MODELLING THE NUMBER OF CUSTOMERS AS A BIRTH AND DEATH PROCESS

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Birth and death may be a better model than Brownian motion for many physical processes, which real options models will increasingly need to deal with. In this paper, we value a perpetual American call option, which gives the monopoly right to invest in a market in which the number of active customers (and hence the sales rate) follows a birth and death process. The problem contains a singular point, and we develop a mixed analytic/numeric method for handling this singular point, based on the method of Frobenius. The method may be useful for other cases of singular points. The birth and death model gives lower option values than the geometric Brownian motion model, except at very low volatilities, which suggests that if a firm incorrectly assumes a geometric Brownian motion process in place of a birth and death process, it may invest too seldom and too late.

1- Introduction

The building of a real options model requires two decisions: firstly what stochastic process to use, and secondly what economic model to assume, for how the stochastic process drives option value (for example by variation of one or more capital values, or of an income stream, and if by the stochastic variation of an income stream, whether through variation of selling price, cost price, or physical quantity, or some mix of these). Most real options authors have used geometric Brownian motion for the stochastic process, though a few
have varied this by including jump or mean-reverting processes\textsuperscript{1}. In this paper we introduce a new application of an alternative class of stochastic process, namely birth and death processes. These have been little studied since Cox and Ross (1976) valued the European option, although loosely related models have been studied by Cox Ingersoll and Ross (1985), and Klumpes and Tippet (2004). Here we introduce the mathematical tools which can value perpetual American call in a general continuous birth and death process. These solution tools may be also useful when singular values arise under other stochastic specifications.

For the second modelling decision, of how the stochastic process drives value, most authors assume that the selling price is the only stochastic variable, and that operating costs, and the potential physical quantity sold per unit of time, are constant. This assumption makes profit a multiple of selling price, perhaps with some deduction for fixed, time-based costs, which must be incurred whether production takes place or not (Dixit and Pindyck, 1994). In such models, the only physical operating decision is the binary one of “to produce” (at 100\% capacity) or “not to produce”. This underestimates the physical complexity of almost any business operation. It ignores the large and well-known literatures of economics, operations management, accounting and even marketing, all of which model physical operating decisions not only in terms of prices, but in terms of continuously variable volumes of physical output, constrained physical output capacity, and optimal allocation of

constrained capacity. Both economic and marketing literatures treat certain special case - under monopolistic competition – in which the operator can make decisions on price itself -at least relative to competitors.

Because the real options literature has largely ignored continuous variation of physical output rate, and the link between physical output rate and profit, many existing “real option” models are in fact far less “real” than they could be, which is a challenge for future research. One aspect of that challenge is to find suitable models for cases where physical quantities are not well-modelled as continuous, and/or whose dynamics are not geometric Brownian motion. Either of these physical properties can be found in many economic processes. The present paper makes a contribution to this development: birth and death processes offer gentler dynamics than geometric Brownian motion, and can be well specified for discrete variables. We will apply the birth and death process to a continuous physical quantity, and we will solve the new problem of valuing a perpetual American call, but we will not explore the more detailed modelling of physical events (e.g. constraints on output rates) or the interaction of stochastic price and stochastic quantity. We apply birth and death models in the limiting case where the population size is a continuous physical variable, but we cite the necessary source references for modelling birth and death processes in the fully discrete case.

Discrete and continuous state versions of birth and death processes have been used in biology and other disciplines to explain the evolution of discrete and quasi-continuous populations (see Cox and Miller (1965); Karlin and Taylor (1975); Grimmett and Stirzaker (1992); Kao (1997); Taylor and Karlin (1998); Ross (2000) for details). In economic situations where a market
consists of a discrete population of active customers, all of whom buy in a relatively homogeneous way, so long as they remain active, and any of whom can enter or leave the market at any time, a birth and death process seems a relevant model for the rate of sales in the market. The births and deaths in question could model the economic lifetimes of human individuals, or the arrival and departure (churn) of corporate customers for a product. Birth and death rates, as properties of the active customer base, can be measured, and used to predict eventual stable states, and these rates can, on occasion, be influenced by marketing actions. Birth and death processes have gentler dynamics than geometric Brownian motion. This can reflect the fact that if the demand for a product is approaching a physical constraint on supply (e.g. for electricity on summer evenings) further changes in the physical supply rate of the product become damped, but excursions in price became less damped. Hence birth and death models are in principle useful either directly or as components within broader models of a variety of real world marketing, physical and financial events.

