Risk Management with Stress Testing: Implications for Portfolio Selection and Asset Pricing

Gordon J. Alexander
University of Minnesota

Alexandre M. Baptista
The George Washington University

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Abstract

Stress Testing (ST) is often used by banks and securities firms to set risk exposure limits. Accordingly, we examine a model with an agent who faces $K$ binding ST constraints and another who does not. We obtain four results. First, the constrained agent’s optimal portfolio exhibits $(K+2)$-fund separation. Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected returns of the risky securities that tends to lower them, thereby increasing the optimal portfolio’s weight in the riskfree security (or the minimum variance portfolio when this security is not available). Third, the market portfolio is inefficient. Fourth, a security’s expected return is affected by both its systematic risk and its idiosyncratic returns in the states used in the constraints. Thus, we provide further motivation to the literature in which security prices are not solely driven by systematic risk.

JEL classification: G11; G12; D81

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1. INTRODUCTION

Over the past ten years, some investors have suffered huge losses due to extreme events. For example, Barings Bank failed in 1995, Long Term Capital Management (LTCM) collapsed in 1998, and Enron went bankrupt in 2001. Furthermore, the terrorist attacks in the U.S. (2001), Spain (2004), and the U.K. (2005) have tremendously affected financial markets.

Since the occurrence of these events, the importance of risk management has been extensively recognized by banks and securities firms when deciding the amount of risk they are willing to take. Moreover, bank regulators now put an emphasis on risk management practices in attempting to reduce the fragility of banking systems.

Of the risk management tools currently available, Value-at-Risk (VaR) and Stress Testing (ST) have emerged as two of the most popular. For example, under the Basle Capital Accord, VaR is used in setting the minimum capital requirement associated with a bank’s exposure to market risk.\(^1\) Furthermore, the Committee on the Global Financial System (2005, pp. 1, 15) of the Bank for International Settlements and Scholes (2000) note that ST is often used by banks and securities firms to set risk exposure limits.

While the previous literature examines the impact of using VaR as a risk management tool on portfolio selection and asset pricing (see, e.g., Alexander and Baptista (2002)), it has yet to similarly explore the impact of using ST constraints.\(^2\) Our paper fills this gap in the literature by providing parsimonious characterizations of (1) optimal portfolios in the presence of ST constraints and (2) equilibrium security expected returns in a two-agent economy where one agent faces these constraints and the other does not.

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\(^1\) For an empirical examination of the VaRs of trading portfolios for a sample of banks, see Berkowitz and O’Brien (2002) and Jorion (2002). VaR is also widely used by fund managers; see, e.g., Hull (2006, p. 435).

\(^2\) An ST constraint is a restriction on the set of portfolios that are available for selection that specifies: (1) the state used in ST and (2) the maximum loss admissible in this state.
An examination of the impact of ST constraints is of particular interest for several reasons. First, as noted earlier, financial institutions often use them to set risk exposure limits. Second, even when a financial institution holds capital in excess of the minimum capital requirements as determined by the Basle Capital Accord, the ST constraints may still be binding. This can happen since ST captures extreme events where losses can be very large. Third, since there is empirical evidence that equity returns have skewness and kurtosis (see, e.g., Harvey and Siddique (2000) and Dittmar (2002)), an ST constraint is a device that can be utilized to control portfolio skewness and kurtosis. Fourth, the use of ST constraints can be motivated by the goal of limiting losses arising from the need to unwind positions in markets that become illiquid as a result of extreme events such as the LTCM collapse. Finally, since the return distributions of portfolios can resemble those of certain options strategies (see, e.g., Merton (1981) and Jagannathan and Korajczyk (1986)), the use of ST constraints can mitigate the option-like features in these distributions.

In investigating the impact of ST constraints, we use Markowitz’s (1952, 1959) mean-variance model. There are important reasons for doing so. First, this model is the cornerstone of portfolio theory. Second, it is widely used in practice to (1) determine optimal asset allocations, (2) measure gains from international diversification, and (3) evaluate portfolio performance. Third, since a set of ST constraints captures some measure of risk beyond variance (i.e., the returns in extreme events), a mean-variance objective function is of interest. Finally, the model has been extensively used in the banking literature.

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3 For a model in which security prices can deviate far from fundamental values, see Shleifer and Vishny (1997).  
4 If the objective function involves expected utility and the agent is restricted by a binding ST constraint, then the agent would not be (by construction) maximizing expected utility. Thus, it is natural to examine an objective function other than expected utility. While beyond the scope of our paper, an important question is whether the use of the mean-variance model with a set of ST constraints provides a better approximation to expected utility maximization than the use of this model without the constraints given, for example, the presence of estimation risk or skewness and kurtosis in portfolio returns.  
5 See, e.g., Hart and Jaffee (1974), Francis (1978), Koehn and Santomero (1980), Kim and Santomero (1988), and
Regardless on whether a riskfree security is available, we obtain four main results. First, the constrained agent’s optimal portfolio exhibits \((K + 2)\)-fund separation, where \(K\) is the number of binding ST constraints.\(^6\) While this result might not seem to be particularly surprising, Alexander and Baptista (2004) have shown that under certain conditions a constrained agent’s optimal portfolio still exhibits two-fund separation in the presence of a binding risk management constraint.

Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected returns of all risky securities that tends to lower them. Hence, the constraints tend to increase the optimal portfolio’s weight in the riskfree security (or the minimum variance portfolio when this security is not available).

Third, the market portfolio is inefficient.\(^7\) The reason why this result holds is that only two of the \(K + 2\) funds required for the constrained agent’s optimal portfolio are efficient.

Fourth, a security’s expected return is affected by both its systematic risk (i.e., beta) and its idiosyncratic returns in each one of those states that are used in the constraints.\(^8\) Specifically, securities with negative (positive) idiosyncratic returns in these states have relatively high (low) expected returns in equilibrium. The intuition for this result is straightforward. Due to the constraints, it is costly (beneficial) for the constrained agent to hold securities with negative (positive) idiosyncratic returns in these states. Accordingly, the agent requires them to have relatively high (low) expected returns.

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\(^6\) Rochet (1992). These papers differ from ours in that they did not examine portfolio selection and equilibrium implications arising from the use of ST constraints.

\(^7\) Our portfolio selection implications of ST constraints can also be extended to the case when the agent has a mean-tracking error variance objective function as in Roll (1992). Here, tracking error refers to the difference between the returns on the portfolio and a given benchmark. However, with a mean-tracking error variance objective function, the unconstrained and constrained optimal portfolios require a position in an additional fund (i.e., the benchmark).

\(^8\) For brevity, the term ‘efficient (inefficient)’ means ‘mean-variance efficient (inefficient).’

\(^8\) A security’s idiosyncratic return in a state is given by its return in this state minus its risk-adjusted return in the state. In our paper, the risk-adjustment is made using either the CAPM of Sharpe (1964), Lintner (1965), and Mossin (1966), or the Black (1972) model.
Next, we illustrate our theoretical results with a simple example. In this example, there are five securities (one of them is riskfree), two agents (each one has half of the wealth in the economy), and two binding ST constraints. Our main findings are as follows. First, since there are two constraints, the constrained agent’s optimal portfolio exhibits four-fund separation. Second, the portfolio’s weights in the two inefficient funds are notable, which cause it to have an efficiency loss of 2.28%.\(^9\) Third, since the market portfolio’s weights in these funds are smaller than those of the constrained agent’s optimal portfolio, the market portfolio has an efficiency loss of only 0.41%. Fourth, the effect of the constraints on the optimal portfolio is identical to that of a downward adjustment in the expected returns of the risky securities ranging from 4.18% to 9.46%. Thus, the weight of the constrained agent’s optimal portfolio in the riskfree security is substantially larger than that of the unconstrained agent’s optimal portfolio. Fifth, the cost of ST as measured by the reduction in the certainty equivalent return incurred by selecting the constrained portfolio is 2.83%. Finally, the component of a security’s expected return that arises from its idiosyncratic returns can be notable. This component is 2.11% for one of the securities that has negative idiosyncratic returns in both of the states used in the constraints. However, the component is −1.02% for one of the other securities since it has positive idiosyncratic returns in these states. In sum, by exploring the implications of ST constraints, we contribute to the literature in which security prices are not solely driven by systematic risk.

