Asset Substitution and Debt Renegotiation∗

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Abstract. We analyze the relationship between asset substitution and capital structure choice. The key to understanding asset substitution is the equity holders’ ex post costs. Asset substitution affects the ex post costs since substitution alters the firm’s future earnings. We obtain a trade-off between decreasing growth rates and increasing volatilities which determines when substitution opportunities destroy ex ante firm value. Debt renegotiation is introduced as a mechanism to avoid asset substitution. Since debt renegotiation influences the equity holders’ ex post costs, debt renegotiation can in some, but not all, cases mitigate the ex ante costs of asset substitution.

Keywords. Optimal capital structure, asset substitution, debt renegotiation.

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Introduction

The traditional trade-off story between tax shield exploitation and default costs is studied extensively in the financial economic literature. In particular, a number of papers have paid much attention to the matter of dynamic capital structure changes and a detailed analysis of possible wealth transfers when the firm defaults on its debt. Another classic corporate finance issue is the asset substitution problem introduced by Jensen and Meckling (1976). The purpose of the present paper is to gather these issues into one model, i.e. to analyze how the firm’s capital structure choice and the problem of asset substitution interact and, in addition, to explore the importance of including debt renegotiation in such a setting.

The framework of the paper is a structural dynamic capital structure model. The setting was initiated by Fischer, Heinkel, and Zechnner (1989), and recently studied in e.g. Goldstein et al. (2001) and Christensen et al. (2002). They use the firm’s instantaneous earnings before interest and taxes as the underlying state variable – an approach also adopted in this paper. Also related to the present paper are Leland (1998) and Ericsson (2000) who incorporate asset substitution and dynamic capital structure into one model. However, both papers only allow the equity holders to change the riskiness (volatility) of the earnings and they do not consider debt renegotiation. Apparently, a pure risk shift seems to be a reasonable first attempt at providing a picture of the relationship between capital structure choice and asset substitution. However, if substitution between a high or low volatility is free of costs, the volatility changes are determined by the equity holders’ risk preferences, i.e. when the equity value function changes between being convex or being concave. Considering equity as a call option on the firm’s earnings, cf. Merton (1974) and Black and Cox (1976), the equity value function is expected to be convex and, hence, the equity holders generally prefer high volatility. Thus, the question is why equity holders do not substitute to the high volatility immediately after the debt issuance? This paper argues – in a simple capital structure model with debt restructuring determined by the equity holders’ incentives – that if only volatility changes are allowed, there is no reason for equity holders to exhibit risk aversion because there are no “costs” imposed on equity holders by changing volatility.

Based on the above, the present paper takes the point of view that substitution involving a lower expected growth rate is not merely an extended framework of the asset substitution problem. Ex post costs, in terms of a lower expected growth rate, are indeed a key to appropriately analyzing the equity holders’ ex post incentives and, hence, asset substitution and its ex ante effects. Thus, in our model we directly impose ex post costs on the equity holders if they employ asset substitution. Ericsson (2000) indirectly imposes ex posts costs because the firm
issues finite maturity debt with a fixed aggregated debt structure and default is triggered by a
cash flow covenant. Leland (1998) imposes ex post costs due to an ex ante determined call of
the debt, which is likely to be incompatible with the equity holders’ ex post incentives.

Our paper demonstrates that because of the additional cost of asset substitution in terms
of a lower expected growth rate, it may not be ex post optimal for the firm’s equity holders to
enforce the asset substitution. Thus, albeit substitution is possible, there are circumstances for
which it has no ex ante costs. However, if the equity holders have sufficient incentives to use their
option to substitute the assets, the ex ante costs increase with the increase in the volatility of
the substituted assets for a fixed decrease in the expected growth rate. Thus, the usual intuition
about asset substitution applies here. In this case, asset substitution can have a devastating
effect on the firm’s exploitation of the tax shield. Examples show that the firm’s exploitation of
the tax advantage of debt decreases by 30% or even more. In the same vein, asset substitution
significantly lowers the firm’s optimal “target” leverage. On the other hand, for a given increase
in volatility, a larger decrease in the expected growth rate makes asset substitution more costly
for the equity holders ex post and, hence, the firm better exploits the tax shield ex ante in this
case.

Since asset substitution can be costly ex ante, it is worthwhile to study the possibility to avoid
asset substitution through a debt renegotiation. That is, since the joint value of debt and equity
is higher if asset substitution is avoided – and the firm’s capital is subsequently re-optimized –
the debt and equity holders have a common interest in avoiding asset substitution. Therefore, we
incorporate the threat of asset substitution in a simple debt renegotiation game. In other words,
debt renegotiation is used as a mechanism with which value between debt and equity holders
can be distributed such that asset substitution is avoided. Compared to the case in which debt
renegotiation cannot be used to avoid asset substitution, debt renegotiation may indeed mitigate
the ex ante costs of substitution. However, if debt renegotiation is possible, the equity holders’
incentives to employ this strategy must be taken into account in general. In fact, the extension
of the equity holders’ strategy space may extend the earnings alternatives for which substitution
plays a role. That is, including asset substitution as a threat in debt renegotiation increases
the equity holders’ incentives to use asset substitution (as a threat) because the ex post costs of
asset substitution are lower in this case. Therefore, compared to the case in which substitution
is not a part of a debt renegotiation, potential asset substitution plays a role for a relatively low
increase in the volatility of the substituted earnings. Thus, albeit debt renegotiation is initially
introduced as a mechanism to mitigate the costs of asset substitution, debt renegotiation may
actually be costly ex ante, because the equity holders cannot credibly commit themselves to
only trigger debt renegotiation for particular (ex ante optimal) project alternatives. Hence, it
may be more or less costly ex ante to have asset substitution as part of the debt renegotiation.

We proceed with the analysis of the firm’s optimal capital structure choice and asset substitution as follows. First we set up a simple dynamic capital structure model with debt renegotiation without asset substitution in section I. That is, we establish a model which can handle the classic trade-off story of weighting the tax shield against default costs. In this basic model a simple debt renegotiation game is used as a debt restructuring mechanism in order to avoid a costly default. The basic model is inspired by the more elaborate debt renegotiation game considered in Christensen et al. (2002) where a successful debt renegotiation results in an optimization of the firm’s capital structure. Alternative debt renegotiation models along the lines of strategic debt service are studied in e.g. Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997), but we do not follow this strand of literature here. With the basic model in place we provide a few numerical results in order to give the main picture of the basic model and the traditional trade-off story. Section II extends the basic model in order to handle the possibility of asset substitution. The equity holders are initially equipped with one option to substitute the firm’s earnings to a pre-specified earnings alternative, i.e. an irreversible substitution is considered. We study how various asset substitution possibilities influence the firm’s optimal capital structure policy. The model is extended in section III to handle asset substitution as part of a debt renegotiation, and we illustrate the implications of the extended debt renegotiation game. Finally, section IV concludes the paper and the appendix at the end contains some technical remarks.

I The Basic Model

Consider a firm with some assets in place. The assets generate a perpetual stream of earnings before interest and taxes (EBIT) denoted as $\xi_t$ at time $t$. The firm is managed by a manager who maximizes the equity holders’ wealth, i.e. incentive problems between the manager and the equity holders are ruled out. Initially, the firm seeks to maximize its value and the firm has an incentive to issue debt due to a favorable tax system. In the following, debt and equity are considered as contingent claims on EBIT.

In order to focus on the effect of potential asset substitution, we restrict attention to a simple debt contract. The debt is perpetual and callable, and the firm promises to pay the debt holders a coupon rate equal to $C$. Hence, the coupon payments cease when the firm’s earnings are either sufficiently low, an event which could be thought of as default, or when the debt is called; the latter occurs when the earnings are sufficiently high. Moreover, debt is issued at par and the cost of issuing new debt is a fraction $k$ of the principal.
Similar to Goldstein et al. (2001), Christensen et al. (2002), and others, we utilize a simplified tax structure as follows. The debt holders pay the tax rate $\tau_i$ on all interest payments received from the firm. Consequently, the debt holders receive $(1 - \tau_i)C$ after tax. The coupon payments are tax deductible for the firm. The earnings before taxes, $\xi_t - C$, are taxed at the corporate tax rate $\tau_c$. This leaves the firm with the net income $(1 - \tau_c)(\xi_t - C)$. As in most of the literature, assume that all net income is immediately paid out as dividends to the equity holders.\(^1\) Thus, with a dividend tax rate of $\tau_d$, the equity holders receive $(1 - \tau_e)(\xi_t - C)$, where $\tau_e = \tau_c + \tau_d - \tau_c \tau_d$ is the effective tax rate from the equity holders’ perspective. Furthermore, the interest rate used for discounting is the after tax (on interest payments) rate, henceforth denoted $r$. In addition, we assume that $r > \mu$ in order to have well defined values.

With the tax structure and debt contract in place, we want to derive the market values of debt and equity. In order to do this, we assume that there exists a pricing measure, $Q$, under which EBIT follows the geometric Brownian motion

$$d\xi_t = \mu\xi_t dt + \sigma\xi_t dW_t, \quad \xi_0 = 1,$$

where $\mu$ is the expected growth rate and $\sigma$ is the volatility (rate) of EBIT. The initial level of EBIT is merely a scaling of the levels of the values because, at capital restructuring times, the market values of debt and equity are positive homogeneous of degree one. Using standard finance theory, e.g. Duffie (2001) or Dixit and Pindyck (1994), the value of debt and the value of equity each satisfy a partial differential equation. Since debt is perpetual there is no time dependence per se in the model and, hence, the market values of debt and equity are the solutions to the ordinary differential equations

$$\frac{1}{2}\sigma^2 \xi^2 D_{\xi \xi}(\xi) + \mu \xi D_{\xi}(\xi) - r D(\xi) + (1 - \tau_i)C = 0$$

and

$$\frac{1}{2}\sigma^2 \xi^2 E_{\xi \xi}(\xi) + \mu \xi E_{\xi}(\xi) - r E(\xi) + (1 - \tau_e)(\xi - C) = 0,$$

respectively. The solutions to equations (2) and (3) are, respectively,

$$D(\xi; \xi_0) = \frac{(1 - \tau_i)C}{r} + \hat{b}_1 \xi^{x_1} + \hat{b}_2 \xi^{x_2},$$

$$E(\xi; \xi_0) = (1 - \tau_e) \left( \frac{\xi}{r - \mu} - \frac{C}{r} \right) + \hat{a}_1 \xi^{x_1} + \hat{a}_2 \xi^{x_2},$$

\(^1\)This is a simplifying assumption used to obtain a tractable model. The optimal dividend policy problem is studied in e.g. Fan and Sundaresan (2000). In an EBIT framework, it is complicated to consider optimal dividend policies because it would involve (at least) one other state variable – accumulated retained earnings.
where $x_1 > 1$ and $x_2 < 0$, see Appendix A. The initial EBIT level is included as an argument in the market values for later use. The unknown constants $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$ must be determined by the appropriate boundary conditions, i.e. the debt and equity values when the debt is called (upper boundary) and at default (lower boundary). In the following, we consider various possibilities of the equity holders’ action when EBIT becomes low and we specify the associated boundary conditions.

