of the variance decomposition of the spread curve, i.e., the variance explained by each one of the state variables for each duration.

---

18 In fact, our estimation procedure uses term structure data and this relation between factor loadings and parameter values as an additional source of information for the parameters that govern the state variables dynamics. So we have the parameters estimates and the factor loadings as a joint output of our estimation.
To summarize the information contained in the last three graphs, it is helpful to average out the variance explained by each factor for each country, as shown in the table below.

### % of variance explained by each state variable (average of all yields)

<table>
<thead>
<tr>
<th>Country</th>
<th>Brazil</th>
<th>Mexico</th>
<th>Colombia</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unobs_USA</td>
<td>1.9%</td>
<td>0.2%</td>
<td>0.3%</td>
<td>59.88%</td>
</tr>
<tr>
<td>CPI_USA</td>
<td>3.2%</td>
<td>0.6%</td>
<td>1.9%</td>
<td>9.91%</td>
</tr>
<tr>
<td>Unemp_USA</td>
<td>17.8%</td>
<td>8.1%</td>
<td>5.0%</td>
<td>30.21%</td>
</tr>
<tr>
<td>Joint</td>
<td>56.0%</td>
<td>25.3%</td>
<td>44.7%</td>
<td>-</td>
</tr>
<tr>
<td>U</td>
<td>15.3%</td>
<td>20.2%</td>
<td>48.1%</td>
<td>-</td>
</tr>
<tr>
<td>Total Publ Debt / GDP</td>
<td>5.8%</td>
<td>1.3%</td>
<td>0.0%</td>
<td>-</td>
</tr>
<tr>
<td>Net Public External Debt / Exports</td>
<td>0.0%</td>
<td>44.2%</td>
<td>0.0%</td>
<td>-</td>
</tr>
</tbody>
</table>

### % of variance explained by each state variable (several yields)

<table>
<thead>
<tr>
<th>Country</th>
<th>Brazil</th>
<th>Mexico</th>
<th>Colombia</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unobs_USA</td>
<td>1.1%</td>
<td>1.59%</td>
<td>2.61%</td>
<td>0.0%</td>
</tr>
<tr>
<td>CPI_USA</td>
<td>4.6%</td>
<td>3.95%</td>
<td>1.86%</td>
<td>1.6%</td>
</tr>
<tr>
<td>Unemp_USA</td>
<td>19.8%</td>
<td>19.43%</td>
<td>14.90%</td>
<td>10.8%</td>
</tr>
<tr>
<td>Joint</td>
<td>38.3%</td>
<td>50.61%</td>
<td>67.28%</td>
<td>21.4%</td>
</tr>
<tr>
<td>U</td>
<td>25.6%</td>
<td>17.74%</td>
<td>9.82%</td>
<td>34.9%</td>
</tr>
<tr>
<td>Total Publ Debt / GDP</td>
<td>10.7%</td>
<td>6.68%</td>
<td>3.53%</td>
<td>2.6%</td>
</tr>
<tr>
<td>Net Public External Debt / Exports</td>
<td>0.0%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

The single most important variable is the joint latent variable, interpreted as international liquidity. Its movements accounts for around 42% of the emerging markets spread. As can be seen in Figure 5, the effect of this variable in all countries and in all yields is negative. This means that an increase in $J_t$ leads to a reduction of all yields in all countries. $J_t$ is more important for Brazil, where it accounts for 56.6% of the movements in the yields, than for Colombia (46.2%) and Mexico (25.6%). It is also interesting to note that the relative importance of this variable increases substantially with the duration for Brazil. It increases slightly with duration for Mexico, and decreases slightly for Colombia.
US variables account for approximately 14% of the movements in the spread term structure. Observe that the effect on emerging market yields is still more important, since US macro variables also affect US yields, which should be added to the spreads. The single most important US variable affecting the spread is the unemployment rate, with a positive relation. This means that when the US economy is doing well, the emerging economy spreads are smaller. The US unemployment rate accounts for 19.2% of the movements in Brazilian spread, 8.6% of the Mexican spread and 5.8% of the Colombian spread. The US CPI inflation accounts for 3.9% in Colombia, 1.2% in Brazil and 0.7% in Mexico. It is somehow surprisingly that, even being closer to the US and having the tightest economic relation (through NAFTA) among countries in our sample, the effects of US variables in Mexico spreads are much less important than in Brazil.