Previous theoretical papers in this literature have recognized the importance of idiosyncratic risk. For example, Levy (1978), Merton (1987), and Malkiel and Xu (2002) develop models in which security expected returns depend on both systematic and idiosyncratic risk.\(^9\) A portfolio’s efficiency loss is the increase in standard deviation arising from selecting it instead of the portfolio on the efficient frontier with the same expected return. All numbers in this paragraph are expressed in annualized terms.
The empirical literature has also examined the importance of idiosyncratic risk. Campbell, Lettau, Malkiel, and Xu (2001) and Malkiel and Xu (2003) show that the risk of individual stocks has noticeably increased over time. Moreover, Goyal and Santa-Clara (2003) provide evidence of a significant positive relation between the \textit{equal}-weighted average stock risk and the market return. In contrast, Bali, Cakici, Yan, and Zhang (2005) find no evidence of a significant relation between the \textit{value}-weighted average stock risk and the market return.

Two features of our work differ from the idiosyncratic risk literature. First, the reason why idiosyncratic returns affect expected returns in our model (i.e., the ST constraints) is, to the best of our knowledge, novel. Second, our model captures the effects of idiosyncratic \textit{returns} in some states (i.e., those used in the ST constraints) on asset pricing while the models in the literature capture the effects of idiosyncratic \textit{risk} on asset pricing.

Since our work derives the effects of a certain measure of risk beyond variance on asset pricing, it is important to emphasize that there is an extensive literature that examines the effects of skewness and kurtosis. In a seminal paper, Kraus and Litzenberger (1976) develop a model that captures the effect of unconditional skewness. Lim (1989) tests their model using stock returns and provides evidence that skewness is priced. More recently, Harvey and Siddique (2000) explore a model where conditional skewness is priced and present empirical evidence that it is helpful in explaining the cross-sectional variation in stock returns. Finally, Dittmar (2002) develops a framework in which agents are averse to kurtosis.

Our work differs from this literature in two respects. First, the importance of a measure of risk beyond variance arises in our model due to the existence of ST constraints while it arises in other models from agents having either a preference for right-skewed portfolios over left-skewed portfolios or an aversion to kurtosis. Second, our model captures the effects of
the returns in just those states used in the ST constraints on asset pricing while other models use the returns in all states to capture the effects of skewness and kurtosis.

Also related to our work are papers that investigate portfolio selection when an agent keeps his or her wealth above a floor. For example, Black and Perold (1992) and Grossman and Vila (1992) examine the case when the floor is non-stochastic, while Grossman and Zhou (1993) and Cvitanic and Karatzas (1995) examine the case when the floor is stochastic. Our work differs from these papers in several important ways. First, we examine the impact of a set of ST constraints rather than that of a floor. Second, we derive equilibrium results in the presence of ST constraints, while the aforementioned papers focus on portfolio selection when an agent keeps his or her wealth above a floor. Finally, we use the mean-variance model rather than the expected utility maximization continuous-time model.

The paper proceeds as follows. Section 2 characterizes optimal portfolios and equilibrium security expected returns when a riskfree security is present. Section 3 examines the case when a riskfree security is absent. Section 4 provides an example that illustrates our theoretical results and Section 5 concludes. The Appendix contains the proofs.

2. THE MODEL

2.1. Securities

Consider an economy where uncertainty is represented by $S$ states. There are $J$ risky securities and a riskfree security with return $R_f$. The returns of the risky securities are given by a $J \times S$ matrix $\mathbf{R}$, with $R_{js}$ denoting the return of security $j$ in state $s$. Let $\mathbf{R}$ be the

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10Note that a set of ST constraints notably differs from the existence of a floor (often also referred to as maximum drawdown). First, while the former restricts the wealth of an agent in some states, the latter restricts it in all states. Second, in the context of our model, a set of ST constraints restricts the wealth of an agent to be equal to or larger than a value that can vary from one state to another, while a floor restricts it to be equal to or larger than a value that does not depend on the state. Alexander and Baptista (2006) examine the portfolio selection implications of a floor in the mean-variance model, but do not investigate its equilibrium implications.

11Nevertheless, Markowitz and van Dijk (2003) find that under certain conditions, the mean-variance model provides a good approximation to multi-period expected utility maximization.
$J \times 1$ expected return vector and $V$ be the $J \times J$ variance-covariance matrix associated with $R$. Let $R_s \equiv [R_{1s} \cdots R_{Js}]^\top$. Suppose that (1) there is no arbitrage;\(^\text{12}\) (2) $\text{rank}(V) = J$ so that there are no redundant securities; and (3) $\text{rank}([1 \ \bar{R} \ R_s \cdots R_{s_{J-2}}]) = J$ for any set of $J - 2$ distinct states, $s_1, \ldots, s_{J-2}$, where $1$ denotes the $J \times 1$ vector $[1 \cdots 1]^\top$.\(^\text{13}\)

A portfolio is a $(J + 1) \times 1$ vector $w$ with $w_{J+1} = 1 - \tilde{w}^\top 1$, where $\tilde{w} \equiv [w_1 \cdots w_J]^\top$.\(^\text{14}\) Let $R_{ws}$ denote the return of portfolio $w$ in state $s$. Let $\bar{R}_w$ and $\sigma^2_w$ denote, respectively, the expected return and variance of $w$.

### 2.2. Agents

There are two agents with a mean-variance objective function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$U(\bar{R}, \sigma) = \bar{R} - \frac{\gamma}{2} \sigma^2,$$

where $\gamma > 0$.\(^\text{15}\) The *unconstrained* agent has $\gamma = \gamma^u$. The *constrained* agent has $\gamma = \gamma^c$, and is restricted to select a portfolio $w$ that satisfies $K$ binding Stress Testing (ST) constraints:

$$R_{ws} \geq -T_s, \ s = s_1, \ldots, s_K,$$

where (i) $1 \leq K \leq J - 2$, (ii) $s_1, \ldots, s_K$ are distinct states, and (iii) $T_{s_1}, \ldots, T_{s_K}$ are possibly distinct bounds in states $s_1, \ldots, s_K$, respectively.\(^\text{16}\)

\(^{12}\) For a characterization of absence of arbitrage, see, e.g., Duffie (2001, Chapter 1).

\(^{13}\) Condition (3) holds *generically* in terms of a probability-theoretic measure of size. Fix the set of all $(J + 1) \times S$ return matrices for which security $J + 1$ is riskfree and conditions (1) and (2) hold. Then, the probability of randomly choosing a return matrix such that condition (3) holds is one. Hence, condition (3) is not particularly restrictive.

\(^{14}\) Note that short-sales are allowed. There are important reasons for doing so. First, there are no analytical results when short sales are disallowed. Second, some money managers hold large short positions (e.g., banks with large trading portfolios and hedge funds). Third, short sales are now easier to execute due to the existence of exchange traded funds and futures on individual stocks.

\(^{15}\) Berk (1997) gives joint conditions on utility functions and return distributions that lead to mean-variance objective functions. For arbitrary distributions, the mean-variance model can be motivated with a quadratic utility function (see Huang and Litzenberger (1988, p. 61)). When no distributional assumption is made, this model can be used as an approximation to utility maximization (see Markowitz (1987, pp. 52-70)).

\(^{16}\) Our results can also be extended to the case when there are several constrained agents with each one of them: (1) having a possibly different $\gamma$ and (2) facing a possibly different set of ST constraints. For an extensive list of events used in ST, see Committee on the Global Financial System (2005, p. 30).
2.3. Portfolio Selection Implications

Next, we characterize the optimal portfolios of unconstrained and constrained agents.

2.3.1. Unconstrained Agent

A portfolio is on the mean-variance boundary if there is no other portfolio with the same expected return and a smaller variance. A portfolio is efficient if it lies on this boundary and has an expected return equal to or greater than $R_f$. Otherwise, the portfolio is inefficient.

Let $A \equiv 1^\top V^{-1} \bar{R}$, $B \equiv \bar{R}^\top V^{-1} \bar{R}$, and $C \equiv 1^\top V^{-1} 1$. Let $w_f \equiv [0 \cdots 0 1]^\top$ and $w_t \equiv \left[ \frac{[V^{-1}(\pi - 1R_f)]^\top}{A - CR_f} 0 \right]^\top$ denote the riskfree and tangency portfolios. Portfolios $w_f$ and $w_t$ are useful to characterize the unconstrained agent’s optimal portfolio.

**Theorem 1.** The unconstrained agent’s optimal portfolio is

$$w^u = (1 - \theta^u_t)w_f + \theta^u_t w_t,$$

where $\theta^u_t \equiv \frac{A - CR_f}{\gamma^u}$.

Theorem 1 says that the unconstrained agent’s optimal portfolio $w^u$ exhibits two-fund separation. Since these funds are $w_f$ and $w_t$, $w^u$ is efficient. Note that if $R_f$ is smaller (larger) than $A/C$, then $w_t$ is efficient (inefficient) and $\theta^u_t$ is positive (negative). Figure 1 illustrates Theorem 1 when $\theta^u_t > 0$. Note that $w_f$, $w_t$, and $w^u$, represented by, respectively, points $f$, $t$, and $u$, lie on the line representing the efficient frontier.

