Durable vs. Disposable Equipment Choice
Under Interest Rate Uncertainty

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Abstract

Considering interest rate uncertainty to be a relevant variable for the firm’s replacement decision problem we show that standard textbook approaches for replacement investment decisions can lead to non-optimal decisions in a stochastic interest rate environment. The problem with traditional approaches is that the real options related with subsequent replacement choices are not considered, i.e., the managers’ flexibility to switch between assets or equipments with different durability and expendability at each renewal time in response to some stochastic feature are completely ignored from the analysis. To overcome and highlight the shortcomings of the traditional approach presented in the academic textbooks we tackle the replacement investment problem in two ways. First, we consider the problem in a deterministic interest rate economy assuming that the only source of uncertainty is a permanent shock to a flat term structure of interest rates at a specified future date. Then, we consider the replacement problem under stochastic interest rates more explicitly in a CIR economy. The resulting formulae are explicit and quite easy to

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implement involving only numerical integration in the stochastic case. The solution to this problem seems to be extremely useful for corporate and public institutions managers when revenue or cost streams are relatively static and investment is driven by interest rate uncertainty since depending on the interest rate levels, interest rate volatility and the optionality to switch between durable and expendable assets at each renewal time managers may prefer to invest in long-lived but more expendable assets instead of short-lived but less costly assets and vice-versa.

*Keywords:* real options, interest rate uncertainty, replacement investment decisions.

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1 Introduction

It is now widely accepted that the traditional discounted cash flow approach to evaluate investment projects cannot properly capture managers’ flexibility to alter and revise their decisions in the future in response to unexpected economic changes. The so-called real options literature has become the new paradigm for investment decisions since it extends the lens in which we may evaluate investments under conditions of uncertainty\(^1\). Managers typically deal with two types of investment decisions: (i) investments in new plant or expansion of existing ones; and (ii) replacement investments. The existing real options literature has focused almost entirely in the evaluation of the first investment decisions type. However, it is important to point out that aggregate annual rate of replacement investment typically exceeds expansion or new investments by a wide margin [Mauer and Ott (1995)]. Since they are so abundant in practice, it is also interesting to analyze the case of replacement investment decisions under conditions of uncertainty.

Periodic replacement investment decisions (as well as expansion and new investments) typically involve a choice between different investment alternatives, i.e., a choice between mutually exclusive investment projects that also have, very frequently, different economic lives and different lump-sum initial costs. If we want to be precise we can distinguish the concepts of physical life, economic life and project life [see, for example, Myers and Majd (1990)]. The physical life of an asset represents its maximum life at which the asset will wear out and it must be replaced. The economic life of an asset is the length of time during which it is being used. The project life is not fixed and it depends on the decision to abandon it. In our case we want to focus our analysis on the replacement problem under stochastic interest rates. Thus, to simplify our setting we will assume that a project is not abandoned before the end of the asset’s physical life, in which its salvage resale value is zero. In others words, it will be assumed that abandonment values are nil in order to concentrate our analysis on the replacement problem exclusively, so that it is not optimal to replace the asset before the end of its physical life. In this case, the physical life, the economic life and the project life will be all equal and we will use its

\(^1\)The seminal book of Dixit and Pindyck (1994) provides an excellent revision of this literature. A complementary survey may be found in Caballero (1999).
names interchangeably throughout this paper.

In practice, firms have to select the best among mutually exclusive alternatives of investment possessing different lives. A number of approaches have been appeared in the literature to tackle such practical problems. Emery (1982) provides some guidelines for the evaluation of mutually exclusive investment alternatives with unequal lives and Pilotte (2000) shows how to evaluate mutually exclusive projects of unequal lives and differing risks and required returns. The most frequently traditional approaches suggested in the literature are\(^2\): (i) equivalent annual annuity (EAA), in which the annuity equivalent to the NPV of each project is computed; (ii) constant chain of replacement or infinite-cycle replacement (ICR), in which the NPV is computed by assuming that each project can be replicated infinitely at constant scale with market conditions remaining unchanged\(^3\); and (iii) least common multiple or finite-cycle replacement (FCR), in which the NPV is computed by assuming that each project is replicated the sufficient number of times until the lowest common multiple of the lives of the alternative projects under consideration is achieved.

It is important to point out that an equipment can be replaced either because its physical life has been achieved or because the current and future expected cash flows no longer justify its use, so that the optimal policy for the firm is to abandon the existing equipment and substitute it with a new one. Thus, replacement decisions are intimately related with abandonment decisions, though abandonment itself does not necessarily have to imply a subsequent replacement. Several studies have recognized the importance of such issue by incorporating the abandonment value in capital budgeting problems. Robichek and Van Horne (1967) are the first to highlight the importance of the option value to abandon a project. However, they assume that the project will be abandoned as soon as its salvage value exceeds the present value of the subsequent future cash flows of the project discounted at an exogenously given “appropriate” discount rate. Dyl and Long


\(^3\)It should be noted that EAA and ICR decision rules are equivalent provided that the investment projects being compared have the same risk level. If they have different risk, however, the EAA approach should not be used.
correctly extend this abandonment decision rule by considering that a firm should not abandon a project the first time the abandonment value exceeds the present value of the remaining cash flows, because there are cases where it may be optimal to wait and abandon in a future date. Thus, the rule is modified to consider the optimal timing of abandonment. Bonini (1977) generalizes these models using a dynamic programming model that includes abandonment possibilities to analyze investment decisions but under conditions of uncertain cash flows. However, in all these earlier models the abandonment decisions are not correctly addressed since an exogenously given cost of capital is viewed as the correct discount rate for the cash flows of the project, meaning that subjective or real world probabilities are being used instead of the appropriate risk-neutral world probabilities.