I.A Equity Holders’ Actions

We consider three variations of the equity holders’ possible actions. In order to get an overview of the various possibilities, the possible strategies of the equity holders are outlined in Figure 1. Initially, the firm issues debt and equity on the financial market. For a sufficiently high level of EBIT, the equity holders call the debt and issue new debt in order to optimally exploit the tax advantage of debt. However, the equity holders’ strategies vary in the case of a low level of EBIT. For a sufficiently low EBIT level, the equity holders decide to enter the “default” stage in the figure. The default stage is one of three cases. In Case I, the equity holders cannot substitute the assets; however, by threatening the debt holders with a costly liquidation, the equity holders can make a debt renegotiation with the debt holders in order to re-optimize the capital structure of the firm. This case is studied in section I.B below.

![Figure 1: Outline of the equity holders' actions and the resulting restructuring of the firm.](image)

The possibility to irreversibly substitute the assets is introduced in Case II analyzed in section II. Here, the equity holders are initially equipped with one “substitution” option. If they exercise this option, the EBIT process is once and for all changed to a process with a lower expected growth rate and a higher volatility. However, if the equity holders want to avoid
a liquidation of the firm, debt service must be as promised in the debt contract. Subsequent
to the substitution the equity holders can restructure the firm’s capital by calling the debt or
renegotiate in order to avoid a costly liquidation, as in Case I. In addition to substituting the
assets, the equity holders still have the possibility of proposing a debt renegotiation – with
liquidation as a threat – in order to re-optimize the capital structure of the firm with the initial
non-substituted assets. However, if it is more profitable for the equity holders to substitute
the assets, they do so. Finally, Case III in section III allows the equity holders to use asset
substitution as well as liquidation as threats in a debt renegotiation.

I.B Dynamic Capital Restructuring without Asset Substitution

We now set up the basic structural model with dynamic debt restructuring, i.e. Case I in Figure 1.
The model is adopted from e.g. Christensen et al. (2002); thus we only outline the main steps of
the model. An important feature in this model is that the debt and equity values are homogenous
of degree one in the initial level of EBIT. To obtain this result in the geometric Brownian motion
environment, one needs to ensure that the payoff rate and the boundary conditions are linear in
the initial level of EBIT; indeed, the models in this paper are set up to satisfy these conditions.
Further details about the homogeneity is given in Appendix B. Homogeneity implies that
\[ \forall \phi > 0 : D(\phi \xi; \phi \xi_0) = D(\xi; \xi_0)\phi , \]
and equivalently for the equity value. In particular, if one sets \( \phi = \frac{1}{\xi_0} \), the initial debt and
equity values can be written as, respectively,
\[ D(\xi_0; \xi_0) = D(1; 1)\xi_0 \quad \text{and} \quad E(\xi_0; \xi_0) = E(1; 1)\xi_0 . \]

Recall that the cost of a debt issuance is the fraction \( k \) of the principal of the debt. The
initial value received by issuing debt and equity is therefore the market value of the firm minus
the debt issuance costs. Thus, the initial firm owners receive
\[ [E(1; 1) + D(1; 1) - kD(1; 1)]\xi_0 = [E(1; 1) + (1 - k)D(1; 1)]\xi_0. \quad (6) \]
The expression in (6) obviously depends on the coupon of the debt. Clearly, the initial firm
owners issue debt with coupon payments such that their value given in (6) is maximized. To fix
the notation, denote the highest possible value of the firm value minus the debt issuance cost
for a given level of EBIT as
\[ A(\xi) = [E(1; 1) + (1 - k)D(1; 1)]\xi \overset{\Delta}{=} A\xi , \quad (7) \]
which later will be referred to as the optimally levered firm value.
We now turn to the conditions for the debt and equity values at the call boundary. For sufficiently high earnings it is beneficial for the firm to increase the debt service in order to better exploit the tax shield. The firm does so by calling the existing debt and then issue new debt. Denote the call boundary as $\xi = u\xi_0$. When the debt is called the firm pays a call premium in addition to the principal of the debt. The call premium is a fraction of the debt’s principal, denoted as $\lambda$. By assumption, the debt is issued at par, hence the principal is $D(\xi_0; \xi_0)$. Thus, the value of the called debt is

$$D(\xi; \xi_0) = (1 + \lambda)D(\xi_0; \xi_0) = (1 + \lambda)D(1; 1)\xi_0.$$  

(8)

When the firm has paid the principal and the call premium to the debt holders, there is no debt in the firm. The equity holders therefore issue new debt in order to maximize their value, which at this instant is the total market value of the firm minus the debt issuance costs, i.e. the equity holders receive the optimally levered firm value. If $C^\ast = c^\ast \xi_0$ is the initial optimal coupon, the optimal coupon at the call boundary is $C^\ast = c^\ast u\xi_0$ due to homogeneity. Thus, the value matching condition for the equity at the call boundary is

$$E(\xi; \xi_0) = \max_C \left\{ E(\xi; \xi) + (1 - k)D(\xi; \xi) \right\} - (1 + \lambda)D(\xi_0; \xi_0)$$

$$= \left\{ E(\xi; \xi) + (1 - k)D(\xi; \xi) \right\}_{C = c^\ast u\xi_0} - (1 + \lambda)D(1; 1)\xi_0$$

$$= [Au - (1 + \lambda)D(1; 1)]\xi_0.$$  

(9)

Consider now the case where the earnings deteriorate to a low level. Eventually it becomes too expensive for the equity holders to service the debt and, hence, the equity holders decide to cease coupon payments. We denote this “default” level of EBIT as $\xi = d\xi_0$. In general, a number of various possibilities can be imagined at default. The most simple case is when the “absolute priority rule” is enforced. This implies that the debt holders must receive their principal before the equity holders receive anything. Alternatively, one can allow for a violation of the absolute priority rule, i.e. the equity holders maintain some value at default; this is often observed empirically, see e.g. Weiss (1990), Eberhart, Moore, and Roenfeldt (1990), and Betker (1995). We adopt this approach here. In addition, following Christensen et al. (2002) and others, we assume that the firm is optimally levered after a default (at the default level of EBIT).\(^2\) Finally, let the default costs be a fraction $\alpha$ of the (new) optimally levered firm value. Thus, at the default level $\xi = d\xi_0$ the optimally levered firm value without default costs is $d\xi_0 A$. The debt and equity holders can renegotiate in order to avoid the liquidation costs, $\alpha d\xi_0 A$.

\(^2\)The default value of the firm, which is distributed to debt and equity holders, has been modeled in various ways in the literature. For instance, Leland (1994) and Goldstein et al. (2001) use the “unlevered” firm value $V_U(\xi) = (1 - \tau_e)\frac{\xi}{r - \tau_e}$ minus default costs to determine the default value of the firm.
Since the focus in the present paper is on asset substitution effects, we adopt a simple splitting rule. Therefore, assume that the equity holders receive the fraction $\gamma$ of the saved liquidation costs and the debt holders receive the rest. A more elaborate debt renegotiation framework can be found in Christensen et al. (2002). With the assumption of the simple debt renegotiation, the boundary conditions for debt and equity at default become

$$D(\xi_1; \xi_0) = D(d\xi_0; \xi_0) = (1 - \alpha\gamma)d\xi_0A,$$

$$E(\xi_1; \xi_0) = E(d\xi_0; \xi_0) = \alpha\gamma d\xi_0A.$$ (11)

Note, that the right-hand side of the boundary conditions (8)-(11) are all linear in the initial EBIT level. Given the restructuring boundaries it is possible to solve for the constants in the debt and equity values in (4) and (5).

**Lemma I.1** With callable debt and the simple debt renegotiation, the equity and debt values can be written as

$$D(\xi_1; \xi_0) = \frac{(1 - \tau_i)c\xi_0}{r} + b_1\xi_0^{1-x_1}\xi^x_1 + b_2\xi_0^{1-x_2}\xi^x_2,$$ (12)

and

$$E(\xi_1; \xi_0) = (1 - \tau_e)(\frac{\xi}{r - \mu} - \frac{c\xi_0}{r}) + a_1\xi_0^{1-x_1}\xi^x_1 + a_2\xi_0^{1-x_2}\xi^x_2,$$ (13)

where the constants $a_1, a_2, b_1, b_2$ depend on the boundary parameters $d$ and $u$ as well as the coupon rate $c$, but not on the initial EBIT level $\xi_0$, see Appendix C.

Finally, the restructuring boundary parameters are determined such that it is ex post incentive compatible for the equity holders to call the debt and to declare default, respectively. To derive the parameters, we apply the smooth-pasting conditions, see e.g. Dixit (1993) and Øksendal and Sulem (2005). From the value matching condition (9) for the call boundary we have

$$\left.\frac{\partial E(\xi; \xi_0)}{\partial \xi}\right|_{\xi = u\xi_0} = A,$$ (14)

and with the value matching condition (11) for the default boundary we have

$$\left.\frac{\partial E(\xi; \xi_0)}{\partial \xi}\right|_{\xi = d\xi_0} = \alpha\gamma A.$$ (15)

The conditions (14) and (15) cannot be solved analytically. However, they are easily solved numerically. The basic model is a natural benchmark for measuring asset substitution costs since it disregards the asset substitution possibility and, hence, provides a better environment for exploiting the tax shield. Therefore, we report a few numerical examples below.
After tax interest rate $r$ 4.5% Call premium $\lambda$ 5%
Effective tax rate $\tau_e$ 50% Liquidation cost $\alpha$ 25%
Interest tax rate $\tau_i$ 35% Bargaining power $\gamma$ 50%
Tax refund rate $\varepsilon$ 50% Expected growth rate $\mu$ 2.5%
Debt issuance cost $k$ 5% Volatility $\sigma$ 20%

Table I: Parameters for the base case.

I.B.1 Numerical results

In addition to the above model specification, we assume that whenever the earnings are lower than the promised debt service, i.e. $\xi < C$, the firm cannot fully deduct interest payments from its tax bill. This is modeled by the effective tax refund parameter $\varepsilon \in [0, 1]$, i.e. the equity holders receive the dividend $(1 - \varepsilon \tau_e)(\xi - C)$ when earnings before taxes are negative.