In this paper we will assume that the business under analysis (in which an investor has a perpetual call option to invest) will sell at a rate of one unit per year to each customer. Thus we can talk interchangeably about the number of customers and the (annual) rate of quantity sold. A customer’s constant rate of purchase is assumed to be infinitely variable within subdivisions of a year.

In finance, continuous state versions of birth and death processes have been used by Cox and Ross (1976) and as we see below, the resulting equations have what is known as a “square root” random disturbance. Cox and Ross (1976) suggest the continuous state birth and death process as an alternative
model for the dynamics of the stock price, and they value a pure vanilla European financial call option. In this paper we reintroduce to the Real Options literature the birth and death model used by Cox and Ross (1976), and point out that its parent discrete birth and death process can handle many interesting “real” discrete processes, such as physical (as opposed to financial) rates of demand, and customer churn. We price the perpetual American call to acquire a process whose income follows a continuous birth and death process, complementing Cox and Ross’s (1976) solution for the European call.

There are several closely related models which share the “square root” feature, but are not identical to birth and death processes, nor derived from them. The well-known Cox Ingersoll and Ross (CIR) model of the interest rate adds mean reversion to the “square root” feature. Both features were selected in order to model stylised facts about the interest rate, namely non-negativity, mean reversion and a variance which increases more slowly than the rate itself.

Biepke, Klumpes and Tippet (2003) developed a general infinite series solution for the CIR model\(^2\), and also for a pure mean reverting model (a design feature selected to permit negative cash flows – a property of many real assets). Klumpes and Tippet (2004) designed a model to combine the property of negative cash flows, with a variance which grows less rapidly as the process moves further from its mean (but retains a positive variance at the mean), and they produced an infinite series solution for this model also. All these models have dynamics which differ somewhat from each other, from birth and death, and from the geometric Brownian motion.

\(^2\) Biepke, Klumpes and Tippet (2003) do not present the particular solution of the options to invest and abandon when the state variable follows a square root process.
The methodological contribution of this paper is to introduce (to the real options literature) a variant mathematical strategy for the solution of differential equations in the presence of singularities, based on the method of Frobenius. This forms a complementary approach to the solution strategy used by Biepke Klumpes and Tippet (2003) and Klumpes and Tippet (2004). The latter construct, for each model, an infinite series solution which holds everywhere except at the singularity point (in those cases where the differential equation contains a singularity). In contrast we construct, for the original birth and death model, a series solution which holds arbitrarily close (operationally, as close as possible) to the singular point\(^3\) (such points need not in general be located at zero, nor be unique). This value permits us to start a numerical solution, which would otherwise be impossible, but we then exploit the superior speed of numerical methods to generate the remainder of the solution domain. We use this technique to value the perpetual American call option to acquire an asset whose income is driven by a birth and death process, thus complementing the Cox and Ross (1976) solution for a European call.

For this solution we study the sensitivity of the investment trigger-level of sales rate to changes in volatility. We also compare some numerical examples of the resulting option values and investment trigger levels of sales, as between a birth and death process, a geometric Brownian motion, and the classic Marshallian investment rule. It is easy to predict qualitatively, from the general form of the respective differential equations, that geometric Brownian

\(^3\) Singularity points often occur in real option problems, and many authors choose to avoid the problems which they pose. This seems undesirable, since most of the important features of the solution are controlled within the neighbourhoods of the singularity points.
motion can give far higher values to call options than the birth and death process. However the birth and death model has no analytic solution, so numerical examples can be helpful in particular cases (even though they are context-specific). These show an effect which is not intuitively obvious, though it is implicit in the differential equations, namely that at very low volatilities the geometric Brownian motion and the birth and death models can give similar values. However at the larger volatilities frequently seen in practice, in the range 0.2 to 0.3, option values are much higher under the geometric Brownian motion assumption than under the birth and death assumption. Hence in cases where a birth and death process would have been a more realistic dynamic model, the use of geometric Brownian motion models could have led to under-investment through post-optimal exercise.