The idiosyncratic political factor accounts for 20.1% of the spread movements in Mexico and for 15.6% of the movements in Brazil. Although proportionally this factor is more important for Mexico than for Brazil, two observations must be made. First, the magnitude of the effect in Brazil is higher than in Mexico, as can be seen comparing the purple lines in Figure 5. Second, as we argued before, some events triggered by Brazilian events are being captured by the variable $J_t$ because there were some contagion, i.e., some joint movements on the yields and thus our model attributed such movements to the joint unobservable variable.$^{19}$

The solvency variables are much more important for Mexico (45%) than for Colombia$^{20}$ and Brazil (less than 6%). We believe that a related result is that the idiosyncratic latent variable (political risk) is much more important for Brazil and Colombia than for Mexico. We conjecture that the cause of these two phenomena is the fact that Mexico is an investment grade country while Brazil and Colombia are not.

### 4.3 Impulse Responses

We present below the impulse response analysis. Each graph depict the response of emerging countries yields to shocks on each state variable. We chose to present only two yields for each country: one year

$^{19}$For Colombia the contribution of this idiosyncratic "political risk" is very high, 46.2% on average. But what can be exacerbating the contribution of this variable in Colombia is the fact that the solvency ratios are in fact quarterly data interpolated to monthly frequency. For this reason we believe that the contribution of the idiosyncratic unobservable is higher than should be and the solvency ratios are lower than they should be.

$^{20}$As in the case of the excessive sensibility to the idiosyncratic political risk factor, the fact that in Colombia solvency ratios are responsible for less than 1% can be understood if we recall that these ratios are actually quarterly and were interpolated to monthly frequency. We believe the importance of these solvency ratios is totally incorporated by the idiosyncratic unobservable variable.
and seven years\textsuperscript{21}.

\textbf{5 Conclusions}

What drives risk-premium of emerging market countries external bonds? More precisely, what are the most important economic factors behind the movements on the term-structure of emerging spread? To answer this question we extended Ang and Piazzesi’s (2003) no-arbitrage term-structure macro model

\textsuperscript{21}Two yields allow us to see movements in the term structure slope. We could have presented movements for the whole curve, but the figures would become cumbersome.
by allowing for the possibility of default, therefore allowing us to use such models to explain emerging market spreads. We presented a discrete time model, which is closely related to Duffie and Singleton (1999).

Theoretically, one of the important results we got is that one of the crucial condition for getting a closed formula is that the stochastic discount factor should be predetermined with respect to default. In our specific application it should not depend on emerging country variables that affect default contemporaneously (as for example, solvency ratios). The emerging markets external debt is an environment that fits well this necessary assumption, once one finds plausible the hypothesis that the stochastic discount factor of the relevant investors (US based, for example) is not affected by the emerging markets risk. Thus, our specification strategy relies on the assumption that the pricing of external bonds of emerging countries reflects the preferences of US investors.

The paper then investigates the determinants of the term structure of emerging market external debt for Brazil, Colombia and Mexico. In particular our methodology allowed us to assess the importance of US macroeconomic factors, country’s solvency ratios and latent variables in determining the risk premium term-structure.

According to a fundamentalist view, the country risk premium should reflect only the country conditions. However, country solvency ratios and idiosyncratic latent variables account for only 21% of the variability of the risk premium for Brazil and Mexico. Then, international liquidity and the state of US macroeconomy are more important than the country condition. One could also object that for big countries as Brazil and Mexico, the international liquidity is in part determined by the state of their economies. Even if one take the extreme view to attribute to the emerging market economy the part explained by their joint latent factor, one is still left with 23% of the variability of Brazil premium explained by US macroeconomic conditions.

This evidence indicates that international financial markets are inefficient, where the country fundamentals are not the most important determinant of their bond prices. The methodology of combining macro variables in a non-arbitrage framework opened a new investigation path for this question, which should generate more research and results under alternative specifications.

6 References


Bekaert, G., S. Cho, and A. Moreno (2005), New Keynesian Macroeconomics and the Term Structure. NBER working paper.