2.3.2. Constrained Agent

Let $F_s \equiv 1^\top V^{-1} R_s$ and $w_s \equiv \left[ \frac{[V^{-1}(R_s - 1R_f)]^\top}{F_s - CR_f} 0 \right]^\top$ where $s = s_1, ..., s_K$. Portfolios $w_{s_1}, ..., w_{s_K}$ are useful to characterize the constrained agent’s optimal portfolio.
Theorem 2. The constrained agent’s optimal portfolio is
\[
\mathbf{w}^c = \left(1 - \theta^c_t - \sum_{k=1}^K \theta^c_{sk}\right) \mathbf{w}_f + \theta^c_t \mathbf{w}_t + \sum_{k=1}^K \theta^c_{sk} \mathbf{w}_{sk},
\] (3)
where \( \theta^c_t \equiv \frac{A - CRf}{\gamma^c}, \theta^c_s \equiv \frac{F_s - CRf}{\gamma^c} \lambda_s \) for \( s = s_1, ..., s_K \), and \( \lambda_{s_1}, ..., \lambda_{s_K} \) are Lagrange multipliers associated with the ST constraints.

Theorem 2 says that the constrained agent’s optimal portfolio \( \mathbf{w}^c \) exhibits \((K + 2)\)-fund separation. Since these funds are \( \mathbf{w}_f, \mathbf{w}_t, \) and \( \mathbf{w}_{s_1}, ..., \mathbf{w}_{s_K} \) where the last \( K \) funds are inefficient, \( \mathbf{w}^c \) is also inefficient.

Figure 1 illustrates Theorem 2 when \( K = 1 \) and \( \theta^c_{s_1} < 0 \). Note that \( \mathbf{w}_{s_1} \) and \( \mathbf{w}^c \), represented by, respectively, points \( s_1 \) and \( c \), lie below the efficient frontier. Using Equation (3), we have
\[
\mathbf{w}^c = (1 - \theta^c_x) \mathbf{w}_f + \theta^c_x \mathbf{w}_x,
\] (4)
where \( \theta^c_x \equiv \theta^c_t + \theta^c_{s_1} \) and \( \mathbf{w}_x \equiv \frac{\theta^c_t \mathbf{w}_t + \theta^c_{s_1} \mathbf{w}_{s_1}}{\theta^c_x} \). Note that \( \mathbf{w}_x \), represented by point \( x \), lies on the dashed hyperbola representing combinations of \( \mathbf{w}_t \) and \( \mathbf{w}_{s_1} \). As Equation (4) implies, \( \mathbf{w}^c \) lies on the dotted line representing combinations of \( \mathbf{w}_f \) and \( \mathbf{w}_x \).

To isolate the impact of the ST constraints on the optimal portfolio from that of \( \gamma \), assume that \( \gamma^u = \gamma^c \). Using Theorems 1 and 2, we have \( \theta^u_t = \theta^c_t \). Thus, the constraints do not affect the optimal portfolio’s weight in fund \( \mathbf{w}_t \). Furthermore, since the fund weights of each of the optimal portfolios sum to one, we have \( \sum_{k=1}^K \theta^c_{sk} = (1 - \theta^c_t) - \left(1 - \theta^c_t - \sum_{k=1}^K \theta^c_{sk}\right) \). That is, the sum of \( \mathbf{w}^c \)’s weights in funds \( \mathbf{w}_{s_1}, ..., \mathbf{w}_{s_K} \) is equal to the difference between the weights of \( \mathbf{w}^u \) and \( \mathbf{w}^c \) in fund \( \mathbf{w}_f \).

The following result further explores the effect of the constraints on the optimal portfolio.
Theorem 3. Suppose that $\gamma^u = \gamma^c$. The constrained agent’s optimal portfolio when the expected return vector is $\overline{R}$ coincides with the unconstrained agent’s optimal portfolio when the expected return is $\overline{R}' \equiv \overline{R} + \sum_{k=1}^{K} \lambda_{s_k} (R_{s_k} - 1R_f)$.

Using Theorem 3, the effect of the ST constraints on the optimal portfolio is identical to that of an adjustment in the expected return vector $\overline{R}$. Since $\lambda_s > 0$ for $s = s_1, \ldots, s_K$, security $j$’s adjusted expected return $\overline{R}'_j$ is smaller (larger) than its expected return $\overline{R}_j$ if $R_{js}$ is smaller (larger) than $R_f$ for $s = s_1, \ldots, s_K$. In practice, a state is chosen to be in an ST constraint only if there are some securities with negative returns in the state. Hence, the case when $R_{js} \geq R_f$ for $j = 1, \ldots, J$ and $s = s_1, \ldots, s_K$ is not plausible.

In the case when $R_{js} < R_f$ for $j = 1, \ldots, J$ and $s = s_1, \ldots, s_K$, the expected returns of all risky securities are adjusted downward. Thus, fund $w_f$ is relatively more attractive. Consequently, the optimal portfolio’s weight in $w_f$ tends to be larger in the presence of the constraints.

Consider now the case when $R_{j_1s} < R_f < R_{j_2s'}$ for some distinct securities $j_1, j_2 \in \{1, \ldots, J\}$ and states $s, s' \in \{s_1, \ldots, s_K\}$. In this case, (1) the expected return of at least one risky security can be adjusted downward and (2) the expected return of at least one risky security can be adjusted upward. Thus, fund $w_f$ can be either relatively more or less attractive. Consequently, there is no clear tendency for the optimal portfolio’s weight in $w_f$ in the presence of the constraints.

2.4. Asset Pricing Implications

Letting $m \in \mathbb{R}^J$ denote the market portfolio and $\sigma_{jm}$ denote the covariance between security $j$ and $m$, security $j$’s beta is $\beta_j \equiv \sigma_{jm}/\sigma_m^2$. Furthermore, let $\varphi$ be the constrained agent’s fraction of the wealth in the economy, where $0 < \varphi < 1$. 
Theorem 4. In equilibrium, the market portfolio is inefficient. Furthermore, security $j$’s expected return is

$$
\bar{R}_j = R_f + \beta_j (\bar{R}_m - R_f) - \sum_{k=1}^{K} [(R_{j s_k} - R_f) - \beta_j (R_{m s_k} - R_f)] \delta_{s_k},
$$

(5)

where

$$
\delta_{s_k} \equiv \lambda_{s_k} \frac{\varphi / \gamma^c}{(1 - \varphi) / \gamma^u + \varphi / \gamma^c}, \; k = 1, ..., K,
$$

(6)

is a positive constant.

Theorem 4 says that in equilibrium the market portfolio is inefficient. The intuition for this result is straightforward. Since (1) the unconstrained agent’s optimal portfolio is efficient, (2) the constrained agent’s optimal portfolio is inefficient, and (3) the market portfolio is a combination of these optimal portfolios, the market portfolio is inefficient. Figure 1 illustrates that when $K = 1$, the market portfolio, represented by point $m$, lies below the efficient frontier on the dashed hyperbola representing combinations of $w_t$ and $w_{s_1}$.

Equation (5) indicates that security $j$’s expected return depends on $K + 2$ terms. The first two terms, $R_f$ and $\beta_j (\bar{R}_m - R_f)$, are identical to those contained in the CAPM.\footnote{The comparisons between the equilibrium of Theorem 4 and the CAPM are made given $R_f$ and $\bar{R}$. In general, however, $R_f$ and $\bar{R}$ in the former will differ in magnitude from $R_f$ and $\bar{R}$ in the latter.} However, the last $K$ terms, which are subtracted from the sum of the first two terms, are not present in the CAPM. Each one of these $K$ terms is given by the product of: (1) security $j$’s idiosyncratic return in state $s_k$, $(R_{j s_k} - R_f) - \beta_j (R_{m s_k} - R_f)$, and (2) the risk premium on the idiosyncratic return in state $s_k$, $\delta_{s_k}$, where $k \in \{1, ..., K\}$. We refer to the sum of these $K$ terms as the idiosyncratic return adjustment.

Since $\delta_{s_k} > 0$ for $k = 1, ..., K$, whether security $j$’s expected return is larger than, equal to, or smaller than that in the CAPM depends on its idiosyncratic returns in states $s_1, ..., s_K$. 
First, if security $j$’s idiosyncratic return is zero in states $s_1, ..., s_K$, then its expected return is equal to that in the CAPM. Second, if security $j$’s idiosyncratic return is (1) either zero or positive in states $s_1, ..., s_K$, and (2) positive for some state $s \in \{s_1, ..., s_K\}$, then its expected return is smaller than that in the CAPM. Third, if security $j$’s idiosyncratic return is (1) either zero or negative in states $s_1, ..., s_K$, and (2) negative for some state $s \in \{s_1, ..., s_K\}$, then its expected return is larger than that in the CAPM. Finally, if security $j$’s idiosyncratic return is negative in some state $s \in \{s_1, ..., s_K\}$ and positive in some state $s' \in \{s_1, ..., s_K\}$, then its expected return can be smaller than, equal to, or larger than that in the CAPM.