Using the insights of the option pricing theory, Kensinger (1980) develops a model which incorporates the option to abandon as a European put option. McDonald and Siegel (1985) presents a methodology for valuing investment projects with an option to temporarily and costlessly shut down production whenever variables costs exceed operating revenues. In this case, the owner of physical capital has a European call option to produce a given commodity. Since he has the right and not the obligation to exercise the option, he can avoid a loss by shutting down the plant if the variable cost of production exceeds the operating revenues at the option’s maturity. Myers and Majd (1990) extend this literature by valuing the option to abandon a project for its salvage value as an American put option, though this more general and correct specification of the abandonment option decision implies the use of numerical techniques to find the solution since there is no closed-form solution for the problem. If one considers a pure abandonment model there is an one-shot project, i.e., a single-cycle investment, that is not expected to be replaced. In order to clarify and distinguish abandonment and replacement models, Howe and McCabe (1983) derive decision rules for determining the optimal asset life for the cases of pure abandonment, infinite-cycle replacement and finite-cycle replacement problems, however without using the richness of the contingent-claims valuation approach.

To our knowledge, there are only three papers that use this framework to model replacement capital budgeting decisions under conditions of uncertainty. Baldwin and
Ruback (1986) study the effect of uncertain inflation on a firm’s choice between investments in fixed assets of different lives by modelling the investment in durable assets as a real option. They conclude that shorter-lived assets have a higher option value than longer-lived assets because the opportunity to switch between alternative machines arrives sooner. Thus, future asset price uncertainty creates a valuable switching option and benefits shorter-lived assets. McLaughlin and Taggart (1992) use the real options framework to compute the opportunity cost of diverting existing, but currently idle, equipment or facilities to a new project. By diverting capacity from a product A to a product B, the firm forgoes the option to produce A immediately, but it acquires an option to replace the diverted capacity. Thus, one component of the opportunity cost involves the analysis of a chain of replacement investment options for a plant. They focus their analysis on output price uncertainty and assume that all replacement plants are equivalent with zero salvage resale value, constant operating cost and a fixed life. Mauer and Ott (1995) analyze a replacement investment problem using a contingent-claims model with maintenance and operation cost uncertainty and study the effects of considering different parameter input values on the optimal replacement policy. Overall, they conclude that the firm’s replacement policy is not highly sensitive to the majority of the input parameters of the model, except for the cost of new asset and the salvage value of an old asset.

It is well known that upon asset replacement problems firms have the option to switch from a more durable but expendable equipment to a cheapest and less durable equipment, and *vice-versa*. Since in this case we have options that may have quite different maturities and unequal lump-sum initial costs, it is expected that interest rate volatility may influence the investment replacement decision. Given that previous work ignores this possibility, we extend the literature of replacement investment decisions by analyzing the impact of interest rate uncertainty on the value of switching between assets with different durability and expendability. Therefore, our paper is also related to a body of literature that examines the investment decision problem under stochastic interest rates, including, among others, Ingersoll and Ross (1992), Ross (1995), Lee (1997), Alvarez and Koskela (2006) and Dias and Shackleton (2005a,b). However, none of these papers have examined
replacement investment decisions under stochastic interest rates\textsuperscript{4}.

The remainder of the paper is organized as follows. Section 2 describes the firm’s problem and analyzes it under deterministic interest rates. To analyze the importance of the optionality effects we introduce interest rate uncertainty in a very simple way by allowing a permanent shock in a flat term structure of interest rates at a specified future date. Section 3 discusses the replacement problem under stochastic interest rates assuming a Cox et al. (1985a,b) economy. Section 4 concludes and presents avenues for future research.

\section{Replacement Problem Under Deterministic Interest Rates}

\subsection{Firm’s Problem Description}

Consider a firm that wants to purchase a production technology (for example, a particular equipment or machine) to produce a given output stream. Assume that two machines are available: a short-lived equipment S and a long-lived equipment L. Equipment S produces a fixed level of output for a given maintenance and operating costs. To use this equipment the firm has to pay a lump-sum cost $I_S$ and additionally it also costs $kdt$ to run at every instant. This equipment has a physical life of $T_S$ years and, as a result, it has to be replaced periodically. Alternatively, the firm can use a more expensive equipment L to produce the same level of output for a given maintenance and operating costs level (assumed to be equal for both equipments, though the problem is easily extended for the case where different maintenance costs are considered). In this case the lump-sum cost to be paid is $I_L$ (with $I_L > I_S$) and it also costs $kdt$ to run at every instant. But this equipment is more durable having a physical life of $T_L$ (with $T_L > T_S$), after which it has also to be replaced. Therefore, the firm’s problem is to choose between long-lived but more expendable and short-lived but less costly mutually investment alternatives.