Appendix A

Proof of Theorem 1

The proof is by induction. We already shown that \( p_t^N = \overline{A}_N + \overline{B}_N X_t \) is valid for \( N = 1 \). Suppose that the log price of a \( N - 1 \) period default-free bond is given by \( p_t^{N-1} = \overline{A}_{N-1} + \overline{B}_{N-1} X_t \). We can use 2 and the state variable dynamics of last section to proof that the log price of an \( N \) period bond is:

\[
p_t^N = E_t [\ln m_{t+1} + p_{t+1}^{N-1}] + \frac{1}{2} \text{var}_t [\ln m_{t+1} + p_{t+1}^{N-1}] \\
= -\frac{1}{2} \lambda_t \lambda_t - \delta_0 - \delta_1 X_t^{us} + \overline{A}_{N-1} + \overline{B}_{N-1} \text{var}_t X_t^{us} + \\
\frac{1}{2} \text{var}_t \left[ -\lambda_t \varepsilon_{t+1} + \overline{B}_{N-1} X_t^{us} \right] \\
= -\frac{1}{2} \lambda_t \lambda_t - \delta_0 + \overline{A}_{us} X_{t-1}^{us} + \overline{B}_{us} X_{t-1}^{us} + \left( \overline{B}_{us} + \Phi^{us} - \delta_1 \right) X_t + \\
\frac{1}{2} \text{var}_t \left[ \overline{B}_{N-1} \Sigma^{us} X_{t-1}^{us} \right] \\
= -\delta_0 + \overline{A}_{N-1} + \overline{B}_{N-1} \left( \mu^{us} - \Sigma^{us} \lambda_0 \right) + \frac{1}{2} \overline{B}_{N-1} \Sigma^{us} \Sigma^{us} \overline{B}_{N-1}^{us} + \\
\left( \Phi^{us} - \Sigma^{us} \lambda_1 \right) X_t^{us}
\]

Thus, substituting \( p_t^N \) by \( \overline{A}_N + \overline{B}_N X_t \) on the left-hand side:

\[
\overline{A}_N + \overline{B}_N X_t = -\delta_0 + \overline{A}_{N-1} + \overline{B}_{N-1} \left( \mu - \Sigma \lambda_0 \right) + \frac{1}{2} \overline{B}_{N-1} \Sigma \Sigma B_{N-1} + \\
\left( \Phi^{us} - \Sigma \lambda_1 \right) X_t^{us}
\]

By equating the coefficients on the left and right hand side we prove the theorem

Proof of Theorem 2

The proof is by induction. We already shown that \( p_t^N = p_t^{us(N)} + \overline{A}_N^n + \overline{B}_N X_t \) is valid for \( N = 1 \). Suppose that the log price of a \( N - 1 \) period defaultable bond is given by \( p_t^{N-1} = p_t^{us(N-1)} + \overline{A}_{N-1} + \\

the terms on Formally, let’s extend the covariance terms, we have a lot of elements of a vector terms. The \( \varepsilon_t^{us} \) are the first \( K_1 \) lines corresponding to the errors on the \( K_1 \) US variables on the \( K = K_1 + K_2 \) line vector \( \varepsilon_t \).

Formally, \( \overline{\varepsilon}_t = \begin{bmatrix} \varepsilon_t^{us} \\ \varepsilon_t^{em} \end{bmatrix} \) where, as before, \( E(\overline{\varepsilon}_t \overline{\varepsilon}_t') = I_{K \times K} \). With no loss of generality we can re-write the terms on \( D'_{1 \times K} = \begin{bmatrix} D^{us'}_{1 \times K_1} & D^{em}_{1 \times K_2} \end{bmatrix} \). Thus,

\[
\text{Cov}(C' \varepsilon_t^{us}, D' \varepsilon_t) = E(C' \varepsilon_t^{us} \varepsilon_t' D)
\]

\[
= E \left( \begin{bmatrix} C' \varepsilon_t^{us} & \varepsilon_t^{us} \\ 1_{1 \times K_1} & K_1 \end{bmatrix} \begin{bmatrix} D^{us} \\ D^{em} \end{bmatrix} \right)
\]

\[
= C' \begin{bmatrix} I_{1 \times K_1} & 0 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} D^{us} \\ D^{em} \end{bmatrix}
\]

\[
= C' D^{us}_{1 \times K_1}
\]