Consider the base case with parameters given in Table I. These values are in line with what is used elsewhere in the literature. Table II reports the results of the basic model: the optimal coupon rate, the default and call boundaries, the optimally levered firm value, and the values of equity and debt. The second-last column shows $TAD$ which is a measure of the tax advantage of debt, see Goldstein et al. (2001). The measure is defined as

$$TAD = \frac{A}{(1 - \tau_e)\tau_i^{-\mu}} - 1,$$

(16)

hence $TAD$ is the relative improvement in the initial value to the firm owners compared to the case where the firm has no debt. The last column in the table reports the (initial) leverage, which is defined as the ratio of the debt value to the firm value, i.e. $Lev = \frac{D(1;1)}{E(1;1) + D(1;1)}$.\footnote{\textsuperscript{3}In later sections with two EBIT processes, we use the initial EBIT process as the denominator in the $TAD$-measure. Also, the leverage reported is the initial debt-to-firm-value ratio.}

In the base case, Table II shows that the optimal coupon rate is $c^* = 0.96$, the optimally levered firm value is $A = 28.2$, and the leverage is 45%. In this case, the “unlevered” firm value is 25 and, thus, the tax advantage of debt is about 13%. In order to get an idea of how the EBIT parameters affect the firm’s capital structure choice, we consider three “substitution” examples with a higher volatility. In two cases we also decrease the expected growth rate to 1.75%, which implies that the unlevered firm value is 18.2. We observe that the base case yields the highest optimally levered firm value. Otherwise, for the cases with the same expected growth rate, the lowest volatility yields the highest firm value and, clearly, a higher volatility has a negative impact on the firm’s leverage and the firm’s exploitation of the tax shield. Finally, a decrease in the expected growth rate significantly decreases the firm value and, in particular, the value of
The results indicate that substitution involving a decrease in the expected growth rate may have a strong impact on the equity holders’ incentives to substitute the assets.

II Asset Substitution

We extend the basic model to include the asset substitution problem. In this section, the equity holders can enforce asset substitution but they cannot use asset substitution as a threat in a debt renegotiation. This is Case II in Figure 1. Numerical results are studied in section II.B. The model is extended to Case III in section III in order to handle asset substitution as a threat in the debt renegotiation.

II.A The Asset Substitution Problem

The basic idea of the paper concerning the asset substitution problem is the following: when the firm has issued debt, the firm can irreversibly alter its earnings to be more volatile and to have a lower expected growth rate. The latter assumption implies that the perpetual (untaxed) value of EBIT decreases.\(^4\) In order to be concrete, suppose that the asset substitution takes place when the earnings decrease to the level \(\xi = d \cdot \xi_0\), where \(0 < d < 1\), and introduce the associated stopping time \(\tau_S = \inf\{t \geq 0 | \xi_t \leq d\xi_0\}\). Thus, before time \(\tau_S\) the assets governed by the firm yield the original (non-substituted) EBIT. After substitution at time \(\tau_S\) the firm’s EBIT evolves with a lower expected growth rate and a higher volatility. Hence, EBIT initially evolves as in (1) as long as \(t < \tau_S\), i.e.

\[d\xi_t = \mu \xi_t dt + \sigma \xi_t dW_t.\]  \(17\)

For \(t > \tau_S\) EBIT evolves according to

\[d\xi_t = \mu_S \xi_t dt + \sigma_S \xi_t dW_t.\]  \(18\)

\(^4\)This value is equal to \(\xi/(r - \mu)\), which increases in the expected growth rate.
The subscript \(S\) indicates that the EBIT process has been substituted. In the EBIT equations (17) and (18) the substitution conditions are that the expected growth rate decreases, \(\mu_S < \mu\), and that the volatility increases, \(\sigma_S > \sigma\).

We continue with the boundary conditions of our dynamic capital structure model with an irreversible substitution possibility. When the firm has issued debt initially, the equity holders consider two strategies instead of paying the contracted coupon, cf. Case II in Figure 1: either they call the debt or they enter the “default” stage. As in the model in section I.B, the equity holders optimally call the outstanding debt after a sufficient increase in EBIT. Immediately following the call, they optimally restructure the firm’s capital structure. Thus, the value matching and incentive compatibility conditions at the call boundary are essentially as in section I.B. Of course, the lower and upper boundaries, \(d\) and \(u\), as well as the value functions \(D(\cdot)\) and \(E(\cdot)\) differ from those in section I.B due to the substitution possibility.

Consider now a decrease in EBIT. As in the Case I model, the equity holders can propose a debt renegotiation offer to the debt holders in order to split the liquidation costs. In addition, the equity holders can now make an irreversible decision to change the evolution of EBIT from (17) to (18). The boundary conditions of the former strategy are as in (10), (11), and (15). For the latter strategy the respective value matching conditions are

\[
D(d\xi_0; \xi_0) = D_S(d\xi_0; \xi_0),
\]

\[
E(d\xi_0; \xi_0) = E_S(d\xi_0; \xi_0).
\]

Thus, the value matching conditions (19) and (20) state that the values before the asset substitution must equal the values after the asset substitution at the moment the asset substitution takes place. But note that substitution per se does not imply that the capital structure has been changed and, therefore, the debt holders must receive their contractual coupon rate \(C\) from the equity holders, otherwise the firm defaults. Since it is the equity holders who decide when to substitute the assets, the substitution level is determined by the smooth-pasting condition

\[
\left. \frac{\partial E(\xi; \xi_0)}{\partial \xi} \right|_{\xi = d\xi_0} = \left. \frac{\partial E_S(\xi; \xi_0)}{\partial \xi} \right|_{\xi = d\xi_0}.
\]

The final step is to derive the values of debt and equity when EBIT follows the substituted process (18) but the firm’s capital structure has not been re-optimized, i.e. \(D_S(\cdot)\) and \(E_S(\cdot)\). As previously, the equity holders can call the debt. This action occurs when \(\xi = u_S\xi_0\). Here, the call boundary conditions are

\[
D_S(u_S\xi_0; \xi_0) = (1 + \lambda)D(1; 1)\xi_0,
\]

\[
E_S(u_S\xi_0; \xi_0) = u_SA^*_S\xi_0 - (1 + \lambda)D(1; 1)\xi_0.
\]
where the optimal call boundary factor $u_S$ is determined by the equity holders’ incentives, i.e. a condition similar to (14) but with the substituted earnings process. Note that the debt holders must be paid the principal settled at the time of the initial debt issuance in (22). Furthermore, when the equity holders issue new debt, the optimally levered firm value per unit of EBIT is $A^*_S$. This is the optimally structured firm value net of restructuring costs when the volatility is high and the drift is low. Therefore, $A^*_S$ is given from Lemma I.1 with $\mu_S$ and $\sigma_S$.

If EBIT decreases sufficiently, $\xi = d_S\xi_0$, the equity holders propose a debt renegotiation in order to avoid liquidation. The conditions are similar to (10), (11), and (15) with $d_S$ and $A^*_S$ replacing $d$ and $A$, respectively.

From the above it follows that the debt and equity value functions $D(\cdot), E(\cdot), D_S(\cdot),$ and $E_S(\cdot)$ all have the same form as in Lemma I.1 since the boundary conditions are all linear in the initial EBIT level. Of course, the constants associated with the value functions $(b_1, b_2, a_1, a_2, b_{1S}, b_{2S}, a_{1S},$ and $a_{2S})$ depend on the boundary parameters $d, u, d_S,$ and $u_S$.

Finally, whether the equity holders choose to substitute the assets or to propose a debt renegotiation depends on the strategy which maximizes their value. Below we numerically analyze the ex ante and ex post effects of the substitution possibility. A discussion of our modeling of asset substitution follows in the proceeding section.

**II.B Numerical Results**

In order to study the effects of asset substitution, we focus on various combinations of the parameters of the substituted earnings process, i.e. changes in expected growth rate $\mu_S$ and volatility $\sigma_S$. Otherwise, we apply the base case parameters from Table I.

The ex ante effects of asset substitution are of central concern. We quantify the cost of the asset substitution possibility as follows. When asset substitution is not possible, the ex ante effect of the firm’s exploitation of the tax shield is quantified by the $TAD$ measure introduced in (16). When asset substitution is possible, it may influence the firm’s ex ante choice of capital structure and, hence, the firm’s capability of being able to exploit the tax advantage of debt. In the following we denote the firm’s tax advantage of debt as $TAS$ when asset substitution is possible.

\[TAS = \frac{A_{\mu,\sigma,\mu_S,\sigma_S}(1 - \tau_e)}{r - \mu} - 1,\]

where $A_{\mu,\sigma,\mu_S,\sigma_S}$ is the firm’s tax advantage of debt when asset substitution is possible.
(a) Loss of tax advantage of debt for various combinations of low expected growth rate and high volatility. The substituted parameters are in percentages.

(b) Equity holders’ optimal strategy at the lower boundary. The substituted parameters are in percentages.

Figure 2: Ex ante loss of the tax advantage of debt and the equity holders’ optimal ex post strategy for various choices of an alternative earnings process. The base case parameters from Table I are used.

in the exploitation of the tax shield relative to the tax shield exploitation if asset substitution is not possible. That is, we introduce the measure

$$\psi(\mu_S, \sigma_S) = -\left( \frac{TAS}{TAD} - 1 \right),$$

which depends on the changes in the expected growth rate and the volatility. Thus, $\psi$ measures the relative loss in the tax advantage of debt when the asset substitution possibility is taken into account and the size of a loss depends on the characteristics of the substituted process. Clearly, $TAS = TAD$ if the equity holders do not have incentives to substitute the assets, i.e. the tax advantage of debt is not hampered and, as a result, $\psi(\mu_S, \sigma_S) = 0$.

Figure 2 reports the results when the equity holders can substitute the assets. We analyze the effects of the substitution possibility as follows. First, for a fixed decrease in the expected growth rate we consider what happens as the volatility increases. Second, we fix an increase in the volatility and consider the effects of a decreasing expected growth rate.

The ex ante costs of asset substitution in the various cases are depicted in Figure 2(a). Consider first the case where the equity holders substitute to an EBIT process with an expected growth rate equal to $\mu_S = 1.75\%$. In this case, the loss of the tax advantage of debt is depicted as the dotted curve in Figure 2(a) for various choices of volatility. The figure illustrates that the volatility must increase to more than 35% before there is any effect of asset substitution. Thus, substitution to an earnings process with a volatility lower than 35% is not ex post optimal for optimally levered firm value $A$ when EBIT initially follows (17) but it can be changed to (18).
the equity holders. Instead, the equity holders propose a debt renegotiation as in section I for a sufficient decrease in the earnings. As a consequence the firm’s capital structure is optimally restructured and, hence, asset substitution never takes place in this case. Of course, at the time of the debt issuance the debt holders correctly infer that substitution is never implemented and, therefore, the firm initially issues debt as if asset substitution was not possible. That is, there are no asset substitution costs and $\psi = 0$. In contrast, if the substituted volatility is higher than 35%, the optimal ex post strategy for the equity holders is to substitute the assets eventually. Anticipating the substitution, the debt holders decrease their valuation of the debt. As a result, the ex ante trade-off between tax shield exploitation and default costs induces the firm to issue less debt, i.e. to choose a lower initial leverage. Clearly, the equity holders become more willing ex post to use asset substitution as the volatility increases, whence it follows that the ex ante effects of a higher volatility is that the costs of asset substitution, $\psi$, increase and that the leverage decreases.\footnote{For this particular example, $\mu_S = 1.75\%$, various volatilities yield the following ex ante costs and leverage $\{(\sigma_S, \psi, \text{Lev}) \text{\%} \} = \{(36, 31, 33), (38, 33, 32), (40, 34, 31), (42, 38, 30)\}$.}

In general, we obtain the same picture for another fixed expected growth rate and, as seen in Figure 2(a), the lower the substituted expected growth rate, the higher must the substituted volatility be, before the substitution alternative is ex post optimal for the equity holders.