The intuition for why securities with negative (positive) idiosyncratic returns in states $s_1, ..., s_K$ have relatively high (low) expected returns in equilibrium is straightforward. Due to the constraints, it is costly (beneficial) for the constrained agent to hold securities with negative (positive) idiosyncratic returns in these states. Accordingly, the agent requires them to have relatively high (low) expected returns.

Using Equation (6), the risk premium on the idiosyncratic return in state $s_k$ depends on two constants: (1) $\lambda_{s_k}$ and (2) $\frac{\varphi/\gamma^c}{(1-\varphi)/\gamma^u + \varphi/\gamma^c}$. The first constant measures the constrained agent’s marginal cost (in terms of utility) arising from marginally decreasing the bound $T_{s_k}$ (i.e., tightening the ST constraint in state $s_k$). The second constant measures the constrained agent’s fraction of the risk-aversion-adjusted wealth in the economy. Note that if $\gamma^u = \gamma^c$, then this constant is equal to $\varphi$.

3. ABSENCE OF A RISKFREE SECURITY

Suppose now that there is no riskfree security. Let $w_a \equiv \frac{V^{-1}1}{C}$ and $w_b \equiv \frac{V^{-1}R}{A}$ denote, respectively, the minimum variance portfolio and the portfolio on the mean-variance boundary with expected return $B/A$. 

3.1. Portfolio Selection Implications

Next, we characterize the optimal portfolios of unconstrained and constrained agents.

3.1.1. Unconstrained Agent

In the absence of a riskfree security, a portfolio is efficient if it lies on the mean-variance boundary and has an expected return equal to or greater than $A/C$ (i.e., that of $w_a$).

**Theorem 5.** The unconstrained agent’s optimal portfolio is

$$w^u = (1 - \theta^u_b)w_a + \theta^u_b w_b,$$  \hspace{1cm} (7)

where $\theta^u_b \equiv \frac{A}{\gamma^u}$.

Theorem 5 says that the unconstrained agent’s optimal portfolio $w^u$ exhibits two-fund separation. Since these funds are $w_a$ and $w_b$, $w^u$ is efficient. Figure 2 illustrates Theorem 5. Note that $w_a$, $w_b$, and $w^u$, represented by, respectively, points $a$, $b$, and $u$, lie on the upper part of the thick hyperbola representing the mean-variance boundary.

3.1.2. Constrained Agent

Let $w_s \equiv \frac{V^{-1}R_s}{F_s}$ for $s = s_1, ..., s_K$. Portfolios $w_{s_1}, ..., w_{s_K}$ are useful to characterize the constrained agent’s optimal portfolio.

**Theorem 6.** The constrained agent’s optimal portfolio is

$$w^c = \left(1 - \theta^c_b - \sum_{k=1}^{K} \theta^c_{s_k}\right)w_a + \theta^c_b w_b + \sum_{k=1}^{K} \theta^c_{s_k} w_{s_k},$$  \hspace{1cm} (8)

where $\theta^c_b \equiv \frac{A}{\gamma^c}$, $\theta^c_s \equiv \frac{E_s}{\gamma^c} \lambda_s$ for $s = s_1, ..., s_K$, and $\lambda_{s_1}, ..., \lambda_{s_K}$ are Lagrange multipliers associated with the ST constraints.
Theorem 6 says that the constrained agent’s optimal portfolio \( w^c \) exhibits \((K + 2)\)-fund separation. Since these funds are \( w_a, w_b \), and \( w_{s_1}, \ldots, w_{s_K} \) where the last \( K \) funds are inefficient, \( w^c \) is also inefficient.

Figure 2 illustrates Theorem 6 when \( K = 1 \) and \( \theta^c_{s_1} < 0 \). Note that \( w_{s_1} \) and \( w^c \), represented by, respectively, points \( s_1 \) and \( c \), lie below the efficient frontier. Using Equation (8), we have

\[
w^c = (1 - \theta^c_x)w_a + \theta^c_x w_x,
\]

where \( \theta^c_x \equiv \theta^c_b + \theta^c_{s_1} \) and \( w_x \equiv \frac{\theta^c_b w_b + \theta^c_{s_1} w_{s_1}}{\theta^c_x} \). Note that \( w_x \), represented by point \( x \), lies on the dashed hyperbola representing combinations of \( w_b \) and \( w_{s_1} \). As Equation (9) implies, \( w^c \) lies on the dotted hyperbola representing combinations of \( w_a \) and \( w_x \).

To isolate the impact of the ST constraints on the optimal portfolio from that of \( \gamma \), assume that \( \gamma^u = \gamma^c \). Using Theorems 5 and 6, we have \( \theta^u_b = \theta^c_b \). Thus, the constraints do not affect the optimal portfolio’s weight in fund \( w_b \). Furthermore, since the fund weights of each of the optimal portfolios sum to one, we have

\[
\sum_{k=1}^{K} \theta^c_{s_k} = (1 - \theta^a_b) - \left(1 - \theta^c_b - \sum_{k=1}^{K} \theta^c_{s_k}\right).
\]

That is, the sum of \( w^c \)’s weights in funds \( w_{s_1}, \ldots, w_{s_K} \) is equal to the difference between the weights of \( w^u \) and \( w^c \) in fund \( w_a \).

The following result further explores the effect of the constraints on the optimal portfolio.

**Theorem 7.** Suppose that \( \gamma^u = \gamma^c \). The constrained agent’s optimal portfolio when the expected return vector is \( \bar{R} \) coincides with the unconstrained agent’s optimal portfolio when the expected return is \( \bar{R}^* \equiv \bar{R} + \sum_{k=1}^{K} \lambda_{s_k} R_{s_k} \).

Using Theorem 7, the effect of the ST constraints on the optimal portfolio is identical to that of an adjustment in the expected return vector \( \bar{R} \). Since \( \lambda_s > 0 \) for \( s = s_1, \ldots, s_K \), security \( j \)’s adjusted expected return \( \bar{R}^*_j \) is smaller (larger) than its expected return \( \bar{R}_j \) if
\( R_{js} \) is smaller (larger) than zero for \( s = s_1, ..., s_K \). Note that the case when \( R_{js} \geq 0 \) for \( j = 1, ..., J \) and \( s = s_1, ..., s_K \) is not plausible since in practice a state is chosen to be in an ST constraint only if there are some securities with negative returns in the state.

In the case when \( R_{js} < 0 \) for \( j = 1, ..., J \) and \( s = s_1, ..., s_K \), the expected returns of all risky securities are adjusted downward. Thus, fund \( w_a \) is relatively more attractive. Consequently, the optimal portfolio’s weight in \( w_a \) tends to be larger in the presence of the constraints.

Consider now the case when \( R_{j_1s} < 0 < R_{j_2s'} \) for some distinct securities \( j_1, j_2 \in \{1, ..., J\} \) and states \( s, s' \in \{s_1, ..., s_K\} \). In this case, (1) the expected return of at least one security can be adjusted downward and (2) the expected return of at least one security can be adjusted upward. Thus, fund \( w_a \) can be either relatively more or less attractive. Consequently, there is no clear tendency for the optimal portfolio’s weight in \( w_a \) in the presence of the constraints.

### 3.2. Asset Pricing Implications

We now characterize the equilibrium.

**Theorem 8.** In equilibrium, the market portfolio is inefficient. Furthermore, security \( j \)'s expected return is

\[
R_j = \hat{R} + \beta_j(R_m - \hat{R}) - \sum_{k=1}^{K}(R_{js_k} - \beta_j R_{ms_k}) \delta_{s_k}, \tag{10}
\]

where

\[
\hat{R} \equiv \left[ -\frac{1 - (1 - \varphi)A/\gamma^u - \varphi A/\gamma^c - \varphi \sum_{k=1}^{K} F_{s_k} \lambda_{s_k}/\gamma^c}{(1 - \varphi)/\gamma^u + \varphi/\gamma^c} \right] \frac{1}{C} \tag{11}
\]

and \( \delta_{s_k} \) is a positive constant as defined in Equation (6).

Theorem 8 says that the market portfolio is inefficient. The intuition for this result is similar to that of Theorem 4. Figure 2 illustrates that the market portfolio, represented by
point \( m \), lies below the efficient frontier on the thin hyperbola representing combinations of \( w^u \) and \( w^c \).