So, in solving the Cov we will assign the \( D^{us} \) to the first \( K_1 \) elements of a vector \( D \), the ones

\[
B_{N-1}^t X_t. \text{ We can use 6 and the state variable dynamics of last section to prove that the log price of an } N \text{ period bond is:}
\]

\[
p_t^{(N)} = E_t \left( \ln m_{t+1} + \ln \Theta_{t+1} + p_{t+1}^{(N-1)} \right) + \frac{1}{2} \text{Var}_t(\ln m_{t+1} + \ln \theta_{t+1} + p_{t+1}^{(N-1)})
\]

\[
= E_t \left( -\frac{1}{2} \lambda'_t \lambda_t - \delta_0 - \delta'_t X_t - \theta_0 - \theta'_t X_t - \beta'_t \varepsilon_{t+1} + p_{t+1}^{(N-1)} + \overline{\tau}_N + \overline{\tau}'_{N-1} X_{t+1} \right) + \frac{1}{2} \text{Var}_t(\lambda'_t \lambda_t - \beta'_t \varepsilon_{t+1} + \overline{\tau}'_{N-1} X_{t+1})
\]

\[
= -\frac{1}{2} \lambda'_t \lambda_t - \delta_0 - \delta'_t X_t - \theta_0 - \theta'_t X_t + \overline{\tau}_N + \overline{\tau}'_{N-1} \Phi X_t + \frac{1}{2} \text{Var}_t(\lambda'_t \lambda_t - \beta'_t \varepsilon_{t+1} + \overline{\tau}'_{N-1} \Sigma \varepsilon_{t+1} + \overline{\tau}'_{N-1} \Sigma \varepsilon_{t+1})
\]
multiplying the US state variables. Having said that, the Cov(...) term on 15 can be substituted by:

$$\text{Cov}(..) = \lambda'_t \beta^{u,s} - \lambda'_t \Sigma^{u,s} \mathbf{B}_{N-1} - \lambda'_t \Sigma^{u,s} \mathbf{B}_{us,N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{us,N-1}$$

Substituting Cov(..) in 15 and then all that in 14 we have:

$$p_t^{(N)} = -\delta_t - \delta'_t \mathbf{X}^{u,s}_t - \theta_t - \theta'_t \mathbf{X}_t + \mathbf{A}_{N-1} + \mathbf{B}_{N-1} \mu^{u,s} + \mathbf{B}_{N-1} \Phi^{u,s} \mathbf{X}^{u,s}_t + \mathbf{A}_{N-1} + \mathbf{B}^{t'}_{N-1} \mu + \mathbf{B}^{t'}_{N-1} \Phi \mathbf{X}_t + \frac{1}{2} \beta'_t \beta + \frac{1}{2} \mathbf{B}_{N-1} \Sigma^{u,s} \mathbf{B}_{N-1} + \frac{1}{2} \mathbf{B}_{N-1} \Sigma^{u,s} \mathbf{B}_{N-1} + \lambda'_t \beta^{u,s} - \lambda'_t \Sigma^{u,s} \mathbf{B}_{N-1} - \lambda'_t \Sigma^{u,s} \mathbf{B}_{us,N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{us,N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{N-1} - \beta^{u,s} \Sigma^{u,s} \mathbf{B}_{us,N-1} \Sigma^{u,s} \mathbf{B}_{us,N-1}$$

Recalling that \( \lambda_t = \lambda_0 + \lambda'_t \mathbf{X}^{u,s}_t \):

$$p_t^{(N)} = \left\{ \mathbf{A}_{N-1} + \mathbf{B}_{N-1} \left( \mu^{u,s} - \Sigma^{u,s} \mathbf{X}^{u,s}_t \right) + \frac{1}{2} \mathbf{B}_{N-1} \left( \mu^{u,s} - \Sigma^{u,s} \mathbf{X}^{u,s}_t \right) - \delta_t \right\} + \left( \mathbf{B}_{N-1} \left( \Phi^{u,s} - \Sigma^{u,s} \lambda'_t \right) - \delta'_t \right) \mathbf{X}^{u,s}_t$$