Similar to Leland (1998) our measure of agency costs is increasing in the substituted volatility. Notably, our results differ in terms of leverage. Leland finds that leverage is slightly increasing in the volatility of the substituted earnings process, whereas we obtain the opposite result, see e.g. footnote 9. The difference is due to the fact that Leland includes the call decision in the firm’s ex ante problem, whereas we let the equity holders’ ex post incentives determine the call. Thus, we obtain a negative relation between initial leverage and ex ante agency costs. Apparently, this seems counterintuitive: if agency costs are high, debt should be beneficial to restrict equity holders and, hence, lower agency costs. This line of reasoning builds on the assumption that debt is used as a financing mechanism for projects with unobservable quality. However, ex ante costs in our setting stem from the firm being less able to exploit the tax shield and, thus, leverage is reversely related to the ex ante agency costs.

We now turn to the second type of analysis, i.e. we fix the substituted volatility and consider various expected growth rates. Suppose $\sigma_S = 40\%$. In this case Figure 2(a) reveals that the cut off level for the expected growth rate is somewhere between 1.5% and 1.75%. That is, if the expected growth rate of the substituted earnings process is lower than 1.5%, then it is not optimal for the equity holders to apply asset substitution and, hence, there are no ex ante costs in this case. On the contrary, when the expected growth rate is higher than 1.75% the
equity holders prefer asset substitution compared to a debt renegotiation (based on default) and the possibility of an asset substitution therefore implies ex ante costs. Albeit the firm’s ex ante value of the tax exploitation is hampered in this case, the costs of asset substitution decreases as the substituted expected growth rate decreases. That is, the larger is the drop in the substituted expected growth rate, the lower is the increase in the $\psi$ curve in Figure 2(a).\textsuperscript{10} A potential substitution to a “worse” earnings process results in a higher tax advantage of debt and, equivalently, a higher initial firm value, because a lower expected growth rate makes it more costly ex post for the equity holders to substitute the assets. This implies that the equity holders are less willing to enforce the asset substitution, i.e. the equity holders accept a larger decrease in EBIT before they trigger the asset substitution. In turn, the debt holders foresee that asset substitution is less likely and increase their valuation of the debt accordingly. Therefore, it is ex ante optimal for the firm to issue more debt and, thus, the firm improves its exploitation of the tax advantage of debt.

The above analysis results in the following conclusion. In general, the equity holders’ optimal ex post strategy depends on the decrease in the expected growth rate relative to the increase in the volatility of the substituted assets. For other parameters than the above mentioned, Figure 2(b) illustrates how this relationship divides the $(\mu_S, \sigma_S)$-space into two regions depending on the equity holders’ optimal ex post strategy. If the increase in the volatility is too small relative to the decrease in the expected growth rate – the lower region in the figure – the equity holders prefer not to substitute the assets, whereas the equity holders’ incentives induce them to use asset substitution when the substitution alternative is in the upper region (labeled Sub in the figure). Hence, in the lower region in Figure 2(b) there are no ex ante costs of asset substitution. In the upper region, however, the equity holders eventually implement the asset substitution and, consequently, asset substitution is ex ante costly. In this case, the ex ante costs – the lower exploitation of the tax advantage of debt – are increasing in the substituted volatility and decreasing in the substituted expected growth rate. The ex ante costs can be high – for instance, $\psi(1.75, 40) = 34\%$ – and, thus, the asset substitution problem can have a devastating effect on the exploitation of the tax shield.

\textbf{II.C Discussion of the Substitution Opportunity}

In this section we discuss our modeling of asset substitution. We assume that substitution is irreversible, i.e. the equity holders have one option to change the firm’s earnings process. Intuitively, if the equity holders could reverse the substitution free of costs, the ex post costs

\textsuperscript{10}When $\sigma_S = 40\%$, we obtain the following ex ante costs, leverage, coupon, and substitution boundary: \{(\mu_S(\%), \psi(\%), Lev(\%), c, d)\} = \{(1.75, 34, 31, 0.64, 0.27), (2.00, 36, 30, 0.62, 0.31), (2.25, 42, 27, 0.56, 0.38)\}. Further results are reported in Table III in Appendix E.
of asset substitution would be lower. We would therefore expect that the equity holders would be relatively more eager to trigger substitution when earnings decrease. However, we choose an irreversible substitution because we find that the analysis of the equity holders’ incentives – the ex post effects – as well as the capital structure choice – the ex ante effects – is more focused in this setting. Moreover, it is not clear which type of modeling is the most “realistic” one. On the one hand, if substitution is merely obtained by, say, financial engineering, then reversibility is perhaps the most reasonable framework. On the other hand, if asset substitution is actually involving a change in the firm’s real investments, then reversibility is perhaps not so appropriate. We adopt the latter approach here as is also done in Ericsson (2000).

Another issue is the information about the asset substitution possibility available to the debt holders, the equity holders, and other parties. Intuitively, if the substitution opportunity is “well known” and if substitution is unwanted from the debt holders’ point of view, the debt holders would want a covenant which prohibit the equity holders from carrying out the substitution. Indeed, if such a contract was available, it would be ex ante optimal to grant this covenant to the debt holders because it would remove the asset substitution problem and, hence, the associated ex ante costs. It is therefore central how we interpret that the alternative earnings process is “well known”. For instance, if it is “well known” that the firm obtains a possibility to substitute its assets by making a real investment, but the ex ante information about the quality of the new investment alternative differs between the various stake holders, then “well known” implies asymmetric information about the earnings process. An interesting direction in this setting would include the possibility that the effect on the firm value of the new investment could be devastating as well as enhancing. Of course, such a model is the key to analyze how a firm’s capital structure can mitigate over- and under-investment problems. Stultz (1990) and Zwiebel (1996) provide discrete time models in this direction, but it is outside the scope of this paper to build such a continuous-time dynamic capital structure model.11

In contrast to asymmetric information stories, the present paper has a narrower scope and focuses on the analysis of tax shield exploitation given an asset substitution problem. Thus, the firm’s real investments are already in place – the EBIT generating machine – and the issue is to quantify how “concerned” the debt holders are about whether or not the equity holders change the firm’s EBIT to something more volatile. Therefore, we take the point of view that

11In addition, in such a setting the firm may issue debt for real investment reasons only, and not just for tax shield exploitation. In this case it is also problematic to assume that the firm’s dividend policy is designed such that the firm at any point in time has zero retained earnings. Furthermore, in an asymmetric information setting it is necessary to specify a manager of the firm and in particular the preferences of the manager and, hence, this opens up for principal-agent problems which are interesting but not considered in this paper.
“well known” means that the alternative process of the earnings is observable to the debt and equity holders but not verifiable by other parties, see e.g. Hart (1995). Consequently, albeit the debt and equity holders have symmetric information about how asset substitution influences the evolution of the EBIT process, it is not possible for the debt holders to go to the court and enforce a debt covenant written on substitution. In this situation the debt holders’ “concern” and, hence, the ex ante firm value, therefore depends on the equity holders’ ex post incentives which either induce or abstain the equity holders from substituting the assets.

A third issue related to the substitution assumption concerns the decrease in the expected growth rate relative to the increase in the volatility. Apparently, a pure risk shifting asset substitution provides us with a reasonable intuition about the relationship between the firm’s capital structure choice and the asset substitution problem. However, if the equity holders’ ex post costs of substituting the assets are sufficiently low, the equity holders’ incentives induce an immediate substitution. In this case, the asset substitution problem implies that the firm settles its capital structure as if the earnings were substituted ex ante. Clearly, this situation is a special case of the asset substitution problem. Notwithstanding, this situation can occur if there are no ex post costs in terms of a decrease in the expected growth rate.¹² In order to support the assertion we establish the following lemma.

**Lemma II.1** Suppose that substitution does not alter the expected growth rate, i.e. \( \mu_S = \mu \).

(i) Consider the model with an irreversible substitution as in (18) and suppose that an upward debt restructuring is not possible. Then the equity holders’ optimal ex post decision is to immediately substitute to the high volatility.

(ii) Consider a simple dynamic capital structure model with default and upward debt restructuring determined by the equity holders. Suppose the equity holders free of costs can make reversible substitutions between a low volatility \( \sigma \) and a high volatility \( \sigma' \). Then it is not optimal for the equity holders to have only one (interior) substitution boundary, \( \xi_S \), when the capital structure is unchanged, such that \( \sigma = \sigma' \), when \( \xi < \xi_S \), and \( \sigma = \sigma, \) when \( \xi > \xi_S \).

**Proof.** See Appendix D. ■

¹²Neither Leland (1998) nor Ericsson (2000) have direct ex post costs in terms of a decrease in the expected growth rate. Leland obtains that the equity holders do not substitute the assets immediately because the call boundary is determined ex ante in his model. Thus, the equity holders are forced to call the debt non-optimally and, hence, there are indirect ex post costs associated with a high volatility. In a framework similar to ours, Ericsson (2000) obtains that immediate substitution can be optimal. However, Ericsson’s default criterion is ex ante determined (a cash flow covenant) and ex post costs are indirectly imposed due to a constant aggregated debt structure based on an exponential retirement and re-issuance of new debt.
The first part of Lemma II.1 applies to the case where the firm issues non-callable debt. Going back to Merton (1974), Black and Cox (1976) and others, the equity holders intuitively have a claim similar to a call option written on the firm’s future earnings. The condition that no upward debt restructuring is possible implies that the firm only restructures its capital following a default and, hence, there are no obstacles to spoil the usual interpretation of equity as a call option. Thus, the equity holders always prefer the highest possible volatility.