\textsuperscript{4}Feldstein and Rothschild (1974) also consider the effect of interest rate changes on replacement investments, but using a deterministic model.
The choice between the two alternatives is based on the present value cost of using each equipment. The firm optimally picks the highest (i.e., least negative) of the two present value costs.

To simplify our analysis, we will consider that the firm only uses one equipment at a time that has to be replaced periodically at the end of its physical life $T$. The firm has the possibility to replace each equipment by an essentially equivalent one, with the same initial investment cost and similar maintenance and operating costs, for a given number of cycles $N$ that, in the limit, can be assumed to be infinite thus resulting in an infinite replacement case. In addition, after the first investment cycle firms may replace the equipment by a similar one or switch to the other, i.e., after each cycle firms have the option to switch between mutually exclusive equipments. It should be noted that in our simplifying setting we are considering that the firm does not have the possibility to expand or contract capacity. We are also ignoring any abandonment option effects in order to simplify our framework and concentrate exclusively on the replacement decision problem. We are assuming that revenue or costs streams are relatively static so that the investment decision is driven by interest rate uncertainty. Thus, the firm’s problem is to choose the best periodic replacement strategy under stochastic interest rates.

2.2 The Determinist Case

Before concentrating on the replacement problem under stochastic interest rates let us begin with an extreme simple case. Suppose that the spot rate is constant at $r_t = r$ for all $t$. In an economy with deterministic interest rates the time $t_0$ price of a default-free discount bond maturing at $T$ is equal to:

$$P(r, t_0, T) = e^{-r(T-t_0)}$$

where it is required that $P(r, T, T) = 1$. Under deterministic interest rates, it turns out that the present value cost (PVC) of a periodic replacement strategy for an equipment with an upfront cost of $I$ and that also costs $kdt$ to run every instant of time can be given by:
\[
PVC(1\text{st cycle of } I) = -I - \int_0^T k e^{-r(t)T} dt \quad (2a)
\]
\[
PVC(2\text{nd cycle of } I) = e^{-rT} \left[ -I - \int_T^{2T} k e^{-r(t)T} dt \right] \quad (2b)
\]
\[
PVC(3\text{rd cycle of } I) = e^{-2rT} \left[ -I - \int_{2T}^{3T} k e^{-r(t)T} dt \right] \quad (2c)
\]

Now generalizing and summing for \( N \) number of cycles gives:

\[
PVC(\text{\( N \) cycles of } I) = \sum_{n=1}^N \left[ e^{-(n-1)rT} \left[ -I - \int_{(n-1)T}^{nT} k e^{-r(t)T} dt \right] \right] \quad (3)
\]

or solving the integral we get an alternative solution given by:

\[
PVC(\text{\( N \) cycles of } I) = \sum_{n=1}^N \left[ e^{-(n-1)rT} \left[ -I - e^{-nrT} (e^{rT} - 1)k \right] \right] \quad (4)
\]

Both solutions (3) and (4) are given as a series form solution. However, it is straightforward to show that these can be changed if one prefers to use a closed-form solution suitable for any possible number of cycles \( N \) and that is not a series form solution. After some calculus we get:

\[
PVC(\text{\( N \) cycles of } I) = \frac{e^{(1-2N)rT}}{(e^{2rT} - 1)r} \times \left[ -(e^{rT} - 1)(e^{2NrT} - 1)k + e^{2NrT}(e^{rT} + 1)(e^{-N\tau T} - 1)rI \right] \quad (5)
\]

Thus, for computing the present value cost for a given number of cycles \( N \), hence resulting in the finite-cycle replacement case, we can use any of the equations (3), (4) or (5) since they will produce the same result. For the infinite-cycle replacement case we can just evaluate the limit of any of the equations as \( N \rightarrow \infty \).

To illustrate the deterministic case through a simple numerical example let us consider two equipments S and L, possessing different initial investment costs and different lives, but having to support the same cost \( k \) to run at every instant, as stated by Table 1, and where \( N \) stands for the number of cycles under the finite-cycle replacement case (FCR).
This results in a least common multiple of $N_i \times T_i = 15$. It is obvious that for the infinite-cycle replacement case $N \to \infty$.

Table 1

Simple numerical computations tell us that, for the case of a finite-cycle replacement, $PVC(T_S, N_S) = -33.87$ and $PVC(T_L, N_L) = -32.27$ for equipments S and L, respectively. Thus, assuming a constant riskless interest rate $r$ (i.e., an opportunity cost of capital) of 2 percent and a like-for-like asset replacement the choice of equipment L is the optimal strategy. Obviously, there may be interest rate levels where choosing equipment S is optimal. We determine the interest rate level, $r_N^*$, that would make the choice of S or L indifferent by equating the two present value costs and solving it in order to $r$, i.e., $PVC(T_S, N_S) = PVC(T_L, N_L)$, which results in $r_N^* = 8.2\%$. Therefore, for $r < r_N^*$ equipment L is the optimal strategy and for $r > r_N^*$ equipment S is the best choice.