Using Theorem 1, \( \{..\} = \mathbf{A}_N + \mathbf{B}_N \mathbf{X}^{u,s}_t = p_t^{u,s(N)} \)

$$p_t^{(N)} = p_t^{u,s(N)} + \mathbf{B}^{t'}_{N-1} \mathbf{X}_t - \mathbf{B}^{t'}_{N-1} \Sigma^{u,s} \mathbf{X}^{u,s}_t$$

Thus, substituting \( p_t^{u,s(N)} \) by \( \mathbf{A}_N + \mathbf{B}^{t'}_{N} \mathbf{X}_t \) on the left-hand side and equating the coefficients:

$$\mathbf{B}^{t'}_{N} = \begin{bmatrix} \mathbf{B}^{u,s}_{N} & \mathbf{B}^{em}_{1xK_{x1}} \end{bmatrix}' = \begin{bmatrix} \mathbf{B}^{u,s}_{N} + \left( \mathbf{B}^{t'}_{N-1} \Phi \right)^{u,s} & \mathbf{B}^{em}_{1xK_{x1}} \mathbf{B}^{em}_{1xK_{x2}} + \left( \mathbf{B}^{t'}_{N-1} \Phi \right)^{em} \end{bmatrix}$$

$$= \mathbf{B}^{t'}_{N} + \mathbf{B}^{t'}_{N-1} \Phi - \mathbf{B}^{u,s}_{N-1} \Sigma^{u,s} \mathbf{X}^{u,s}_t$$

(16)
\[
A^*_N = A^*_1 + A^* N_{-1} + B^*_N - N_{-1}(\mu + \frac{1}{2} \Sigma \Sigma B^*_N) - B^{uuu}_N \Sigma^u_0 - \beta^u \Sigma^u B^*_N - 1 - \beta_1^u B^*_N + B^*_N \Sigma^u \Sigma^u B^{uuu}_N
\]

(17)

Appendix B

The space-state representation.

The dynamics of the state variables are given by equation (7): \( Y_t = \mu + \phi Y_{t-1} + \Sigma \xi_t \), where the state variables are:

\[
Y_t = \begin{bmatrix}
U^{US}_t \\
\pi_t \\
y_t \\
J^{EM}_t \\
U^{Bra}_t \\
D/Y^{Bra}_t \\
Dx/Ex^{Bra}_t \\
U^{Mex}_t \\
D/Y^{Mex}_t \\
Dx/Ex^{Mex}_t \\
U^{Col}_t \\
D/Y^{Col}_t \\
Dx/Ex^{Col}_t
\end{bmatrix}
\]

The observation equation is \( X_t = A^* + HY_t + Nu_t \) where \( X_t \) are the observable variables,

\[
X_t = \begin{bmatrix}
\pi_t \\
y_t \\
D/Y^{Bra}_t \\
Dx/Ex^{Bra}_t \\
D/Y^{Mex}_t \\
Dx/Ex^{Mex}_t \\
D/Y^{Col}_t \\
Dx/Ex^{Col}_t \\
\text{spread}^{1}_{\text{months}}(t) \\
\text{spread}^{1}_{\text{years}}(t) \\
\text{spread}^{2}_{\text{years}}(t) \\
\text{spread}^{3}_{\text{years}}(t) \\
\text{spread}^{7}_{\text{years}}(t) \\
\text{spread}^{10}_{\text{years}}(t)
\end{bmatrix}
\]

\( Nu_t \) are the observation errors. The constants are defined as follows,
\[
A^* = \begin{bmatrix}
0 \\
A_{\text{Bra}} \\
A_{\text{Mex}} \\
A_{\text{Col}}
\end{bmatrix}, \quad \text{where,} \quad A^i = \begin{bmatrix}
A_{i\text{months}}^i \\
A_{i\text{year}}^i \\
A_{i\text{2years}}^i \\
A_{i\text{3years}}^i \\
A_{i\text{5years}}^i \\
A_{i\text{7years}}^i \\
A_{i\text{10years}}^i
\end{bmatrix}
\text{for } i = \text{Bra, Mex, Col}
\]

And,
\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
B_{\text{Bra}} \\
B_{\text{Mex}} \\
B_{\text{Col}}
\end{bmatrix}
\]