The intuition behind part (ii) is the following. Asset substitution occurs when the equity holders’ incentives induce them to change the volatility. This is equivalent to a change in the equity holders’ risk preferences, i.e. the equity holders change the riskiness of the earnings when their risk preferences changes. Since the presence of risk seeking equity holders implies that the equity value function is convex, and the opposite holds when the equity value function is concave, the question is when the convexity of the equity value function changes. Since equity is intuitively similar to a call option, the equity holders prefer high volatility. This is central in the asset substitution problem because equity holders are generally risk seeking, whereas debt holders are generally risk averse. Hence, from an ex ante perspective of the asset substitution problem the appropriate question is why the equity holders do not substitute to the high volatility immediately after the debt issuance. At least we have to ask why the equity holders should be risk averse – and therefore choose a low volatility – when earnings are sufficiently high. In a (simple) dynamic capital structure model with reversible volatility shifts there is no obvious reason for equity holders to exhibit risk aversion because there are no “costs” imposed on equity holders if they change the volatility. In particular, since an upward debt restructuring is an opportunity for the equity holders to capitalize on an increased exploitation of the tax shield, the equity holders are clearly not risk averse for high earnings close to the upward restructuring boundary. Thus, if we do not add any costs of substitution, the equity holders cannot credibly abstain from always preferring high volatility ex post. Rational debt holders obviously foresee the immediate substitution and, thus, price their debt claim accordingly. That is, the change to the high volatility must effectively be carried out immediately in equilibrium.

Generally, changes in the equity holders’ risk preferences cannot be excluded, but in the above setting the equity holders’ risk preferences are convex close to default as well as the call boundary and, hence, the equity holders prefer high volatility close to the restructuring boundaries. It follows that it is not optimal for the equity holders to have only one substitution point \( \xi_S \) such that volatility is low when earnings are above this point.\(^{13}\)

\(^{13}\)One could imagine other types of models where the equity holders may be risk averse for high earnings. For instance, if the firm issues convertible debt, then the debt holders have an option to substitute their debt claims to equity. In this case, it may be possible that the “old” equity holders find that a conversion dilutes their claim.
With the above examples in mind, the present paper takes the point of view that substitution involving a lower expected growth rate is not merely an extended framework of the asset substitution problem. Indeed the paper considers ex post costs in terms of a lower expected growth rate as a key in order to appropriately analyze asset substitution and its ex ante effects.

III Avoiding Asset Substitution via Debt Renegotiation

The previous section demonstrated that asset substitution can be ex ante costly. A mechanism which likely mitigates the ex ante costs of asset substitution is debt renegotiation where the debt and equity holders renegotiate with the purpose of avoiding the asset substitution. Intuitively, debt renegotiation is firm value improving because the firm’s assets are not substituted, which in turn benefits the debt holders and, hence, the firm can better exploit the tax shield. However, the debt renegotiation possibility need not only be ex ante value enhancing because the equity holders’ ex post costs of threatening with an asset substitution strategy are also affected.

We extend the model from section II in order to include asset substitution as a threat in a debt renegotiation. This is Case III in Figure 1. As in section II, the firm issues callable debt and the call conditions are basically as in the previous sections. The extension concerns the case of a low EBIT level where the equity holders enter the “default” stage. As previously, the equity holders either propose a debt renegotiation in order to avoid liquidation or they consider an asset substitution where the evolution of the current earnings (17) is substituted to the process (18). The conditions for the former strategy are described (10), (11), and (15) in the basic model in section I. However, in contrast to the model in section II, the equity holders now use asset substitution as a threat in a debt renegotiation. The debt renegotiation game is illustrated in Figure 3. The debt renegotiation process is initiated by the equity holders, who propose a debt renegotiation to the debt holders. If the debt holders accept the proposal, the equity holders pay the offered value to the debt holders and the firm’s capital is subsequently optimally restructured. On the other hand, if the debt holders reject the equity holders’ debt renegotiation offer, the equity holders immediately make an asset substitution, but they continue to service the debt as promised (if it is optimal to do so). This case corresponds to the substitution case in section II.A. Therefore, the debt and equity values are basically equal to $D_S(\cdot)$ and $E_S(\cdot)$, respectively, but the coupon and, hence, the post-substitution boundaries – $(d_S, u_S)$ – differ.

Since we have established the rejection values of debt and equity we can now determine the exact debt renegotiation offer proposed by the equity holders. The idea of the debt renegotiation is to avoid the asset substitution and to split the associated gain. The gain of avoiding the asset sufficiently for them to prefer a low volatility for high earnings.
substitution is the optimally levered firm value of the restructured firm with the high drift and low volatility earnings process minus the firm value if the asset substitution takes place. The debt renegotiation is initiated when $\xi = d\xi_0$ and at this point the optimally levered firm value is equal to

$$E(d\xi_0; d\xi_0) + (1 - k)D(d\xi_0; d\xi_0) = Ad\xi_0,$$

and the post asset substitution firm value is

$$E_S(d\xi_0; \xi_0) + D_S(d\xi_0; \xi_0).$$

Thus, the debt renegotiation gain ($RG$) is

$$RG(d\xi_0; \xi_0) = Ad\xi_0 - (E_S(d\xi_0; \xi_0) + D_S(d\xi_0; \xi_0)). \quad (25)$$

The equity holders’ proposal splits the gain in (25) between the debt and equity holders. Similar to the basic model, Case I in Figure 1, the equity holders’ bargaining power in the renegotiation process is represented by the parameter $\gamma \in [0, 1]$. Thus, the equity holders receive the fraction $\gamma$ of the debt renegotiation gain and the debt holders receive the rest of the gain. Hence, neither the debt holders nor the equity holders are worse off if the renegotiation results in a restructuring rather than an asset substitution. Given this distribution of values, the value matching condition at the lower restructuring boundary for debt is

$$D(d\xi_0; \xi_0) = (1 - \gamma)RG(d\xi_0; \xi_0) + D_S(d\xi_0; \xi_0), \quad (26)$$

and for the equity value we have

$$E(d\xi_0; \xi_0) = \gamma RG(d\xi_0; \xi_0) + E_S(d\xi_0; \xi_0). \quad (27)$$

Since it is the equity holders who propose the debt renegotiation, the EBIT level that triggers renegotiation is derived by solving the smooth pasting condition imposed by (27). Using (25) and rearranging terms yields the condition

$$\left. \frac{\partial E(\xi; \xi_0)}{\partial \xi} \right|_{\xi = d\xi_0} = \gamma A + (1 - \gamma) \left. \frac{\partial E_S(\xi; \xi_0)}{\partial \xi} \right|_{\xi = d\xi_0} - \gamma \left. \frac{\partial D_S(\xi; \xi_0)}{\partial \xi} \right|_{\xi = d\xi_0}. \quad (28)$$
Altogether, when the equity holders propose a debt renegotiation, they either use asset substitution or liquidation as a threat. In the former case, the conditions at the lower boundary are those in (26), (27), and (28), whereas the conditions in the latter case are similar to those in section II.A. The actual conditions therefore depend on the threatening strategy which maximizes the equity holders’ ex post value.

Our choice of the debt renegotiation game is, of course, one of many alternatives. For instance, debt could be serviced strategically or the debt holders could make take-it-or-leave-it offers as in e.g. Mella-Barral and Perraudin (1997) or Mella-Barral (1999). Our renegotiation game resets the firm’s capital structure in one go. In contrast, strategic debt service models impose temporary debt service write downs, i.e. the debt contract is not formally resettled. Moreover, if debt holders have the right to trigger debt renegotiation, the timing of debt renegotiation is determined by the debt holders’ incentives and not – as in the present paper – the equity holders’ incentives. In principle, the present model could be changed to consider this situation. However, with the purpose of avoiding asset substitution, we find that it is most reasonable to grant the renegotiation initiative to the equity holders.14

Another alternative debt renegotiation game is to have a sequence of offers by the equity holders and responses by the debt holders. One could then add costs of the debt renegotiation game such that each round of the debt renegotiation is costly either directly – for instance by a fractional reduction in the earnings – or indirectly due to a limited number of renegotiation rounds. The latter approach is pursued in e.g. Christensen et al. (2002). Either way, extending the debt renegotiation game in this direction would partly help to endogenize the relative bargaining power between the debt holders and the equity holders. Such extensions are interesting, however, they also complicate the model further. We choose a simple debt renegotiation game in order to focus on the primary effects of asset substitution. Clearly, the results of the model depend on the particular debt renegotiation game and, naturally, the previous as well as the following numerical analysis must be considered with this in mind.

III.A Numerical Results

In this section we study the effects of the extended debt renegotiation game. We analyze substitution as in section II.B. The ex ante costs of substitution are depicted in Figure 4(a). Suppose first the substituted earnings process has an expected growth rate $\mu_S = 1.75\%$. This case is exhibited as the dashed-dotted graph in the figure. Here, the tax advantage of debt

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14 In modeling terms, if the debt holders have the renegotiation initiative, the smooth pasting condition for the debt renegotiation boundary must be associated to the value matching condition for the debt holders.
is unaffected when the substituted volatility is lower than 25% and in this case, the equity holders optimally threaten with liquidation. As the volatility increases, substitution becomes more attractive. If the volatility is higher than 25%, the equity holders optimally threaten with asset substitution and the higher is the volatility, the higher is the ex ante costs of asset substitution. On the other hand, consider a fixed substituted volatility, for instance, \( \sigma_S = 40\% \). In this case, it is less costly for the equity holders to employ asset substitution as the decrease in the expected growth rate gets smaller. That is, the equity holders find asset substitution ex post more attractive for higher expected growth rates. Consequently, the tax advantage of debt and, hence, the optimally levered firm value decrease in the substituted expected growth rate, i.e. the ex ante costs of substitution increase. These results are qualitatively as in the case where asset substitution is possible but not a part of the debt renegotiation game. The intuition underlying the basic effects of asset substitution is therefore as explained in section II.

We now elaborate on how the difference in the equity holders’ possibility to apply asset substitution affects the firm’s capital structure choice. Consider the case of a fixed substituted expected growth rate equal to 1.75%. For this particular case, we illustrate the results in Figure 4(b) for various choices of substituted volatilities. Here, we compare the loss of the tax advantage of debt in the case where asset substitution is included in the debt renegotiation (dotted curve) with the case where it is not included (full drawn curve). As depicted in Figure 4(b), asset substitution plays a role for much lower substituted volatilities when the equity holders can threaten with asset substitution. Similar to the discussion concerning asset substitution in section II.B, the key to understanding this result is to figure out how the equity holders’ ex post costs of implementing asset substitution are affected by including asset substitution in the debt renegotiation.

When the equity holders can include asset substitution in the debt renegotiation, the equity holders’ ex post costs of asset substitution are based on the values on the off-the-equilibrium-path in the debt renegotiation game. That is, the equity holders’ ex post costs depend on the value of debt as well as the value of equity if substitution takes place. Thus, there is a dichotomy in the decomposition of the equity holders’ ex post costs. On the one hand, the ex post costs increase in the “negativity” of the substitution alternative – that is, the decrease in the expected growth rate compared to the increase in the volatility – as in the previous model in section II. On the other hand, if a substitution takes place, also the debt value decreases in the negativity of the substitution alternative. However, this effect is beneficial for the equity holders because the off-the-equilibrium-path debt value, \( D_S \), is the “base” payment which the debt holders must receive, cf. condition (26). Thus, the lower off-the-equilibrium-path value of debt decreases the equity holders’ ex post costs of the asset substitution strategy. When the
(a) Loss of tax advantage of debt for various combinations of low expected growth rate and high volatility.