Now considering the infinite-cycle replacement case we have $PVC(T_S, \infty) = -111.61$ and $PVC(T_L, \infty) = -105.06$ for equipments S and L, respectively. This confirms that L is the optimal choice for the assumed deterministic interest rate. The indifferent interest rate level, $r^*_{\infty}$, that makes $PVC(T_S, \infty) = PVC(T_L, \infty)$ is $r^*_{\infty} = 8.8\%$. Hence, equipment L is the optimal strategy for $r < r^*_{\infty}$ and equipment S is the optimal choice for $r > r^*_{\infty}$.

Summing up, we may conclude that assuming a like-for-like asset replacement in an economy with determinist interest rates long-lived but more expendable assets are always the best strategy for lower interest rates, whereas at times of higher interest rates firms are induced to buy short-lived but less expendable assets.

Using, for example, equation (5) it is possible to determinate the comparative statics (interpreted in absolute value terms) of the present value cost function under deterministic interest rates, $PVC(r, T, I, k, N)$, with respect to its five variables, e.g., the interest rate $(r)$, the physical life of the equipment $(T)$, the initial investment cost of the equipment $(I)$, the running cost at every instant of time $(k)$ and the number of cycles of the equipment $(N)$. The respective comparative statics are (see appendix A for additional details):

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\textsuperscript{5}Obviously, there are some parameter sets where the choice of one of the assets is always optimal, independently of the level of $r$, since there is no interception point.
\[
\frac{\partial PVC(\cdot)}{\partial r} < 0; \quad \frac{\partial PVC(\cdot)}{\partial T} \geq 0; \quad \frac{\partial PVC(\cdot)}{\partial I} > 0; \quad \frac{\partial PVC(\cdot)}{\partial k} > 0; \quad \frac{\partial PVC(\cdot)}{\partial N} > 0.
\]

However, even without such formulae it is quite intuitive to achieve such conclusions. It turns out that the present value cost function will present higher negative values (i.e., it will rise in absolute value terms) as \( I, k \) and \( N \) rise. On the contrary, the present value cost function will produce lower negative values (i.e., it will fall in absolute value terms) as \( r \) rises due to the time value of money effect. Finally, the comparative static of the present value cost with respect to \( T \) is indeterminate since it can rise or fall as \( T \) rises, depending essentially on the values of \( r \) and \( N \).

### 2.3 Optionality Effects

It is important to point out that all the traditional approaches usually entails a common assumption by assuming that the subsequent project is essentially the same as the current project. For example, it is clear that the two standard approaches for choosing between mutually exclusive projects with different initial costs and unequal lives that we have used in a deterministic interest rate environment, use an implicit assumption of a like-for-like asset replication. Thus, if it is optimal to choose equipment \( S \), with a \( T \)-year life, today, it is assumed that it will be optimal to choose again \( S \) in \( T \) years time. Since there is no uncertainty attached to subsequent rounds of investment under this formulation, the firm’s choice between short-lived or long-lived equipments can be made once and for all. This means that the firm may choose an investment programme of all short-lived or all long-lived equipments at the time of its initial investment decision, i.e., the firm makes a once-and-for-all choice of equipment durability. This is clearly a very strong assumption since it is assumed that several important variables, such as output price, interest rates, initial investment costs, etc., will remain unchanged over time. The problem with the so-called traditional approaches is that the real options related with subsequent replacement choices are not considered, i.e., the managers’ flexibility to switch between equipments with different durability and expendability in response to some stochastic feature are completely ignored from the analysis.
Before concentrating the analysis using a stochastic interest rate model it is possible to use a simple numerical example to illustrate the importance of uncertainty on the optimal replacement investment decision by introducing interest rate uncertainty in a very simple way. Until now, we are assuming a deterministic interest rate environment where the spot rate is constant at \( r_t = r \) for all \( t \). Let us assume that \( r = 8\% \). Let us also assume that after three years interest rates may suffer a permanent change to either 6% or 10% each with equal probability of occurrence. Afterwards, the interest rates will remain constant at either 6% or 10%, so that there is no further uncertainty beyond three years. If the firm chooses equipment S today, it will face a choice between equipments S and L in three years time, where interest rates can be at either 6% or 10% with equal probability. Since after that date there is no further uncertainty, a standard approach such as the infinite-cycle replacement method is appropriate to take the decision at that time. Similarly, if equipment L is chosen now the firm will face a choice between S and L in five years time, where interest rates are either 6% or 10% with equal probability. Once again, there is no further uncertainty beyond that date and the infinite-cycle replacement approach is used for taking the decision at that date.

Now we need to determine the optimal choice after the first cycle of investment. Since there is no further uncertainty beyond the first cycle date the infinite-cycle replacement method is appropriate to compare the PVC of equipments S and L. Assuming that interest rates suffer a permanent shock at the end of three years and increase to 10% we have \( \text{PVC}(T_S, \infty) = -25.04 \) and \( \text{PVC}(T_L, \infty) = -25.29 \). This indicates that the optimal choice in three years time is to choose equipment S if interest rates are at this level. Now, if interest rates have a permanent change and decrease to 6% we have \( \text{PVC}(T_S, \infty) = -39.43 \) and \( \text{PVC}(T_L, \infty) = -38.51 \). Thus, the optimal choice is to choose equipment L in the next cycle of investment if interest rates are at this level.