Equation (10) indicates that security \( j \)'s expected return depends on \( K+2 \) terms. The first two terms are related to those in the Black model.\(^{18}\) However, the last \( K \) terms, which are subtracted from the sum of the first two terms, are not present in the Black model. Each one of these \( K \) terms is given by the product of: (1) the idiosyncratic return of security \( j \) in state \( s_k \), \( R_{jsk} - \beta_j R_{msk} \), and (2) the risk premium on the idiosyncratic return in state \( s_k \), \( \delta_{sk} \), where \( k \in \{1, \ldots, K\} \).\(^{19}\)

Since \( \delta_{sk} > 0 \) for \( k = 1, \ldots, K \), securities with negative (positive) idiosyncratic returns in states \( s_1, \ldots, s_K \) have relatively high (low) expected returns in equilibrium. The intuition for this result is similar to that presented for Theorem 4.

4. EXAMPLE

In this section, we illustrate our theoretical results using a simple example.

4.1. Securities

There are four risky securities (\( j = 1, 2, 3, 4 \)) and a riskfree security (\( j = 5 \)). All securities are assumed to be in positive net supply.\(^{20}\) Table 1 presents their expected returns and standard deviations. For simplicity, assume that the correlation coefficient between the returns of each pair of distinct risky securities is equal to 0.4. As explained shortly, the constrained agent faces two ST constraints which use states \( s_1 \) and \( s_2 \). Hence, Table 1

\(^{18}\)In the Black model, \( \bar{R}_j = \bar{R}_{zc(m)} + \beta_j [\bar{R}_m - \bar{R}_{zc(m)}] \), where \( \bar{R}_{zc(m)} \equiv -(\gamma'' - A) / C \) is the expected return on the portfolio on the mean-variance boundary with zero covariance with the market portfolio. Note that if \( \varphi = 0 \), then the first two terms of Equation (10) coincide with those in the Black model.

\(^{19}\)The return of security \( j \) in state \( s \) predicted by the Black model is \( \beta_j R_{ms} + (1 - \beta_j)R_{zc(m)s} \), where \( R_{zc(m)s} \) is the return of portfolio \( zc(m) \) in state \( s \). Note that \( (1 - \beta_j)R_{zc(m)s} \) is small if (1) \( \beta_j \) is close to 1 or (2) \( R_{zc(m)s} \) is close to zero. Thus, for simplicity, in defining security \( j \)'s idiosyncratic return in state \( s \), we use \( \beta_j R_{ms} \) as the return of security \( j \) in state \( s \) predicted by the Black model.

\(^{20}\)An example with similar results can be provided where the riskfree security is assumed to be in zero net supply.
also provides the security returns in these states. These returns are notably negative, but nevertheless plausible. For example, the Nasdaq index declined by 27.1% and 22.9% in, respectively, October 1987 and November 2000. It is important to emphasize that the qualitative results in our example do not depend on the assumptions that are imposed on (1) the existence of a riskfree security, (2) the distribution of security returns, and (3) the number of ST constraints.

4.2. Agents

The unconstrained and constrained agents have the objective function defined by Equation (1) with $\gamma = \gamma^u = \gamma^c = 3$. The constrained agent faces two ST constraints:

$$R_{w_s} \geq -T_s, \ s = s_1, s_2,$$

where $T_{s_1} = 3\%$ and $T_{s_2} = 4\%$. The reason why we have $T_{s_2} > T_{s_1}$ is that the returns of three out of the four risky securities in state $s_2$ are larger in absolute terms than those in state $s_1$.

4.3. Portfolio Selection Implications

Table 2 presents the optimal portfolios of unconstrained and constrained agents.

4.3.1. Unconstrained Agent

The first row of panel (a) indicates that the unconstrained agent’s optimal portfolio $w^u$ is characterized by weights $\theta^u_f = -15.93\%$ and $\theta^u_t = 115.93\%$ in, respectively, funds $w_f$ and $w_t$. The first row of panel (b) shows that $w^u$’s weights in the risky securities range from 23.79% (security 3) to 33.31% (security 1). The first row of panel (d) says that $w^u$’s expected

---

21We consider an investment horizon of one month. Nevertheless, with the exception of the returns and idiosyncratic returns in states $s_1$ and $s_2$, all the numbers in the tables are reported in annualized terms.

22This is a reasonable value of $\gamma$ as noted by, for example, Sharpe (1987, p. 94).
return and standard deviation are, respectively, 14.14% and 18.38%. Moreover, \( w^u \)'s returns in states \( s_1 \) and \( s_2 \) are notably negative.

### 4.3.2. Constrained Agent

The second row of panel (a) indicates that the constrained agent's optimal portfolio \( w^c \) is characterized by weights \( \theta^c_f = 59.13\% \), \( \theta^c_t = 115.93\% \), \( \theta^c_{s_1} = -45.48\% \), and \( \theta^c_{s_2} = -29.58\% \) in, respectively, funds \( w_f \), \( w_t \), \( w_{s_1} \), and \( w_{s_2} \). Note that the weights of \( w^u \) and \( w^c \) in fund \( w_t \) are identical. Furthermore, \( w^c \)'s weight in fund \( w_f \) is not only notably larger than \( w^u \)'s, it is also positive.\(^{23}\) The intuition for this result is simple. Since \( w_f \)'s returns in states \( s_1 \) and \( s_2 \) are positive, a relatively large weight in \( w_f \) is useful to meet the ST constraints. Further intuition can be seen in panel (c), which provides the adjustment in expected returns with the same effect on the optimal portfolio as the constraints. Observe in line 2 that the expected return of all risky securities are notably adjusted downward. Since \( w_f \) is now relatively more attractive, the constraints cause the optimal portfolio’s weight in \( w_f \) to increase substantially.

The second row of panel (b) shows that \( w^c \)'s weights in the risky securities range from \(-11.23\% \) (security 1) to \(30.27\% \) (security 2). Note that \( w^c \)'s weights in securities 1, 3, and 4 are notably smaller than those of \( w^u \), while that in security 2 is slightly larger. The intuition for these results is simple. The returns of securities 1, 3, and 4 in states \( s_1 \) and \( s_2 \) are larger in absolute terms than those of security 2. Thus, the ST constraints force a reduction of the weights in securities 1, 3, and 4. Panel (c) provides further intuition. The largest adjustment in the expected return occurs for securities 1 \((-9.46\% \), 3 \((-4.83\% \), and 4 \((-4.64\% \). Consequently, the ST constraints lead to a considerable reduction of the weights in these securities. Since security 1 has a relatively small adjusted expected return \((5.24\% \),

\(^{23}\)Note that the riskfree security is in positive net supply, as assumed earlier.
$w^c$'s weight in this security is negative. In contrast, since security 2 has a relatively large adjusted expected return (9.02%), $w^c$'s weight in this security is positive.

The second row of panel (d) says that $w^c$’s expected return and standard deviation are, respectively, 7.24% and 8.15%. Thus, they are notably smaller than those of $w^u$. It can be seen that $w^c$’s returns in states $s_1$ and $s_2$ meet the ST constraints. Since $w^c$ is inefficient, of particular interest is its efficiency loss, denoted by $\rho_{w^c}$, which we measure in terms of standard deviation. That is, $w^c$’s efficiency loss is the increase in standard deviation arising from selecting $w^c$ instead of the efficient portfolio with the same expected return. Note that $w^c$’s efficiency loss is $\rho_{w^c} = 2.28\%$.

**4.3.3. Cost of Stress Testing**

Next, we assess the cost of ST. In doing so, we measure the cost of ST by the reduction in the certainty equivalent return incurred by the agent who selects the constrained optimal portfolio instead of the unconstrained optimal portfolio. Since the certainty equivalent returns of the unconstrained and constrained optimal portfolios are, respectively, 9.07% and 6.24%, the cost of ST is 2.83%. That is, the protection provided by imposing the ST constraints is procured at the cost of a reduction in the certainty equivalent return of 2.83%.

**4.4. Asset Pricing Implications**

Suppose that the constrained agent’s fraction of the wealth in the economy is $\varphi = 50\%$.