(b) Loss of tax advantage of debt when $\mu_S = 1.75\%$ in two cases: No Renegotiation (NR) of asset substitution or with renegotiation.

(c) Equity holders’ optimal strategy at the lower boundary in the Case II model (Sub) and the Case III model (Reneg). $(\mu, \sigma)$ indicate the initial EBIT parametrization.

Figure 4: Ex ante loss of the tax advantage of debt and the equity holders’ optimal ex post strategy at default for various combinations of substitution possibilities (shown in percentages).
equity holders cannot use asset substitution as a threat, the decreasing effect of the ex post costs is not present. Therefore, the equity holders’ ex post costs of asset substitution is lower in the case where the equity holders threaten with substitution and, hence, the equity holders are more inclined to apply the asset substitution strategy in this case. That is, asset substitution plays a role for additional combinations of alternative earnings processes and, hence, the firm’s ex ante capital structure choice is influenced by asset substitution in more circumstances when asset substitution is included in the debt renegotiation game.

The above discussion shows that the firm’s capital structure choice is influenced in more cases, but it is not immediate how the firm’s ex ante value is affected when asset substitution is a part of the debt renegotiation game. Similar to the case in section II, the ex ante negative effect of asset substitution is that the debt holders foresee the potential asset substitution and, thus, the firm’s exploitation of the tax shield is hampered. However, in the case of debt renegotiation there is also a mitigating effect of asset substitution. This is due to the fact that the debt holders realize that when the equity holders initiate the debt renegotiation, the debt holders receive a fraction of the gain of optimizing the capital structure of the firm and avoiding the substitution to the alternative earnings process. Therefore, the debt holders are less worried about asset substitution in this case, i.e. the debt holders increase their evaluation of the debt claim and, hence, the firm increases its exploitation of the tax shield.

Albeit debt renegotiation has a mitigating effect on the ex ante costs of asset substitution, the effect on the firm’s exploitation of the tax advantage of debt is not unanimous when asset substitution is included in the debt renegotiation game. For a concrete example, consider again Figure 4(b), where we fix the expected growth rate to $\mu_S = 1.75\%$. When substitution implies only a minor increase in the new volatility, $\sigma_S < 25\%$, then it is never optimal for the equity holders to substitute the assets. In this case asset substitution has no influence on the firm’s capital structure and, thus, the ex ante costs are zero. Suppose instead that the substituted volatility is moderately higher, $25\% < \sigma_S < 35\%$. Here, the equity holders only take the asset substitution opportunity into account if they can use asset substitution as a threat in a debt renegotiation. Therefore, the firm must also take cognizance of asset substitution when the capital structure is set and, thus, debt renegotiation hampers the initial firm value in this case. For instance, suppose the volatility can be substituted to $\sigma_S = 30\%$. Then there are no ex ante costs if substitution is not a part of the debt renegotiation, whereas the ex ante costs are $\psi(1.75, 30) = 2.3\%$ in the extended debt renegotiation game. Finally, consider a high volatility such that the equity holders employ asset substitution even if it is not a part of a debt renegotiation, i.e. $\sigma_S > 35\%$ in Figure 4(b). In this case, the above mentioned mitigating effect lowers the ex ante costs of asset substitution. For instance, if $\sigma_S = 40\%$, then the ex ante costs of
asset substitution are \( \psi(1.75, 40) = 34\% \) in the Case II model. However, since asset substitution is always employed for such high volatilities, the extension of the debt renegotiation game only has positive ex ante effects and as a consequence, the ex ante costs of asset substitution decreases to \( \psi(1.75, 40) = 10.2\% \).

Finally, we illustrate the equity holders’ optimal use of liquidation versus substitution. Figure 4(c) depicts two curves which separate the \((\mu_S, \sigma_S)\)-plane. Consider first the lowest curve, denoted “Reneg”, which corresponds to the Case III model. If the expected growth rate and volatility of the substituted assets are below the Reneg-curve, the equity holders optimally use liquidation as a threat. In this case there are no ex ante costs of asset substitution. Otherwise, the equity holders use asset substitution as a threat in the debt renegotiation in order to obtain a restructuring of the firm’s capital. In this case, asset substitution implies ex ante costs for the firm. Consider now the highest curve in Figure 4(c), denoted “Sub”. This curve stems from the Case II model, i.e. substitution is not included in the debt renegotiation, and it corresponds to the curve in Figure 2(b). As previously argued, asset substitution plays a role for a relatively low increase in the volatility, given that substitution can be used as a threat in the debt renegotiation and in accordance herewith, the Reneg-curve is positioned below the Sub-curve in Figure 4(c). However, if the debt renegotiation game does not contain the asset substitution threat, asset substitution is only important above the Sub-curve. For combinations of expected growth rates and volatilities above the Sub-curve, the equity holders always use the asset substitution strategy. Therefore, the extended debt renegotiation game is ex ante beneficial in this region. However, in the intermediate region – above the Reneg-curve but below the Sub-curve – the firm only needs to take asset substitution into account if the equity holders can use substitution as a threat and, thus, debt renegotiation is ex ante costly in this case.

We close our analysis by hypothesizing about the firm’s possibility to mitigate the ex ante costs of asset substitution. Albeit the present model does not allow the firm to choose between types of debt, one could imagine that the firm chooses between two different types of debt: one type is hard to renegotiate with respect to avoiding asset substitution, the other type is easy to renegotiate. For instance, the former type is publicly traded debt, the latter type is bank debt. The firm realizes ex ante the equity holders’ ex post opportunity to substitute the assets. If the asset substitution opportunity is above the Sub-curve in Figure 4(c), then the firm issues bank (renegotiable) debt in order to mitigate the asset substitution problem. If the substitution opportunity is in the intermediate Reneg-Sub region in Figure 4(c), the firm restricts the equity holders’ ex post strategy space by issuing public (hard renegotiable) debt. Finally, for a sufficient decrease in the expected growth rate of the substitution opportunity, the equity holders do not have strong enough incentives to substitute the asset and, thus, the firm is indifferent between the
two types of debt. To summarize, the results of the model predict that for severe (highly risk increasing) asset substitution opportunities, the firm should consider debt classes which allow for an ex post re-establishment of the equity holders’ incentives to avoid asset substitution. In contrast, for modest asset substitution opportunities, the firm should set up its capital structure such that there is a credible commitment on the equity holders’ ex post costs of substitution. Very ex post costly asset substitution opportunities are of no concern for the firm, since it is too expensive for the equity holders to employ the substitution. Intuitively, we thus find that for ex ante “very bad” asset substitution opportunities, debt renegotiation is beneficial since it is a mechanism with which the equity holders credibly abstain from substituting the assets, whereas renegotiation is not beneficial for “not very bad” substitution opportunities because renegotiation increases the equity holders’ ex post power too much.

IV Conclusion

This paper studies the interplay between the traditional trade-off theory and asset substitution. We include asset substitution in two different ways. First, the equity holders irreversibly substitute the firm’s assets when the earnings of the firm deteriorate. Second, the equity holders use asset substitution as a threat in a debt renegotiation. Importantly, the substitution and capital restructuring triggering decisions are made in accordance with the equity holders’ incentives.

A central element is that asset substitution increases volatility and decreases the expected growth rate of the earnings. This implies a trade-off between the decrease in the expected growth rate versus the increase in the volatility. The key to understanding this trade-off is the equity holders’ ex post costs of applying asset substitution which in turn determine the firm’s ex ante costs. The equity holders’ ex post costs of substitution decrease in volatility and, thus, ex ante costs of asset substitution increase. However, if the increase in the volatility is low relative to the decrease in the expected growth rate, the equity holders never employ asset substitution and, hence, asset substitution has no ex ante effects. Therefore, asset substitution need not always harm the firm’s exploitation of the tax shield, but if asset substitution is ex post optimal for the equity holders, the ex ante costs are significant. Debt renegotiation based on substitution threats is considered as a mechanism for mitigating the costs. The equity holders’  

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15 One could, of course, consider how the firm optimally exploits the tax advantage of debt in a simple dynamic capital structure model if it can choose how renegotiable the debt is. In order to keep the focus on the asset substitution problem we do not explore this extension further. A particular example concerning tax shield optimization is studied in Hackbart et al. (2004). Another interesting extension is to add imperfect information about the asset substitution such that the exact substitution opportunity is not known prior to the debt issuance, but all agents have the same priors about the opportunity.
ex post costs of substitution are partly offset since asset substitution is also ex post costly for the debt holders. Consequently, additional combinations of volatility and expected growth rates influence the firm’s capital structure. Debt renegotiation mitigates ex ante costs for “very bad” substitution opportunities, but it increases the costs for ”moderate” substitution alternatives. Thus, we hypothesize that the firm may design its debt according to its perception of the substitution opportunity. Altogether, our model shows that asset substitution has important consequences for a firm’s capital structure choice and that debt renegotiation in some cases help to mitigate the ex ante costs of asset substitution.

Appendix

A The solution to the differential equation

We here verify that the solution to the equations (2) and (3) are as given in equations (4) and (5). Consider a claim which continuously pays the rate $A \xi + B$ to the holder of the claim until a boundary is hit. Denote the price of this claim as $F(\xi; \xi_0)$. Then, from standard finance theory, the arbitrage-free price $F$ must satisfy the differential equation

\[
\frac{1}{2} \sigma^2 \xi^2 F_{\xi \xi}(\xi; \xi_0) + \mu \xi F_{\xi}(\xi; \xi_0) - r F(\xi; \xi_0) + A \xi + B = 0,
\]

until a boundary is hit. The conjecture is that the solution to the differential equation is

\[
F(\xi; \xi_0) = \frac{A}{r - \mu} \xi + \frac{B}{r} + \hat{k}_1 \xi^{x_1} + \hat{k}_2 \xi^{x_2},
\]

where $\hat{k}_1$ and $\hat{k}_2$ are constants determined by the boundary conditions. Furthermore, the constants $x_1$ and $x_2$ are the positive and negative root, respectively, of the fundamental quadratic

\[
Q(x) = \frac{1}{2} \sigma^2 x^2 + (\mu - \frac{1}{2} \sigma^2) x - r,
\]

i.e.

\[
x_1 = \frac{-(\mu - \frac{1}{2} \sigma^2) + \sqrt{(\mu - \frac{1}{2} \sigma^2)^2 + 2 r \sigma^2}}{\sigma^2}, \quad x_2 = \frac{-(\mu - \frac{1}{2} \sigma^2) - \sqrt{(\mu - \frac{1}{2} \sigma^2)^2 + 2 r \sigma^2}}{\sigma^2}.
\]