Armed with this information it is easy to illustrate our point. If equipment S is chosen

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\(^6\text{We are assuming now that } r = 8\% \text{ instead of the previously used 2\% to illustrate our point since the indifferent interest rate level under the infinite-cycle replacement case, } r^*_\infty, \text{ that makes } \text{PVC}(T_S, \infty) = \text{PVC}(T_L, \infty) \text{ is } r^*_\infty = 8.8\%. \text{ It should be remembered that equipment L is the optimal strategy for } r < r^*_\infty \text{ and equipment S is the optimal choice for } r > r^*_\infty. \text{ Thus, without uncertainty and considering } r = 8\% \text{ equipment L would be the optimal choice.} \)
today the PVC of such decision, $PVC^*_S$, is given by the PVC of the first cycle of investment in equipment S plus the PVC of the optimal choice in three years time, that is given by:

$$
PVC^*_S = PVC_S[0.08, 3, 5, 1, 1] + 0.5 \times \left[ PVC_S[0.10, 3, 5, 1, \infty] + \right.
$$

$$
PVC_L[0.06, 5, 7.5, 1, \infty] \times e^{-0.08 \times 3} = -32.66
$$

For the case of equipment L we have:

$$
PVC^*_L = PVC_L[0.08, 5, 7.5, 1, 1] + 0.5 \times \left[ PVC_S[0.10, 3, 5, 1, \infty] + \right.
$$

$$
PVC_L[0.06, 5, 7.5, 1, \infty] \times e^{-0.08 \times 5} = -32.92
$$

The results indicate that $PVC^*_S < PVC^*_L$. We have concluded that in a deterministic interest rate with $r = 8\%$ equipment L would be the optimal choice. However, by recognizing the possibility or optionality to switch between equipments at subsequent future dates has changed the optimal decision from equipment L to equipment S. Thus, ignoring optionality at each renewal time the firm chooses the long-lived but more expendable equipment L if interest rates remain constant at $r = 8\%$. Once interest rate uncertainty is introduced in a very simplified form, and thus optionality at future renewal dates is considered, the firm chooses the short-lived but less expendable equipment S. This is an intuitively appealing result since the value of the switching option increases as the life of the equipment decreases because the firm is allowed to switch sooner. Therefore, the option value component of the short-lived equipment exceeds the option value component of the long-lived equipment. This simple numerical example points out that corporate and public institutions managers must be advised that the standard textbook rules usually used for replacement investment decisions can lead to gross errors of analysis in a stochastic interest rate environment.
3 Replacement Problem Under Stochastic Interest Rates

In the previous section we consider the problem in a very simplified way by assuming that the only source of uncertainty is a permanent shock in a flat term structure of interest rates at a specified future date. Now, we will consider the replacement problem under stochastic interest rates more explicitly by using the so-called CIR term structure model [see Cox et al. (1985a,b)] since it is one of the models that satisfies most of the desirable properties expected from a term structure model and due to its simplicity and analytical tractability.

3.1 Term Structure Model Description

Consider a CIR economy in which $E^Q_{t_0}$ denotes the expectation under the martingale probability measure $Q$ (or risk-neutral measure $Q$), at time $t = t_0$, with respect to the risk-adjusted process for the instantaneous interest rate that can be written as the following stochastic differential equation:

$$dr_t = [\kappa \theta - (\lambda + \kappa) r_t] dt + \sigma \sqrt{r_t} dW^Q_t$$

(6)

where $\kappa$ is the parameter that determines the speed of adjustment (reversion rate), i.e., it measures the intensity with which the interest rate is drawn back towards its long-run mean, $\theta$ is the long-run mean of the instantaneous interest rate (asymptotic interest rate), $\sigma$ is the volatility of the process, $\lambda$ is the ”market” risk parameter (positive premiums will exist if $\lambda < 0$)$^7$, $r_t$ is the instantaneous interest rate and $dW^Q_t$ is a standard Brownian motion under $Q$. The price at time $t = t_0$ of a zero coupon bond maturing at time $T$ is then equal to:

$$P(r, t_0, T) = E^Q_{t_0} \left[ e^{-\int_{t_0}^{T} r(s) \, ds} \right] = A(t_0, T) e^{-B(t_0, T) r(t_0)}$$

(7)

$^7$It should be emphasized that although risk premiums for interest rates may be introduced, they cannot be observed or measured separately [see the detailed discussion in Dias and Shackleton (2005a, section 2) as well as the references contained therein].
where

\[ A(t_0, T) = \left[ \frac{2\omega e^{[(\kappa + \lambda + \omega)(T-t_0)]/2}}{(\omega + \kappa + \lambda)(e^{\omega(T-t_0)} - 1) + 2\omega} \right]^{2\sigma^2/\sigma^2} \]  

(8a)

\[ B(t_0, T) = \frac{2(e^{\omega(T-t_0)} - 1)}{(\omega + \kappa + \lambda)(e^{\omega(T-t_0)} - 1) + 2\omega} \]  

(8b)

\[ \omega = \left( (\kappa + \lambda)^2 + 2\sigma^2 \right)^{1/2} \]  

(8c)