Following this introduction we present in section II the main characteristics of the birth and death process, and we compare its dynamics with those of geometric Brownian motion. In section III we value the call option to invest in a market where the quantity sold follows a birth and death process and we explain the unusual solution method required. Section IV compares results of assuming birth and death processes versus geometric Brownian motion. Section V concludes.

II – A New Stochastic Process for Modelling a Physical Quantity Sold

Birth and death processes are a type of Markov process in discrete levels and continuous time, which are often used to model the evolution of populations. The combination of birth and death allows the population size to fluctuate,
instead of only increasing, as it does in the simple birth process, or only decreasing, as it does in the simple death process (see Cox and Miller (1965); Karlin and Taylor (1975); Grimmett and Stirzaker (1992); Kao (1997); Taylor and Karlin (1998); and Ross (2000)). In the simple birth and death process it is assumed that each individual, independently of all other individuals, gives birth to new individuals, one at a time, at rate $\phi$. Each individual is liable to die, and the lifetime of each individual has an exponential distribution with parameter $\nu$.

Although birth and death processes are mainly used to describe the evolution of populations in discrete levels, diffusion limits exist, at which we can use birth and death to describe the evolution of a positive random variable whose changes in state are quantized arbitrarily finely at $4\delta$. In many situations there is no practical difference between a finely quantized discrete case and a fully continuous random variable (e.g. real world stock prices move discretely, in ticks, not along the continuum assumed by the Black-Scholes equation), but birth and death processes give rather different dynamics from those of geometric Brownian motion. We will model only the continuous case, which does not use the birth and death parameters $\phi$ and $\nu$ explicitly but readers, who require these parameters for marketing purposes, or for modelling the discrete case, can consult the references above.

Let $Q_t$ denote the actual count of active customers measured at time $t$, and let the dynamics of the actual count of active customers be explained by a

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4 For details on the diffusion limit of a simple birth and death process see Cox and Ross (1976) and Feller (1959).
simple birth and death process. In the limit $Q_t$ is continuous, and the stochastic
differential equation for its changes is:

\begin{equation}
\begin{array}{c}
dQ_t = \mu Q_t dt + \sqrt{Q_t} \sigma dz \\
\end{array}
\end{equation}

(1)

where $\mu$ denotes the drift, $\sigma$ the standard deviation and $dz$ the increment of a
Wiener process. For obvious reasons this diffusion limit of the birth and death
process is sometimes called a square root process, and as noted it is closely
related to the mean-reverting Cox, Ingersoll, Ross (1985) model of the interest
rate. The square root process can have advantages for modelling certain sorts of
physical events. For example a shortage of physical supply may produce an
explosion in price (well captured by a geometric Brownian motion) but a
relatively modest increase in the quantity sold, as captured by a birth and death
process. We look to future models to treat both price and quantity, along with
operational constraints on quantity. Conversely discrete birth and death events
take place as jumps, and on a small-scale these can be far more volatile than
either the continuous birth and death process or geometric Brownian motion.

The square root process contrasts with (2), the stochastic differential
equation of a geometric Brownian motion:

\begin{equation}
\begin{array}{c}
dQ_t = \mu_{GBM} Q_t dt + Q_t \sigma_{GBM} dz_{GBM} \\
\end{array}
\end{equation}

(2)

where $\mu_{GBM}$ is the drift, $\sigma_{GBM}$ is the standard deviation and $dz_{GBM}$ is the
increment of a Wiener process.
The physical interpretation of the dynamics of the birth and death process described by equation (1) differs significantly from those of the more usual geometric Brownian motion. In the birth and death process each unit of quantity sold is the act of a single customer, which is stochastically independent of acts by all of the others, i.e. any customer can enter independently of another, and he can leave at any time (independently of the others). In geometric Brownian motion, it is not necessary (and may not be meaningful) to specify a number of customers, but all customers face identical multiplicative forces for change, which vary over time, and change identically for all of them. A natural consequence of the square root specification is that although in both processes volatility increases as the state variable increases, the volatility increases less in the birth and death process. This is illustrated in Figure 1:

![Figure 1 – \( Q_0=100, \mu=\mu_{GBM}=1\%, \sigma=\sigma_{GBM}=50\% \)](image)

Figure 1 shows five discrete realizations of geometric Brownian motion (left hand side) and the birth and death process (right hand side) for one hundred small time increments of the order 0.001. Looking at the vertical scale of the

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5 Assuming, without any loss of generality, that one customer purchases one unit.
Figures, we see as expected that the range of values of $Q_{100}$ is higher in the geometric Brownian motion than in the birth and death process. In fact in a birth and death process the growth rate of the state variable actually decreases with the state variable, while in geometric Brownian motion the growth rate of the state variable is constant. This can be seen if we rearrange equations (1) and (2) to represent the stochastic differential equations as changes in the growth rate of $Q$, i.e. $\frac{dQ}{Q}$.

\begin{align*}
(1-a) \quad \frac{dQ}{Q} &= \mu dt + Q^{-1/2} \sigma dz \\
(2-a) \quad \frac{dQ}{Q} &= \mu_{GBM} dt + \sigma_{GBM} d\zeta_{GBM}
\end{align*}

In this example both models have identical and constant drifts and volatilities so we can see that the growth rate (with regard to $Q$) is constant under geometric Brownian motion (2-a), and is decreasing under the birth and death process (1-a). This characteristic of the birth and death process seems intuitively to be more realistic for modelling the physical quantities of goods sold in many markets. As we have noted, it may be desirable to specify separate models for physical quantity and price, though these separations will only have non-trivial effects if one variable has a constraint which is not shared by the other. Here we model total profit as a linear multiple of a positive birth and death variable, and we are not modelling separate dynamics for quantity.
and price, nor modelling non-trivial interactions between them, in the generation of income.

**III – The Option to Invest in a Monopoly Market**

We now derive the value and the optimal exercise rule for a perpetual real call option to invest in a market where the sales rate (whether physically or financially driven) follows a birth and death process, taking the viewpoint of a risk neutral monopoly investor. We assume that the profits per unit equal one, and that one unit per annum is sold to each customer, therefore $Q_t$, which is the actual count of active customers measured at time $t$, represents both the annual rate of physical quantity sold and the annual rate of money profit. To avoid introducing further dynamics, we assume that each active customer consumes his or her single unit per annum continuously, at a uniform rate throughout the year, and the year is infinitely divisible. Henceforth we will consider a general time $t$ and therefore drop the time index.

We are interpreting the model as a non-geometric Brownian motion dynamic process for physical sales $Q$, while holding price constant. Alternatively like Biepke, Klumpes and Tippett (2003) we can interpret $Q$ as an index of price, assuming physical quantity is fixed.

Let $C(Q)$ denote the value of a perpetual call option to invest in a monopoly market where the, annualized rate of sale $Q$, follows a stochastic birth and death process as in (1). We construct a portfolio formed by a long position in the option and a short position in $\Delta$ units of $Q$. Any change in this portfolio can be explained by:
where \((r-\mu)Q\Delta t\) represents the fact that a trade in the demand rate \(Q\) must earn the rate \(r\) for the risk neutral investor, which is subdivided between capital appreciation at some rate \(\mu\) and the dividend at the rate \(r-\mu\). Expanding \(dC(Q)\) using Ito’s Lemma:

\[
\frac{dC(Q)}{dQ} = \frac{dC(Q)}{dQ} dQ + \frac{1}{2} \frac{d^2C(Q)}{dQ^2} dQ^2
\]