**4.4.1. Market Portfolio**

The third row of panel (a) indicates that the market portfolio $m$ is characterized by weights $\theta_t = 147.87\%$, $\theta_{s_1} = -29.01\%$, and $\theta_{s_2} = -18.86\%$ in, respectively, funds $w_t$, $w_{s_1}$, and $w_{s_2}$. The third row of panel (b) shows that $m$’s weights in the risky securities range
from 14.08% (security 1) to 37.04% (security 2). The third row of panel (d) says that \( m \)'s expected return and standard deviation are, respectively, 12.53% and 15.88%. Thus, they are smaller (larger) than those of \( w^u \) (\( w^c \)). Moreover, \( m \)'s returns in states \( s_1 \) and \( s_2 \) are smaller (larger) in absolute terms than those of \( w^u \) (\( w^c \)). Finally, \( m \)'s efficiency loss is only \( \rho_m = 0.41\% \). The reason why we obtain this small efficiency loss is that \( m \)'s weights in funds \( w_s \) and \( w_{s_2} \) are substantially smaller than that in fund \( w_t \).

4.4.2. Security Expected Returns

Panel (a) of Table 3 shows the risk premia. First, the CAPM’s risk premium is \( \bar{R}_m - R_f = 8.53\% \). Second, the risk premium of the idiosyncratic return in state \( s_1 \) is \( \delta_{s_1} = 10.90\% \). That is, a security’s expected return increases by 10.90 basis points per percentage point of its idiosyncratic return in state \( s_1 \). Third, the risk premium of the idiosyncratic return in state \( s_2 \) is \( \delta_{s_2} = 5.96\% \). That is, a security’s expected return increases by 5.96 basis points per percentage point of its idiosyncratic return in state \( s_2 \).

Panel (b) presents characteristics of the risky securities. The first row shows that their betas range from 0.76 (security 4) to 1.20 (security 2). The second and third rows provide their idiosyncratic returns in states \( s_1 \) and \( s_2 \). While securities 1 and 4 have negative idiosyncratic returns in both states, security 2 has positive ones. Security 3 has a negative idiosyncratic return in state \( s_1 \), but a positive one in state \( s_2 \).

Panel (c) decomposes security expected returns in four terms. The first three rows provide the two terms contained in the CAPM and their sum, i.e., the CAPM expected return. As expected, this sum is higher when the security beta is higher. The next three rows provide the two terms that are not present in the CAPM and their sum, i.e., the idiosyncratic return adjustment. Since securities 1 and 4 have negative idiosyncratic returns in states \( s_1 \) and
In sum, the idiosyncratic return adjustments can be notable.

5. CONCLUSION

Over the past ten years, some investors have suffered huge losses due to extreme events (e.g., the Barings Bank, LTCM, and Enron failures). In order to prevent such losses, ST is now commonly utilized as a risk management tool. For example, the Committee on the Global Financial System (2005, pp. 1, 15) of the Bank for International Settlements and Scholes (2000) note that ST is often used by banks and securities firms to set risk exposure limits. Accordingly, our paper examines a simple model to explore the portfolio selection and asset pricing implications of ST constraints.

Regardless on whether a riskfree security is available, we obtain four main results. First, the constrained agent’s optimal portfolio exhibits \((K + 2)\)-fund separation, where \(K\) is the number of binding ST constraints. Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected returns of all risky securities that tends to lower them. Hence, the constraints tend to increase the optimal portfolio’s weight in the riskfree security (or the minimum variance portfolio when this security is not available). Third, the market portfolio is inefficient. Fourth, a security’s expected return is
affected by both its systematic risk (i.e., beta) and its idiosyncratic returns in each one of those states that are used in the constraints. Specifically, securities with negative (positive) idiosyncratic returns in these states have relatively high (low) expected returns in equilibrium.

Next, we illustrate our theoretical results with a simple example. In this example, there are five securities (one of them is riskfree), two agents (each of them has half of the wealth in the economy), and two binding ST constraints. Our main findings are as follows. First, since there are two constraints, the constrained agent’s optimal portfolio exhibits four-fund separation. Second, the portfolio’s weights in the two inefficient funds are notable, which cause it to have an efficiency loss of 2.28%. Third, since the market portfolio’s weights in these funds are smaller than those of the constrained agent’s optimal portfolio, the market portfolio has an efficiency loss of only 0.41%. Fourth, the effect of the constraints on the optimal portfolio is identical to that of a downward adjustment in the expected returns of the risky securities ranging from 4.18% to 9.46%. Thus, the weight of the constrained agent’s optimal portfolio in the riskfree security is substantially larger than that of the unconstrained agent’s optimal portfolio. Fifth, the cost of ST as measured by the reduction in the certainty equivalent return incurred by selecting the constrained portfolio is 2.83%. Finally, the component of a security’s expected return that arises from its idiosyncratic returns can be notable. This component is 2.11% for one of the securities that has negative idiosyncratic returns in both of the states used in the constraints. However, the component is −1.02% for one of the other securities since it has positive idiosyncratic returns in these states. In sum, by exploring the implications of ST constraints, we provide further motivation to the literature in which security prices are not solely driven by systematic risk (see, e.g., Campbell, Lettau, Malkiel, and Xu (2001) and Goyal and Santa-Clara (2003)).
APPENDIX

Proof of Theorem 1. Observe that $\tilde{w}^u$ solves

$$\max_{w \in \mathbb{R}^J} R_f + w^\top (\bar{R} - 1R_f) - \frac{\gamma^u}{2} w^\top Vw. \quad (12)$$

A first-order condition for $\tilde{w}^u$ to solve problem (12) is

$$\bar{R} - 1R_f - \gamma^u V\tilde{w}^u = 0. \quad (13)$$

Since $\text{rank}(V) = J$, Equation (13) implies that

$$\tilde{w}^u = \frac{V^{-1}(\bar{R} - 1R_f)}{\gamma^u}. \quad (14)$$

Using the definition of $w_t$ and Equation (14), we have Equation (2). □

Proof of Theorem 2. Note that the ST constraints are assumed to bind. Hence, $\tilde{w}^c$ solves

$$\max_{w \in \mathbb{R}^J} R_f + w^\top (\bar{R} - 1R_f) - \frac{\gamma^c}{2} w^\top Vw$$

s.t. $w^\top (R_s - 1R_f) = -T_s - R_f, \ s = s_1, ..., s_K. \quad (15)$$

A first-order condition for $\tilde{w}^c$ to solve problem (15) subject to constraints (16) is

$$\bar{R} - 1R_f - \gamma^c V\tilde{w}^c + \sum_{k=1}^K \lambda_{s_k} (R_{s_k} - 1R_f) = 0, \quad (17)$$

where $\lambda_{s_1}, ..., \lambda_{s_K}$ are Lagrange multipliers associated with these constraints. Since $\text{rank}(V) = J$, Equation (17) implies that

$$\tilde{w}^c = \frac{V^{-1}(\bar{R} - 1R_f) + \sum_{k=1}^K \lambda_{s_k} V^{-1}(R_{s_k} - 1R_f)}{\gamma^c}. \quad (18)$$

Using the definition of $w_t, w_{s_1}, ..., w_{s_K}$, and Equation (18), we have Equation (3).

We now find $\lambda_{s_1}, ..., \lambda_{s_K}$. Premultiplying Equation (18) by $(R_s - 1R_f)^\top$ and using Equation (16), we have

$$-T_s - R_f = \frac{I_s + \sum_{k=1}^K \lambda_{s_k} I_{ss_k}}{\gamma^c}, \ s = s_1, ..., s_K, \quad (19)$$
where \( I_s \equiv (R_s - 1R_f)^\top V^{-1}(\overline{R} - 1R_f) \) and \( I_{sk} \equiv (R_s - 1R_f)^\top V^{-1}(R_{sk} - 1R_f) \). It follows from Equation (19) that

\[
\sum_{k=1}^{K} \lambda_{sk} I_{sk} = -[I_s + (T_s + R_f) \gamma^{c}] , \quad s = s_1, ..., s_K. \tag{20}
\]

Let \( X_1 \equiv [\lambda_{s_1} \cdots \lambda_{s_K}]^\top , \quad U_1 \equiv [R_{s_1} - 1R_f \cdots R_{s_K} - 1R_f] , \quad Y_1 \equiv U_1^\top V^{-1} U_1 , \) and \( Z_1 \equiv [-[I_s + (T_s + R_f) \gamma^{c}] \cdots -[I_{s_K} + (T_{s_K} + R_f) \gamma^{c}]]^\top \). Using Equation (20), we have \( Y_1 X_1 = Z_1 \). Hence, \( X_1 = Y_1^{-1} Z_1 \). ■

**Proof of Theorem 3.** The desired result follows from Equations (14) and (18). ■

**Proof of Theorem 4.** We begin by showing that the market portfolio is inefficient. Using Equations (2) and (3), we have

\[
m = \left(1 - \sum_{k=1}^{K} \pi_{sk}\right) \tilde{w}_t + \sum_{k=1}^{K} \pi_{sk} \tilde{w}_{sk}, \tag{21}
\]

where

\[
\pi_s = \frac{\varphi \theta^{c}_s}{(1 - \varphi) \theta^{u}_t + \varphi \left(\theta^{c}_t + \sum_{k=1}^{K} \theta^{c}_{sk}\right)} , \quad s = s_1, ..., s_K. \tag{22}
\]

Since (1) \( \pi_s \neq 0 \) for \( s = s_1, ..., s_K \), (2) portfolio \( w_s \) is inefficient for \( s = s_1, ..., s_K \), and (3) \( \text{rank}([1 \quad \overline{R} \quad R_{s_1} \cdots R_{s_K}]) = K + 2 \), Equation (21) implies that \( m \) is also inefficient.