### 3.2 The Stochastic Case

In this framework, the present value cost of a periodic replacement strategy for an equipment with an upfront cost of \( I \) and that also costs \( kdt \) to run at every instant of time can be given by:

\[
PVC(1\text{st cycle of } I) = -I - \mathbb{E}_0^Q \left[ \int_0^T k e^{-\int_0^r r(s) ds} dt \right] \]  

(9a)

\[
PVC(2\text{nd cycle of } I) = \mathbb{E}_0^Q \left[ e^{-\int_0^T r(s) ds} \left[ -I - \mathbb{E}_T^Q \left[ \int_T^{2T} k e^{-\int_0^r r(s) ds} dt \right] \right] \right] \]  

(9b)

\[ \vdots \]

Using the tower law of conditional expectations for the PVC of the second cycle of \( I \) we have:

\[
PVC(2\text{nd cycle of } I) = -I \mathbb{E}_0^Q \left[ e^{-\int_0^T r(s) ds} \right] - k \mathbb{E}_0^Q \left[ \int_T^{2T} e^{-\int_0^r r(s) ds} dt \right] \]  

(10)

and applying Fubini’s theorem to the second expectation operator gives:

\[
PVC(2\text{nd cycle of } I) = -I \mathbb{E}_0^Q \left[ e^{-\int_0^T r(s) ds} \right] - k \mathbb{E}_0^Q \left[ e^{-\int_0^r r(s) ds} \right] dt \]  

(11)

Now using equation (7) we can state the PVC of the second cycle of \( I \) as:

\[
PVC(2\text{nd cycle of } I) = -I P(r, 0, T) - k \int_T^{2T} P(r, 0, 2t) dt \]  

(12)

Generalizing and summing for \( N \) number of cycles gives:
\[ PVC(N \text{ cycles of } I) = -I \sum_{n=1}^{N} P(r, 0, (n-1)T) - k \sum_{n=1}^{N} \int_{(n-1)T}^{nT} P(r, 0, nt) \, dt \quad (13) \]

For computing the present value cost under the finite-cycle replacement case we use equation (13) for a given number of cycles \( N \). In order to perform numerical analysis we use the base case parameter values for equipments S and L that have been used in the deterministic case. Once again, the least common multiple for the finite-cycle replacement case is \( N_i \times T_i = 15 \). The infinite-cycle replacement case is obtained by evaluating the limit of the equation as \( N \to \infty \). For the stochastic case we will need to use other parameters for the CIR model. We suppose a CIR diffusion with an instantaneous interest rate with a mean level of 200 basis points (\( \theta = 0.02 \)), a mean reversion rate \( \kappa = 0.25 \), a volatility parameter \( \sigma = 0.10 \) and a “market” risk parameter \( \lambda = 0 \) [see Duffie and Singleton (2003, pg. 67)]. Table 2 summarizes these parameters. We will consider an initial interest rate level with \( r(0) = 1\% < \theta \) as our base case interest rate. In addition, we will also analyze the case where \( r(0) = 4\% > \theta \) for comparative purposes.

[Insert Table 2 Here]

Starting with the finite-cycle replacement case under stochastic interest rates in a CIR economy and using the base case parameter values presented above, we conclude that \( PVC(T_S, N_S) = -32.75 \) and \( PVC(T_L, N_L) = -32.21 \) for equipments S and L, respectively. In this case, the interest rate level that would make the choice of S or L indifferent is \( r^*_N = 8.42\% \). Thus, assuming a like-for-like asset replacement with a finite horizon, determined by the least common multiple of the assets’ physical lives we conclude that the optimal choice is to still invest in the long-lived equipment L for lower initial interest rate levels while firms optimally choose the short-lived equipment S for higher interest rate levels although the threshold has moved. For the infinite-cycle replacement case we have \( PVC(T_S, \infty) = -105.39 \) and \( PVC(T_L, \infty) = -100.36 \) for equipments S and L, respectively. This confirms the previous conclusion.

In order to get a better understanding of the impact of interest rate uncertainty on replacement investments we perform some numerical analysis to illustrate the effect of
different CIR parameter inputs, e.g., $\kappa$, $\theta$, $\sigma$ and $\lambda$, on the present value cost function and on the optimal replacement decision. Table 3 shows the present value cost for equipments S and L for both the finite-cycle replacement and infinite-cycle replacement cases with $r(0) = 0.01 < \theta$ and Table 4 reveals the same type of results but for the case of $r(0) = 0.04 > \theta$. The initial interest rate level, $r(0)^*_{\infty}$, that would make the choice of S or L indifferent is also presented by equating the two present value costs and solving it in order to $r(0)$, i.e., $PVC(T_S, N_S) = PVC(T_L, N_L)$. Now computing the indifferent interest rate level, $r(0)^*_{\infty}$, that makes the two alternatives essentially equivalent under the infinite-cycle replacement case is much more difficult since the computation time increases sharply. Since it is not expected that its computation provides any commensurate improvement for the analysis we will only present the results for the finite case.