Since $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$, and $\sigma^2 > 0$, one gets that $x_1 > 1$ and $x_2 < 0$. The first and second derivative of $F$ are

\[
F_{\xi}(\xi; \xi_0) = \frac{A}{r - \mu} + x_1 \hat{k}_1 \xi^{x_1 - 1} + x_2 \hat{k}_2 \xi^{x_2 - 1},
\]

and

\[
F_{\xi \xi}(\xi; \xi_0) = x_1 (x_1 - 1) \hat{k}_1 \xi^{x_1 - 2} + x_2 (x_2 - 1) \hat{k}_2 \xi^{x_2 - 2}.
\]
Inserting (30)-(32) into the left-hand side of equation (29) yields
\[
\frac{1}{2} \sigma^2 \xi F_{\xi\xi}(\xi) + \mu \xi F_{\xi}(\xi) - r F(\xi) + A \xi + B
\]
\[
= \frac{1}{2} \sigma^2 x_1(x_1 - 1) \hat{k}_1 \xi^{x_1} + \frac{1}{2} \sigma^2 x_2(x_2 - 1) \hat{k}_2 \xi^{x_2} + \mu \frac{A}{r - \mu} \xi + \mu x_1 \hat{k}_1 \xi^{x_1} + \mu x_2 \hat{k}_2 \xi^{x_2}
\]
\[
- r \frac{A}{r - \mu} \xi - B - r \hat{k}_1 \xi^{x_1} - r \hat{k}_2 \xi^{x_2} + A \xi + B
\]
\[
= \left( \frac{1}{2} \sigma^2 x_1(x_1 - 1) + \mu x_1 - r \right) \hat{k}_1 \xi^{x_1} + \left( \frac{1}{2} \sigma^2 x_2(x_2 - 1) + \mu x_2 - r \right) \hat{k}_2 \xi^{x_2}
\]
\[
+ \left( \left( \mu - r \right) \frac{A}{r - \mu} + A \right) \xi - B + B
\]
\[
= 0,
\]
because, by definition, \( Q(x_1) = 0 \) and \( Q(x_2) = 0 \).

\section{Homogeneity}

An important feature of the modeling framework is that the price functions have some nice homogeneity features. Since this has already been discussed in Christensen et al. (2002), we only outline the derivation of the homogeneity properties.

Firstly, expand the notation of the price of the \( F \)-claim from Section A such that the arguments include the continuous pay off rate, the boundaries, \( \xi \) and \( \bar{\xi} \), and the values received at the boundaries, \( F \) and \( \bar{F} \):
\[
F(\xi; \xi_0) = F(\xi; \xi_0, A, B, \xi, \bar{\xi}, F, \bar{F}).
\]
(33)

In general, the value of a claim can be expressed as the present value of receiving the continuous pay off plus the present value of the value received at the boundaries. Consider first a perpetual claim that pays \( A \xi + B \) to the holder. Denote the value of this claim as \( G(\xi; A, B) \). Then, e.g. by calculating the (risk neutral) expected value, one has
\[
G(\xi; A, B) = \frac{A}{r - \mu} \xi + \frac{B}{r}.
\]
(34)

Now define \( p_d(\xi; \xi_0, 0, 0, \xi, 1, \bar{\xi}, 0) \) as the value of a claim that pays 1 when the lower boundary is hit, conditioning on not hitting the upper boundary before. Then \( p_d \) must satisfy
\[
p_d(\xi; \xi_0, 0, 0, \xi, 1, \bar{\xi}, 0) = k_1 \xi^{x_1} + k_2 \xi^{x_2}, \quad p_d(\bar{\xi}; \xi_0) = 0, \quad p_d(\xi; \xi_0) = 1.
\]
Similarly, define \( p_u \) as the price of the claim that pays 1 at the upper boundary if the lower boundary has not already been hit. Simple algebra then shows that the prices are
\[
p_d(\xi; \xi_0, 0, 0, \xi, 1, \bar{\xi}, 0) = \frac{\xi^{x_2} \xi^{x_1} + \xi^{x_1} \xi^{x_2}}{\Omega},
\]
(35)
\[
p_u(\xi; \xi_0, 0, 0, \xi, 0, \xi, 1) = \frac{\xi^{x_2} \xi^{x_1} - \xi^{x_1} \xi^{x_2}}{\Omega},
\]
(36)
where

$$\Omega = \xi^{x_2}B^x_1 - \xi^{x_1}B^x_2. \quad (37)$$

Using the prices of the three claims, one can now write the price of the $F$-claim as

$$F(\xi; \xi_0, A, B, \xi_2, \xi, F, \xi, \bar{F}) = G(\xi; A; B) + (F - G(\xi; A; B))p_d(\xi; \xi_0, 0, 0, \xi_1, 1, \xi, 0) + (\bar{F} - G(\xi; A; B))p_u(\xi; \xi_0, 0, 0, \xi_0, \xi, 1). \quad (38)$$

In order to show the homogeneity property of the $F$-claim, consider first the homogeneity properties of the $G$, $p_d$, and the $p_u$-claims. Fix any $\phi > 0$. Then, simply by inserting, it follows that

$$G(\phi \xi; A, \phi B) = \phi G(\xi; A, B),$$
$$p_d(\phi \xi; \phi \xi_0, 0, 0, \phi \xi, \phi 1, \phi \xi, \phi 0) = p_d(\xi; \xi_0, 0, 0, \xi, 1, \xi, 0),$$
$$p_u(\phi \xi; \phi \xi_0, 0, 0, \phi \xi_2, \phi 0, \phi \xi, \phi 1) = p_u(\xi; \xi_0, 0, 0, \xi_0, \xi, 1),$$

i.e. $G$ is positive homogeneous of degree one, and $p_d$ as well as $p_u$ are positive homogeneous of degree zero. Hence,

$$F(\phi \xi; \phi \xi_0, A, \phi B, \phi \xi_2, \phi F, \phi \xi, \phi F)$$
$$= G(\phi \xi; A; B) + (\phi F - G(\phi \xi; A; B))p_d(\phi \xi; \phi \xi_0, 0, 0, \phi \xi, \phi 1, \phi \xi, \phi 0)$$
$$+ (\phi \bar{F} - G(\phi \xi; A; B))p_u(\phi \xi; \phi \xi_0, 0, 0, \phi \xi_2, \phi 0, \phi \xi, \phi 1),$$

i.e.

$$F(\phi \xi; \phi \xi_0, A, \phi B, \phi \xi_2, \phi F, \phi \xi, \phi F)$$
$$= \phi G(\xi; A; B) + (\phi F - \phi G(\xi; A; B))p_d(\xi; \xi_0, 0, 0, \xi_1, 1, \xi, 0)$$
$$+ (\phi \bar{F} - \phi G(\xi; A; B))p_u(\xi; \xi_0, 0, 0, \xi_0, \xi, 1)$$
$$= \phi F(\xi; \xi_0, A, B, \xi_2, \xi, F, \xi),$$

Now consider $F$ at the initial level of EBIT with $\phi = \frac{1}{\xi_0}$. Then

$$F(\xi_0; \xi_0, A, B, \xi_2, \xi, F, \xi, \bar{F}) = \xi_0 F(\frac{1}{\xi_0}; \frac{1}{\xi_0}, A, \frac{1}{\xi_0} B, \frac{1}{\xi_0} F, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \bar{F})$$
$$= \xi_0 F(1; 1, A, \frac{1}{\xi_0} B, \frac{1}{\xi_0} F, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \bar{F}).$$

Since $F$ is either debt or equity, $F(1; 1, A, \frac{1}{\xi_0} B, \frac{1}{\xi_0} F, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \xi, \frac{1}{\xi_0} \bar{F})$ is independent of the initial level of EBIT because $B$, the boundaries, and the value received at the boundaries are all linear in $\xi_0$ when debt and equity is considered.

29
C Proof of Lemma I.1

In this section we argue how the constants in the debt and equity value can be derived. From (4) and (5) it is already known that

\[ D(\xi; \xi_0) = \frac{(1 - \tau_i)c\xi_0}{r} + b_1\xi^{x_1} + b_2\xi^{x_2} \] (39)

and

\[ E(\xi; \xi_0) = (1 - \tau_e)\left(\frac{\xi}{r - \mu} - \frac{c\xi_0}{r}\right) + \hat{a}_1\xi^{x_1} + \hat{a}_2\xi^{x_2}. \] (40)

We first rewrite the constants \( \hat{b}_1, \hat{b}_2, \hat{a}_1, \) and \( \hat{a}_2 \) to \( b_1\xi_0^{1-x_1}, b_2\xi_0^{1-x_2}, a_1\xi_0^{1-x_1}, \) and \( a_2\xi_0^{1-x_2}, \) respectively. Here, \( b_1, b_2, a_1, \) and \( a_2 \) should be independent of the initial level of EBIT. Now the debt and equity values are

\[ D(\xi; \xi_0) = \frac{(1 - \tau_i)c\xi_0}{r} + b_1\xi_0^{1-x_1}\xi^{x_1} + b_2\xi_0^{1-x_2}\xi^{x_2} \] (41)

and

\[ E(\xi; \xi_0) = (1 - \tau_e)\left(\frac{\xi}{r - \mu} - \frac{c\xi_0}{r}\right) + a_1\xi_0^{1-x_1}\xi^{x_1} + a_2\xi_0^{1-x_2}\xi^{x_2}. \] (42)

Using the boundary conditions (8)–(11) one gets

\[
\begin{align*}
D(u\xi_0; \xi_0) &= (1 + \lambda)D(1; 1)\xi_0 \\
E(u\xi_0; \xi_0) &= [Au - (1 + \lambda)D(1; 1)]\xi_0 \\
D(d\xi_0; \xi_0) &= (1 - \alpha\gamma)d\xi_0A, \\
E(d\xi_0; \xi_0) &= \alpha\gamma d\xi_0A,
\end{align*}
\]

i.e.

\[
\begin{align*}
(1 - \tau_i)\frac{c\xi_0}{r} + b_1\xi_0^{1-x_1}(u\xi_0)^{x_1} + b_2\xi_0^{1-x_2}(u\xi_0)^{x_2} \\
= (1 + \lambda)[(1 - \tau_i)\frac{c}{r} + b_1 + b_2]\xi_0, \\
(1 - \tau_e)\left(\frac{u\xi_0}{r - \mu} - \frac{c\xi_0}{r}\right) + a_1\xi_0^{1-x_1}(u\xi_0)^{x_1} + a_2\xi_0^{1-x_2}(u\xi_0)^{x_2}, \\
= \left\{(1 - \tau_e)\left(\frac{1}{r - \mu} - \frac{c}{r}\right) + a_1 + a_2 + (1 - k)[(1 - \tau_i)\frac{c}{r} + b_1 + b_2]\right\}u\xi_0, \\
(1 - \tau_i)\frac{c\xi_0}{r} + b_1\xi_0^{1-x_1}(d\xi_0)^{x_1} + b_2\xi_0^{1-x_2}(d\xi_0)^{x_2}, \\
= (1 - \alpha\gamma)\left\{(1 - \tau_e)\left(\frac{1}{r - \mu} - \frac{c}{r}\right) + a_1 + a_2 + (1 - k)[(1 - \tau_i)\frac{c}{r} + b_1 + b_2]\right\}d\xi_0,
\end{align*}
\]
and

\[
(1 - \tau_e) \left( \frac{d\xi_0}{r - \mu} - \frac{c\xi_0}{r} \right) + a_1 \xi_0^{1-x_1}(d\xi_0)^{x_1} + a_2 \xi_0^{1-x_2}(d\xi_0)^{x_2} = \\
\alpha \gamma \left\{ (1 - \tau_e) \left( \frac{1}{r - \mu} - \frac{c}{r} \right) + a_1 + a_2 + (1 - k)(1 - \tau_e) \frac{c}{r} + b_1 + b_2 \right\} d\xi_0.
\]

Note that in the above four equations, the initial EBIT level cancels out, i.e. there is no dependence on $\xi_0$. Hence, the constants $b_1, b_2, a_1$, and $a_2$ are independent on $\xi_0$. Furthermore, the above equation system is linear in $b_1, b_2, a_1$, and $a_2$. Thus, by simple but tedious calculations one obtains the debt and equity values given the boundary constants $d$ and $u$.