Next, we show that Equation (5) holds. Using Equation (21) and the definitions of \( \tilde{w}_t \) and \( \tilde{w}_{sk} \), we have

\[
[\sigma_{1m} \cdots \sigma_{Jm}]^\top = Vm = \left(1 - \sum_{k=1}^{K} \pi_{sk}\right) \left(\frac{\overline{R} - 1R_f}{A - CR_f}\right) + \sum_{k=1}^{K} \pi_{sk} \left(\frac{R_{sk} - 1R_f}{F_{sk} - CR_f}\right). \tag{23}
\]

It follows from Equation (23) that

\[
\sigma_{m}^2 = m^\top Vm = \left(1 - \sum_{k=1}^{K} \pi_{sk}\right) \left(\frac{\overline{R}_m - R_f}{A - CR_f}\right) + \sum_{k=1}^{K} \pi_{sk} \left(\frac{R_{ms_k} - R_f}{F_{sk} - CR_f}\right). \tag{24}
\]
The fact that $\sigma_{jm} = \beta_j \sigma_m^2$ and Equations (23) and (24) imply that

$$
\overline{R}_j = R_f + \beta_j (R_m - R_f) - \sum_{k=1}^{K} [(R_{js_k} - R_f) - \beta_j (R_{ms_k} - R_f)] \left( \frac{\pi_{s_k}}{1 - \sum_{k=1}^{K} \pi_{s_k}} \right) \left( \frac{A - CR_f}{F_{s_k} - CR_f} \right).
$$

(25)

Using Equations (22) and (25), we have Equation (5). Note that $\delta_{s_k} > 0$ since $\lambda_{s_k} > 0$.

Proof of Theorem 5. Observe that $w^u$ solves

$$
\max_{w \in \mathbb{R}^J} \quad w^\top \overline{R} - \frac{\gamma^u}{2} w^\top V w
$$

s.t. $w^\top 1 = 1$.

(26)

(27)

A first-order condition for $w^u$ to solve to problem (26) subject to constraint (27) is

$$
\overline{R} - \gamma^u V w^u + \lambda 1 = 0,
$$

(28)

where $\lambda$ is the Lagrange multiplier associated with this constraint. Since $\text{rank}(V) = J$, Equation (28) implies that

$$
w^u = \frac{V^{-1} \overline{R} + \lambda V^{-1} 1}{\gamma^u}.
$$

(29)

Using the definition of $w_a$ and $w_b$, and Equation (29), we have Equation (7).

We now find $\lambda$. Premultiplying Equation (29) by $1^\top$ and using constraint (27), we have

$$
1 = \frac{A + \lambda C}{\gamma^u}.
$$

Hence, $\lambda = \frac{\gamma^u - A}{C}$. ■

Proof of Theorem 6. Note that the ST constraints are assumed to bind. Hence, $w^c$ solves

$$
\max_{w \in \mathbb{R}^J} \quad w^\top \overline{R} - \frac{\gamma^c}{2} w^\top V w
$$

s.t. $w^\top 1 = 1$

$$
w^\top R_s = -T_s, \quad s = s_1, ..., s_K.
$$

(30)

(31)

(32)
A first-order condition for $w^c$ to solve problem (30) subject to constraints (31) and (32) is

$$\overline{R} - \gamma^c V w^c + \lambda 1 + \sum_{k=1}^K \lambda_{s_k} R_{s_k} = 0,$$

where $\lambda, \lambda_{s_1}, ..., \lambda_{s_K}$ are Lagrange multipliers associated with these constraints. Since $\text{rank}(V) = J$, Equation (33) implies that

$$w^c = V^{-1} \overline{R} + \lambda 1 + \sum_{k=1}^K \lambda_{s_k} V^{-1} R_{s_k}. \tag{34}$$

Using the definition of $w_a, w_b, w_{s_1}, ..., w_{s_K}$ and Equation (34), we have Equation (8).

We now find $\lambda, \lambda_{s_1}, ..., \lambda_{s_K}$. Premultiplying Equation (34) by $1^\top$ and using constraint (31), we have

$$1 = A + \lambda C + \sum_{k=1}^K \lambda_{s_k} F_{s_k}. \tag{35}$$

It follows from Equation (35) that

$$\lambda C + \sum_{k=1}^K \lambda_{s_k} F_{s_k} = \gamma^c - A. \tag{36}$$

Premultiplying Equation (34) by $R_s^\top$ and using constraint (32), we have

$$-T_s = L_s + \lambda F_s + \sum_{k=1}^K \lambda_{s_k} L_{s_{s_k}}, \ s = s_1, ..., s_K, \tag{37}$$

where $L_s \equiv R_s^\top V^{-1} \overline{R}$ and $L_{s_{s_k}} \equiv R_s^\top V^{-1} R_{s_{s_k}}$. It follows from Equation (37) that

$$\lambda F_s + \sum_{k=1}^K \lambda_{s_k} L_{s_{s_k}} = -T_s \gamma^c - L_s, \ s = s_1, ..., s_K. \tag{38}$$

Let $X_2 \equiv [\lambda \ \lambda_{s_1} \ \cdots \ \lambda_{s_K}]^\top$, $U_2 \equiv [1 \ R_{s_1} \ \cdots \ R_{s_K}]$, $Y_2 \equiv U_2^\top V^{-1} U_2$, and $Z_2 \equiv [\gamma^c - A \ - (T_{s_1} \gamma^c + L_{s_1}) \ \cdots \ -(T_{s_K} \gamma^c + L_{s_K})]^\top$. Using Equations (36) and (38), we have $Y_2 X_2 = Z_2$. Hence, $X_2 = Y_2^{-1} Z_2$. [Proof of Theorem 7.]

The desired result follows from Equations (29) and (34). [Proof of Theorem 7.]
Proof of Theorem 8. We begin by showing that the market portfolio $m$ is inefficient.

Using Equations (7) and (8), we have

$$m = \left(1 - \pi_b - \sum_{k=1}^{K} \pi_{sk}\right) w_a + \pi_b w_b + \sum_{k=1}^{K} \pi_{sk} w_{sk}, \quad (39)$$

where

$$\pi_b = (1 - \varphi)^{\alpha} + \varphi \beta, \quad (40)$$
$$\pi_s = \varphi \beta, \quad s = s_1, \ldots, s_K. \quad (41)$$

Since (1) $\pi_s \neq 0$ for $s = s_1, \ldots, s_K$, (2) portfolio $w_s$ is inefficient for $s = s_1, \ldots, s_K$, and (3) $\text{rank}([1 \quad \bar{R} \quad R_{s_1} \quad \cdots \quad R_{s_K}]) = K + 2$, Equation (39) implies that $m$ is also inefficient.

Next, we show that Equation (10) holds. Using Equation (39) and the definitions of $w_a$, $w_b$, and $w_{sk}$, we have

$$[\sigma_{1m} \quad \cdots \quad \sigma_{jm}]^\top = Vm = \left(1 - \pi_b - \sum_{k=1}^{K} \pi_{sk}\right) \frac{1}{C} + \pi_b \frac{\bar{R}}{A} + \sum_{k=1}^{K} \pi_{sk} \frac{R_{sk}}{F_{sk}}. \quad (42)$$

It follows from Equation (42) that

$$\sigma_m^2 = m^\top Vm = \left(1 - \pi_b - \sum_{k=1}^{K} \pi_{sk}\right) \frac{1}{C} + \pi_b \frac{\bar{R}_m}{A} + \sum_{k=1}^{K} \pi_{sk} \frac{R_{msk}}{F_{sk}}. \quad (43)$$

The fact that $\sigma_{jm} = \beta_j \sigma_m^2$ and Equations (42) and (43) imply that

$$\bar{R}_j = - \left(1 - \pi_b - \sum_{k=1}^{K} \pi_{sk}\right) \frac{A}{C} + \beta_j \left[\bar{R}_m - \left(1 - \pi_b - \sum_{k=1}^{K} \pi_{sk}\right) \frac{A}{C}\right]$$
$$- \sum_{k=1}^{K} (R_{jsk} - \beta_j R_{msk}) \left(\frac{\pi_{sk}}{\pi_b}\right) \frac{A}{F_{sk}}. \quad (44)$$