[Insert Table 3 Here]

[Insert Table 4 Here]

Panel A of both tables examines the effect of speed of adjustment ($\kappa$) on the present value cost function and on the optimal replacement investment policy. The results show that the comparative static of the PVC with respect to $\kappa$ has a mixed effect. In addition, increasing the $\kappa$ parameter reveals that the interest rate threshold can rise or fall. Panel B of the tables shows the effect of long-run mean ($\theta$) on the present value cost function and on the optimal replacement investment policy. The results indicate that the current interest rate level $r(0)$ is crucial for the optimal decision and there is a $\theta$ level that determines the optimal replacement policy. The PVC function decreases in absolute value terms as $\theta$ rises. The interest rate threshold also decreases as $\theta$ rises. Panel C of the tables presents the effect of instantaneous volatility ($\sigma$) on the present value cost function and on the optimal replacement investment policy. The PVC function increases in absolute value terms as the volatility rises. As Figure 1 highlights there is a tendency for an increase in the interest rate threshold as the volatility starts rising. Panel D of the tables reveals the effect of "market" price of risk ($\lambda$) on the present value cost function and on the optimal replacement investment policy. The PVC function decreases in absolute value terms and the interest rate threshold falls as $\lambda$ rises.
Other results not included here (available upon request from the corresponding author) show that using other CIR parameter values for modelling the instantaneous interest rate may lead to negative interest rate thresholds for different levels of $\kappa$, $\sigma$ and $\lambda$ in addition to the previously discussed $\theta$ parameter. This clearly implies that one strategy is never preferred (the long-lived equipment in this case). Thus, the optimal selection rule determining the choice between short-lived and long-lived equipments under interest uncertainty may differ significantly from the selection rule under certainty.

Since CIR rates do not go to negative values it is important to explain more carefully why we can end up with such values. To determine the interest rate thresholds we use the root-finding function FindRoot in Mathematica. We can give FindRoot bounds on the region in which we want it to look for solutions. In this case it makes sense to use 0 as the lower bound ($r_{\text{min}}$) and a very large value as the upper bound ($r_{\text{max}}$) to avoid negative interest rates, and then choose a starting point ($r_{\text{start}}$) between the two bounds. If we do this we get no solution. This explains why one strategy is never preferred. Thus, the negative interest rate thresholds are only obtained if we do not impose any bounds in the numerical root-finding function. Although the resulting negative rates do not have any economic significance in a CIR economy they may be helpful when one wants to plot some graphs.

Now determining analytically the comparative statics (interpreted in absolute value terms) of the present value cost function under stochastic interest rates, where $PVC(\cdot)$ stands for $PVC(\kappa, \theta, \sigma, \lambda, r, T, I, k, N)$, is much more difficult. Numerical analysis indicates that the comparative statics with respect to the five variables that are common to both the deterministic and stochastic interest rate economies have the same behaviour in both cases, thus resulting in:

$$\frac{\partial PVC(\cdot)}{\partial r} < 0; \quad \frac{\partial PVC(\cdot)}{\partial T} \geq 0; \quad \frac{\partial PVC(\cdot)}{\partial I} > 0; \quad \frac{\partial PVC(\cdot)}{\partial k} > 0; \quad \frac{\partial PVC(\cdot)}{\partial N} > 0.$$ 

\textsuperscript{8}For example, using $\kappa = 0.2339$, $\theta = 0.0808$, $\sigma = 0.0854$ and $\lambda = 0$ as the base case parameter values obtained from the empirical work of Chan et al. (1992).
For the four CIR parameters, numerical analysis indicates that they have the following comparative statics effect (interpreted in absolute value terms):

\[
\frac{\partial PV_C(\cdot)}{\partial \kappa} \geq 0; \quad \frac{\partial PV_C(\cdot)}{\partial \theta} < 0; \quad \frac{\partial PV_C(\cdot)}{\partial \sigma} > 0; \quad \frac{\partial PV_C(\cdot)}{\partial \lambda} < 0.
\]

4 Conclusions

To overcome and highlight the shortcomings of traditional approaches presented in the academic textbooks we tackle the investment replacement problem in two ways. First, we consider the problem in a deterministic interest rate economy assuming that the only source of uncertainty is a permanent shock in a flat term structure of interest rates at a specified future date. Then, we consider the replacement problem under stochastic interest rates more explicitly in a CIR economy. The resulting formulae are explicit and quite easy to implement involving only numerical integration in the stochastic case. The solution to this problem seems to be extremely useful for corporate and public institutions managers when revenue or cost streams are relatively static and investment is driven by interest rate uncertainty on the cost side since depending on the interest rate levels, interest rate volatility and the optionality to switch between durable and expendable assets at each renewal time managers may prefer to invest in long-lived but more expendable assets instead of short-lived but less costly assets and vice-versa.