Substituting (1) and (4) into (3) and collecting like terms:

\[
\frac{1}{2} \sigma^2 Q \frac{d^2C(Q)}{dQ^2} dt + \mu Q \frac{dC(Q)}{dQ} dt - \mu Q \Delta t - (r-\mu)Q\Delta t + \mu Q\frac{dC(Q)}{dQ} dz - \mu Q \Delta dz
\]

If \(\Delta = \frac{dC(Q)}{dQ}\) the two random terms and the two trend terms in the previous equation disappear yielding:

\[
\frac{1}{2} \sigma^2 Q \frac{d^2C(Q)}{dQ^2} dt - (r-\mu)Q \frac{dC(Q)}{dQ} dt
\]

Since there is no randomness in this portfolio, it should earn the risk free rate of return. Thus:
Dividing (6) by \( dt \) and rearranging, we obtain the differential equation for the value \( C(Q) \) of the opportunity to invest:

\[
\frac{1}{2} \sigma^2 Q \frac{d^2 C(Q)}{dQ^2} dt - (r - \mu)Q \frac{dC(Q)}{dQ} dt = r \left[ C(Q) - \frac{dC(Q)}{dQ} Q \right] dt
\]

We now solve this differential equation to determine the trigger value of \( Q \), denoted by \( Q^* \), at which the monopolist will exercise his perpetual call option to invest. We will determine \( Q^* \) as a function of the volatility \( \sigma \), treating the variables \( r \) and \( \mu \) as known and fixed.

**III.A – The Solution Method**

Since (7) has no closed form solution, we propose the following innovative variant of a solution method. The equation is singular at \( Q=0 \) (the highest derivative is multiplied by \( Q \)) and simple integration methods fail. To show this (7) can be presented as:

\[
\frac{d^2 C(Q)}{dQ^2} + \frac{2}{\sigma^2} \frac{\mu}{dQ} - \frac{2r}{\sigma^2 Q} C(Q) = 0
\]
At $Q=0$ the term $-\frac{2r}{\sigma Q} C(Q)$ fails to be analytical, thus no solution exists at zero and the equation is very unstable in the region close to zero. Contrary to what is sometimes urged, such singular points need not invariably be avoided. Many differential equations have singular points, and the choice of appropriate solutions is often determined near those points (Simmons, 1991). Thus, it is useful to construct an analytical solution for use near $Q=0$. This can be used to calculate the values of $C(Q)$ and its derivative at a point arbitrarily close to $Q=0$ (but not at $Q=0$). A numerical routine can then be employed to extend the solution to higher values of $Q$.\(^6\) We now derive a solution method which has not been explicitly pointed out in the existing real options literature, and which may find uses beyond the present model.

We begin by introducing the scaled variable $\xi$, defined by $Q = \sigma^2 \xi$, in place of $Q$. Then equation (7) can be written as:

\[(7-b) \quad \frac{1}{2} \xi \frac{d^2 C(\xi)}{d\xi^2} + \mu \xi \frac{dC(\xi)}{d\xi} - rC(\xi) = 0\]

Notice that this form of the differential equation does not explicitly contain the volatility parameter $\sigma$. Thus we can construct a universal solution of this equation that applies for any value of $\sigma$. This is true for both the analytical

\(^6\) This approach is common when solving equations with singularity problems. In principle the solution used near the singularity points can be used for all the solution points, nevertheless it is quicker (in terms of computer speed) to use the series approximation only near the singularity and then use a numerical method to find the solution further away from the singularity points.
solution around $Q=0$ and any numerical extensions of it. The parameter $\sigma$ still affects the final solution, however, since it appears in the boundary conditions. 

Having determined the universal solution, we apply the boundary conditions, and can thus determine the trigger value $Q^*$ for a specific value of $\sigma$, and also observe how the trigger $Q^*$ depends in general on the value of $\sigma$.