Using Equations (40), (41), and (44), we have Equation (10). Note that $\delta_{sk} > 0$ since $\lambda_{sk} > 0$. ■
REFERENCES


Figure 1. Optimal portfolios and the market portfolio in the presence of a riskfree security

This figure illustrates the unconstrained and constrained agents’ optimal portfolios ($w^u$ and $w^c$) and the market portfolio ($m$), represented by, respectively, points $u$, $c$, and $m$, in the presence of a riskfree security. Also shown are $w_f$, $w_t$, $w_{s_1}$, and $w_x$, represented by, respectively, points $f$, $t$, $s_1$, and $x$. The line represents combinations of $w_f$ and $w_t$ (i.e., the efficient frontier). The dashed hyperbola represents combinations of $w_t$ and $w_{s_1}$. The dotted line represents combinations of $w_f$ and $w_x$. 
Figure 2. Optimal portfolios and the market portfolio in the absence of a riskfree security

This figure illustrates the unconstrained and constrained agents’ optimal portfolios ($w^u$ and $w^c$) and the market portfolio ($m$), represented by, respectively, points $u$, $c$, and $m$, in the absence of a riskfree security. Also shown are $w_a$, $w_b$, $w_{s_1}$, and $w_x$, represented by, respectively, points $a$, $b$, $s_1$, and $x$. The thick hyperbola represents combinations of $w_a$ and $w_b$ (i.e., the mean-variance boundary). The dashed hyperbola represents combinations of $w_b$ and $w_{s_1}$. The dotted hyperbola represents combinations of $w^c$ and $w_x$. The thin hyperbola represents combinations of $w^u$ and $w^c$. 
Table 1. Parameters used in the example

This table presents the parameters used in the example of Section 6. There are four risky securities \((j = 1, 2, 3, 4)\) and a riskfree security \((j = 5)\). The expected returns \((\bar{R}_j, j = 1, \ldots, 5)\) and standard deviations \((\sigma_j, j = 1, \ldots, 5)\) are annualized, while the returns in states \(s_1\) and \(s_2\) \((R_{js_1} \text{ and } R_{js_2}, j = 1, \ldots, 5)\) are not. All numbers are reported in percentage terms.

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{R}_j)</td>
<td>14.70</td>
<td>13.20</td>
<td>12.00</td>
<td>10.80</td>
<td>4.00</td>
</tr>
<tr>
<td>(\sigma_j)</td>
<td>24.25</td>
<td>22.52</td>
<td>20.78</td>
<td>17.32</td>
<td>0.00</td>
</tr>
<tr>
<td>(R_{js_1})</td>
<td>-27.00</td>
<td>-11.00</td>
<td>-14.00</td>
<td>-12.00</td>
<td>0.33</td>
</tr>
<tr>
<td>(R_{js_2})</td>
<td>-29.00</td>
<td>-14.00</td>
<td>-14.00</td>
<td>-16.00</td>
<td>0.33</td>
</tr>
</tbody>
</table>
Table 2. Optimal portfolios and the market portfolio

This table presents the unconstrained and constrained agents’ optimal portfolios \((w^u\) and \(w^c\)) and the market portfolio \((m)\). Panel (a) provides the portfolios’ fund weights. Panel (b) provides the portfolios’ security weights. Panel (c) provides the adjustment in expected returns with the same effect on the optimal portfolio as the ST constraints. Panel (d) provides summary statistics on the optimal portfolios and the market portfolio. All numbers are reported in percentage terms.

(a) Fund weights

<table>
<thead>
<tr>
<th></th>
<th>(\theta_f)</th>
<th>(\theta_t)</th>
<th>(\theta_{s_1})</th>
<th>(\theta_{s_2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w^u)</td>
<td>-15.93</td>
<td>115.93</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(w^c)</td>
<td>59.13</td>
<td>115.93</td>
<td>-45.48</td>
<td>-29.58</td>
</tr>
<tr>
<td>(m)</td>
<td>-</td>
<td>147.87</td>
<td>-29.01</td>
<td>-18.86</td>
</tr>
</tbody>
</table>

(b) Security weights

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w^u)</td>
<td>33.31</td>
<td>27.81</td>
<td>23.79</td>
<td>31.02</td>
<td>-15.93</td>
</tr>
<tr>
<td>(w^c)</td>
<td>-11.23</td>
<td>30.27</td>
<td>13.92</td>
<td>7.91</td>
<td>59.13</td>
</tr>
<tr>
<td>(m)</td>
<td>14.08</td>
<td>37.04</td>
<td>24.05</td>
<td>24.83</td>
<td>-</td>
</tr>
</tbody>
</table>

(c) Adjustment in expected returns

<table>
<thead>
<tr>
<th>(j)</th>
<th>(\bar{R}_j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_{s_1}(R_{js_1} - R_f) + \lambda_{s_2}(R_{js_2} - R_f))</td>
<td>-9.46</td>
<td>-4.18</td>
<td>-4.83</td>
<td>-4.64</td>
<td></td>
</tr>
<tr>
<td>(\bar{R}_j)</td>
<td>5.24</td>
<td>9.02</td>
<td>7.17</td>
<td>6.16</td>
<td></td>
</tr>
</tbody>
</table>

(d) Summary statistics

<table>
<thead>
<tr>
<th>(j)</th>
<th>(\bar{R}_w)</th>
<th>(\sigma_w)</th>
<th>(R_{ws_1})</th>
<th>(R_{ws_2})</th>
<th>(\rho_w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w^u)</td>
<td>14.14</td>
<td>18.38</td>
<td>-19.16</td>
<td>-21.90</td>
<td>-</td>
</tr>
<tr>
<td>(w^c)</td>
<td>7.24</td>
<td>8.15</td>
<td>-3.00</td>
<td>-4.00</td>
<td>2.28</td>
</tr>
<tr>
<td>(m)</td>
<td>12.53</td>
<td>15.88</td>
<td>-14.22</td>
<td>-16.61</td>
<td>0.41</td>
</tr>
</tbody>
</table>
Table 3. Risk premia, security characteristics, and decomposition of security expected returns

Panel (a) provides the risk premia. Panel (b) presents security characteristics. Panel (c) decomposes security expected returns. With the exception of the idiosyncratic returns in states $s_1$ and $s_2$, all numbers are annualized. With the exception of $\beta_j$, all numbers are reported in percentage terms.

(a) Risk premia

<table>
<thead>
<tr>
<th></th>
<th>$\overline{R_m} - R_f$</th>
<th>$\delta_{s_1}$</th>
<th>$\delta_{s_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8.53</td>
<td>10.90</td>
<td>5.96</td>
</tr>
</tbody>
</table>

(b) Security characteristics

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_j$</td>
<td>1.01</td>
<td>1.20</td>
<td>0.94</td>
<td>0.76</td>
</tr>
<tr>
<td>$(R_{js_1} - R_f) - \beta_j (R_{ms_1} - R_f)$</td>
<td>-12.67</td>
<td>6.12</td>
<td>-0.63</td>
<td>-1.33</td>
</tr>
<tr>
<td>$(R_{js_2} - R_f) - \beta_j (R_{ms_2} - R_f)$</td>
<td>-12.27</td>
<td>5.98</td>
<td>1.62</td>
<td>-3.53</td>
</tr>
</tbody>
</table>

(c) Decomposition of security expected returns

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_f$</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td>$\beta_j (\overline{R_m} - R_f)$</td>
<td>8.59</td>
<td>10.22</td>
<td>8.03</td>
<td>6.44</td>
</tr>
<tr>
<td>CAPM expected return</td>
<td>12.59</td>
<td>14.22</td>
<td>12.03</td>
<td>10.44</td>
</tr>
<tr>
<td>$[(R_{js_1} - R_f) - \beta_j (R_{ms_1} - R_f)]\delta_{s_1}$</td>
<td>-1.38</td>
<td>0.67</td>
<td>-0.07</td>
<td>-0.15</td>
</tr>
<tr>
<td>$[(R_{js_2} - R_f) - \beta_j (R_{ms_2} - R_f)]\delta_{s_2}$</td>
<td>-0.73</td>
<td>0.35</td>
<td>0.10</td>
<td>-0.21</td>
</tr>
<tr>
<td>Idiosyncratic return adjustment</td>
<td>-2.11</td>
<td>1.02</td>
<td>0.03</td>
<td>-0.36</td>
</tr>
</tbody>
</table>

| $\overline{P_{ij}}$ | 14.70 | 13.20 | 12.00 | 10.80 |