Overall, we may conclude that ignoring interest rate uncertainty by using the simplifying, but unrealistic, assumption of constant interest rates may produce errors of analysis. Hence, for cases where revenue or cost streams are relatively static and investment is driven by interest rate uncertainty it is important to analyze the effects of interest rate volatility on replacement decisions, even if uncertainty is at very low levels. Our results suggest that the optimal selection rule determining the choice between short-lived and long-lived equipments under interest uncertainty may differ significantly from the selection rule under certainty. Moreover, the analysis indicates that each replacement investment decision must be modelled on a case by case basis, by identifying the relevant determinants of the decision to replace and specifying the stochastic environment that is more appropriated for each case as well as the embedded option features.
It should be noted that we do not determine explicitly the value of switching between durable and expendable assets in a CIR economy. We know that the investment opportunity to be taken at the end of the first cycle of investment is a contingent-claim whose payoff includes the payoff function of an option on the minimum of two mutually exclusive investment risky assets. More specifically, if the firm chooses equipment S initially with a life of \( T_S \) years it gets an option to choose between S and L after \( T_S \) years. But if the firm chooses equipment L with a life of \( T_L \) years it receives an option to choose between S and L after \( T_L \) years. The valuation of these options are extremely complex since each replacement option involves further subsequent options, i.e., a compound option valuation problem. But even without focusing on the compound option valuation problem we can illustrate our point by considering the case of a simple replacement option that have to be made at the end of the physical life of each asset. By doing this we would be able to determine explicitly the value of switching between durable and expendable assets in a CIR economy. This will be left for future work. Other possibilities for future research include the option to abandon before the end of the physical life of the asset and thus the determination of the optimal replacement timing. Also, throughout this paper we have considered no technological uncertainty. Thus, another interesting avenue for future work is the consideration of a technological breakthrough (following a Poison process with a constant intensity parameter \( \lambda \geq 0 \) that would measure the probability per unit time that such a breakthrough occurs) as well as the examination of the applicability of these techniques in the presence of tax effects.
Appendix A: Comparative Statics for the Replacement Problem Under Deterministic Interest Rates

The purpose of this appendix is to present the comparative statics (interpreted in absolute value terms) of the present value cost function under deterministic interest rates, \(PVC(r, T, I, k, N)\), with respect to its five variables, e.g., the interest rate \(r\), the physical life of the equipment \(T\), the initial investment cost of the equipment \(I\), the running cost at every instant of time \(k\) and the number of cycles of the equipment \(N\).

\[
\frac{\partial PVC(r, T, I, k, N)}{\partial r} = \frac{e^{(1-2N)rT}}{(-1 + e^{2rT})^2 r^2} \left[ -e^{2N r T} \left(1 + e^{r T}\right)^2 I r^2 T \times \right.
\]
\[
\left. \left[ -1 + e^{-N r T} + e^{-N r T} \left(-1 + e^{r T}\right) N \right] + k \left(-1 + e^{r T}\right)^2 \times \right]
\]
\[
\left[ -1 + e^{(1+2N)rT} + r(1 - 2NT) + e^{2N r T} \left(1 - r T\right) - e^{r T} \left(1 + 2N r T\right) \right] < 0 \tag{A.1}
\]

\[
\frac{\partial PVC(r, T, I, k, N)}{\partial T} = \frac{e^{(1-2N)rT}}{(-1 + e^{2rT})^2} \left[ -\left(-1 + e^{r T}\right)^2 k \times \right.
\]
\[
\left. \left[ -1 + e^{2N r T} + 2N + 2e^{r T} N \right] - e^{2N r T} \left(1 + e^{r T}\right)^2 I r \times \right]
\]
\[
\left[ -1 + e^{-N r T} + e^{-N r T} \left(-1 + e^{r T}\right) N \right] \geq 0 \tag{A.2}
\]

\[
\frac{\partial PVC(r, T, I, k, N)}{\partial I} = \frac{e^{r T} \left(-1 + e^{-N r T}\right)}{-1 + e^{r T}} > 0 \tag{A.3}
\]

\[
\frac{\partial PVC(r, T, I, k, N)}{\partial k} = \frac{e^{r T} \left(-1 + e^{-2N r T}\right)}{(1 + e^{r T}) r} > 0 \tag{A.4}
\]

\[
\frac{\partial PVC(r, T, I, k, N)}{\partial N} = \frac{e^{(1-2N)rT}}{-1 + e^{2rT}} \left[ 2k T \left(1 - e^{r T}\right) - e^{N r T} \left(1 + e^{r T}\right) I r T \right] > 0 \tag{A.5}
\]
Table 1: Base case parameter values for the deterministic interest rate economy.

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<tr>
<th>Parameter</th>
<th>Equipment S</th>
<th>Equipment L</th>
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Table 2: Base case parameter values for the CIR economy.

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<th>Parameter</th>
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<tr>
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Table 3: Present value cost for equipments S and L for the FCR and ICR cases with $r(0) = 0.01 < \theta$. CIR base case parameters: $\kappa = 0.25$, $\theta = 0.02$, $\sigma = 0.10$ and $\lambda = 0$.

<table>
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<th>Parameter Value</th>
<th>$PVC(T_S, N_S)$</th>
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<th>$PVC(T_S, \infty)$</th>
<th>$PVC(T_L, \infty)$</th>
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Table 3: (continued)

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Table 4: Present value cost for equipments S and L for the FCR and ICR cases with $r(0) = 0.04 > \theta$. CIR base case parameters: $\kappa = 0.25$, $\theta = 0.02$, $\sigma = 0.10$ and $\lambda = 0$.

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<th>Parameter Value</th>
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<th>$PVC(T_L, \infty)$</th>
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Figure 1: Interest rate threshold as a function of interest rate volatility. CIR parameters: $\kappa = 0.25$, $\theta = 0.02$ and $\lambda = 0$. 
References