**D Proof of Lemma II.1**

Consider the first part of the lemma and let $d\xi_0$ denote the substitution boundary. Define the associated stopping (first hitting) time $\tau_S = \inf\{t \geq 0 | \xi_t \leq d\xi_0\}$. Then the condition of the first part of the lemma is that

\[
(\text{drift rate, volatility}) = \begin{cases} \\
(\mu, \sigma), & t \leq \tau_S \\
(\mu, \sigma_S), & t > \tau_S 
\end{cases}
\]

Given the coupon $c\xi_0$ we know from Lemma I.1 that the equity value function can be written on the form

\[
E(\xi; \xi_0) = (1 - \tau_e) \left( \frac{\xi}{r - \mu} - \frac{c\xi_0}{r} \right) + a_1 \xi_0^{1-x_1}\xi^{x_1} + a_2 \xi_0^{1-x_2}\xi^{x_2}, \quad t \leq \tau_S \tag{43}
\]

\[
E_S(\xi; \xi_0) = (1 - \tau_e) \left( \frac{\xi}{r - \mu} - \frac{c\xi_0}{r} \right) + a_1 \xi_0^{1-x_1}\xi^{x_1} + a_2 \xi_0^{1-x_2}\xi^{x_2}, \quad t > \tau_S, \tag{44}
\]

where $\varepsilon = 1$ is assumed for simplicity.

Suppose that upward debt restructuring is excluded. In order for the boundary associated with the irreversible substitution to be optimal, we must have value matching and smooth pasting. The value matching condition on the substitution boundary yields the condition

\[
a_1 \xi_0^{1-x_1}(d\xi_0)^{x_1} + a_2 \xi_0^{1-x_2}(d\xi_0)^{x_2} = a_1 \xi_0^{1-x_1}\xi^{x_1} + a_2 \xi_0^{1-x_2}\xi^{x_2},
\]

i.e.

\[
a_1 d^{x_1} + a_2 d^{x_2} = a_1 \xi_0^{1-x_1}\xi^{x_1} + a_2 \xi_0^{1-x_2}\xi^{x_2}, \tag{45}
\]

and the smooth pasting condition yields

\[
x_1 a_1 d^{x_1} + x_2 a_2 d^{x_2} = x_1 a_1 \xi_0^{1-x_1}\xi^{x_1} + x_2 a_2 \xi_0^{1-x_2}\xi^{x_2}. \tag{46}
\]
Now, when an upward debt restructuring is excluded, the value of an option to restructure upward has value 0 whence it follows that the constants $a_1$ and $a_{1S}$ are 0 (or, equivalently, since $x_1 > 1$ and $x_{1S} > 1$, ruling out speculative bubbles require the constants to be 0). The value matching condition (45) therefore becomes

$$a_2d^{F_2} = a_{2S}d^{F_{2S}},$$

and inserting into (46) yields

$$x_2a_{2S}d^{F_{2S}} = x_{2S}a_{2S}d^{F_{2S}},$$

and thus there is redundancy in the boundary conditions. That is, $d = 0$ or $d = 1$: either the equity holders never substitute the assets or they substitute immediately. Suppose substitution never occurs. It follows from e.g. Christensen et al. (2002) that default implying an equity value equal to 0 occurs at $\xi$, where

$$\frac{\xi}{r - \mu} = \frac{x_2}{x_2 - 1} \frac{c\xi_0}{r}.$$

Now, if the equity holders ex post surprisingly decide to substitute the volatility, then the coupon remains the same but the default level is

$$\frac{\xi_S}{r - \mu} = \frac{x_{2S}}{x_{2S} - 1} \frac{c\xi_0}{r}.$$

The implicit function theorem yields that $x_2$ increases in $\sigma$. Whence it follows that $\xi > \xi_S$. Thus

$$E_S(\xi; \xi_0) = E(\xi; \xi_0) > E_S(\xi_S; \xi_0) = 0,$$

since equity is increasing in EBIT. This contradicts the assumption that substitution does not occur and, hence, the equity holders substitute the volatility immediately.

We now turn to part (ii) of the lemma. Assume for simplicity that the effective tax refund rate is $\varepsilon = 1$. Suppose for a moment that there is a volatility interval $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ and that the firm’s capital structure is unchanged in the continuation region $(\xi, \overline{\xi})$ (with the coupon rate $C = c\xi_0$). In this case, the equity holders’ problem is to continuously choose $\sigma$ in order to maximize their value and a change of $\sigma$ imposes no costs. In this setting the equity value function is the solution to the Hamilton-Jacobi-Bellman equation

$$\sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \left\{ \frac{1}{2} \sigma^2 \xi^2 E_\xi(\xi; \xi_0) + \mu \xi E_\xi(\xi; \xi_0) - r E(\xi; \xi_0) + (1 - \tau_c)(\xi - c\xi_0) \right\} = 0.$$

Now, since the Hamilton-Jacobi-Bellman equation is affine in $\sigma^2$ it follows that the control is bang-bang in $\sigma^2$. But since only $\sigma \geq 0$ is admissible it follows that $\sigma$ is a bang-bang control and
that

\[ \sigma = \begin{cases} \overline{\sigma}, & \text{if } E_{\xi \xi}(\xi; \xi_0) > 0 \\ \underline{\sigma}, & \text{if } E_{\xi \xi}(\xi; \xi_0) < 0, \end{cases} \tag{47} \]

because \( \frac{1}{2} \xi^2 > 0 \). Thus, the equity holders may without loss of generality just as well have access to the end points of the volatility interval, i.e. \( \underline{\sigma} \) and \( \overline{\sigma} \), rather than the full interval. This is the case in e.g. Leland (1998).

Consider the requirement in (47) in relation to the capital restructuring boundaries. Close to default (the lower restructuring boundary) the equity holders are optimally risk seeking in order to avoid default (and they have limited liability), i.e. the equity holders choose \( \sigma = \overline{\sigma} \) close to default. This should be uncontroversible since it is at the heart of the asset substitution problem that equity holders want to increase risk close to default (and, hence, that the equity value function is convex). Consider the problem close to an upward restructuring. Here, we employ the usual interpretation from real options analysis and note that the equity value function is separable into three parts, see e.g. Dixit and Pindyck (1994). Part one is the fundamental component, i.e. the present value of perpetually receiving the payoff stream \( \xi - c\xi_0 \). This part is affine and thus weakly convex in \( \xi \). Part two is the value of the option to change (increase) the volatility when earnings decrease, and part three is the value of the option of making an upward debt restructuring. The key is that the equity holders have a “long” position in the two option components and, hence, the value of each option component is positive and convex. (The option values are power functions with powers \( x_2 < 0 \) and \( x_1 > 1 \), respectively, whence it follows that both of the option components are convex in \( \xi \).) Consequently, as the sum of convex functions is a convex function, the equity value function is convex in \( \xi \) and, hence, \( \overline{\sigma} \) is the optimal solution when the earnings are sufficiently close to the upward restructuring boundary.

In order to conclude, recall that the premise of part (ii) of the proposition is that there is not just a single (interior) substitution boundary, \( \xi_S \in (\overline{\xi}, \underline{\xi}) \). Suppose, in contrast, that there is only one substitution boundary, \( \xi_S \). According to condition (47) substitution to the low volatility therefore requires that \( \forall \xi \in (\xi_S, \overline{\xi}) : E_{\xi \xi}(\xi; \xi_0) < 0 \), i.e. the equity value function changes from being convex to being concave when the substitution takes place. However, as argued above the equity value function is convex close to default as well as close to an upward capital restructuring and, hence, for a volatility change to take place there must be an (at least one) interval strictly in the interior of \( (\xi_S, \overline{\xi}) \) where the equity value is concave in earnings. It follows that a single substitution point, \( \xi_S \), such that equity holders choose \( \overline{\sigma} \) when \( \xi \in (\underline{\xi}, \xi_S) \) and choose \( \underline{\sigma} \) when \( \xi \in (\xi_S, \overline{\xi}) \), cannot be the optimal solution, as asserted. □
Table III: Results without and with asset substitution as a threat, Case II and Case III respectively, where the assets can be substituted to $(\mu_S, \sigma_S)$. The base case parameters from Table I are used. $TAS$ is the tax advantage of debt, $Lev$ is leverage, and $\psi$ is the ex ante costs of asset substitution, i.e. the relative loss in the tax advantage of debt.

### Some additional numerical results

In this section, we report some further details related to the examples in the text. The first column in Table III indicates the corresponding model and the characterization of the substituted earnings process. For instance, Case II (1.75, 40) in the fourth row means that the numbers to the right (in the fourth row) are results from the Case II model, i.e. the model in section II with asset substitution, but with debt renegotiation only concerning liquidation. In this case, the equity holders have the possibility to substitute the firm’s asset such that the firm’s EBIT subsequent to the substitution is characterized by a geometric Brownian motion with expected growth rate $\mu_S = 1.75\%$ and volatility $\sigma_S = 40\%$. The resulting numbers are (from left to right): the optimal coupon rate, the ex post optimal substitution boundary, the ex post optimal call boundary, the (initial) optimal levered firm value, the initial value of equity, and the initial value of debt. These numbers are relative to the initial EBIT level $\xi_0 = 1$. The last three numbers are measured in percentages and they represent: the tax advantage of debt, the leverage, and the ex ante costs of asset substitution.
References