Equation (7-b) does not have an explicit solution. Therefore we seek an analytic solution for application near $Q=0$. We can find one in the form of a Frobenius series\(^7\), namely (where $c$ represents the roots of the indicial equation defined below):

$$C(\xi) = \sum_{n=0}^{\infty} a_n \xi^{n+c}$$

The convergence of the power series $C(\xi)$ is given by the limit of the radius of convergence $\rho$, where the radius of convergence is defined as: $\rho = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$. If $\rho=0$, then the series diverges for all $\xi \neq 0$; if $0 < \rho < \infty$, then the series converges if $|\xi| < \rho$ and diverges if $|\xi| > \rho$; if $\rho = \infty$, then the series converges for all $\xi$. Even if the limit defined above does not exist, there will always be a number $\rho$ such that one of the three alternatives defined above holds.

To make this series specific to the problem at hand, the Frobenius series is substituted into (7-b) to obtain, after rearrangement:

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\(^7\) See Edwards and Penney, 1985, for a detailed exposition of the Frobenius method.
We now compare the coefficients of the different powers of $\xi$. If $n = -1$ (first term only) we obtain the indicial equation:

$$\frac{1}{2} a_0 c (c - 1) = 0$$

Since $a_0 \neq 0$ (the first term must exist) this determines the possible values (roots) of the index $c$. According to the Frobenius method, equation (7-b) has two solutions. The first solution corresponds to the larger root of the indicial equation. In other words, the first solution is obtained by substituting the larger root of $c$ in the Frobenius series. The second solution exists only if the difference between the two roots is neither zero, nor a positive integer. Here there are two roots, $c=0$ and $c=1$. Therefore the indicial equation for the lower power of $\xi$ is a failing Frobenius case, meaning that the smaller root of this equation does not lead to an acceptable solution of the differential equation (7-b) near the value $\xi = 0$. This is because the two roots differ by an integer. When the roots differ by an integer, the Frobenius method guarantees the existence of the solution to the differential equation, in the form of a Frobenius series, only for the highest root, in this case $c=1$. If the two roots are equal, the differential equation has only one solution, but this is not true in the present case, so a second solution must be found. In this case, the root $c=1$ gives a solution in the form of a Frobenius series, and the second solution,
corresponding to the root $c=0$, will consist of a Frobenius series for $c=0$ plus the logarithm of the equation solution for the highest root, $c=1$. Therefore, in such cases the first of two independent solutions can be constructed based on the $c=1$ root. The second independent solution is constructed from a series solution based on $c=0$, plus the term $\ln(\xi)^*(\text{first solution})$. That is (where $C_1(\xi)$ and $C_2(\xi)$ denote the first and second solution)\(^8\):

\[
C_1(\xi) = \sum_{n=0}^{\infty} a_n \xi^{n+1}, \quad \text{and} \quad C_2(\xi) = \ln(\xi)C_1(\xi) + \sum_{n=0}^{\infty} l_n \xi^n
\]

We now impose a boundary condition on $C(\xi)$ to specify the solution more closely.

As the state variable $Q$ goes to zero (and hence $\xi \to 0$), the value of the option to invest has necessarily to decrease. When the state variable reaches zero, the option to invest is worthless (so $C(\xi)$ must equal 0)\(^9\). This is automatically satisfied by the $C_1(\xi)$ function (notice that if $\xi = 0$ all the terms of $C_1(\xi)$ will also equal zero), but the $C_2(\xi)$ contains a constant term $l_0$. (Notice that the first term of $C_2(\xi)$, i.e. when $n=0$, is undefined if $\xi = 0$). Therefore, to satisfy this boundary condition the $C_2(\xi)$ term must be removed from the solution and the value $C(\xi)$ must be a multiple of $C_1(\xi)$ only.

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\(^8\) Where $l_n$ denotes the coefficients of the series when $c=0$.

\(^9\) Birth and death rates, in simple birth and death processes, are linear in the size of the population; therefore zero is an absorbing state. If the size of the population is ever zero, it will stay at zero forever, unless immigration is allowed.
We now find the solution $C(\xi)$. There is an arbitrary constant in this solution, the value of $a_0$, which we call $A$ (looking at equation (11) below, we see that all terms have factors of $a_0$). Rather than carry a factor of $A$ throughout our working, we calculate the particular function $C_h(\xi)$ that has $a_0=1$, i.e. $C_h(\xi) = C(\xi) \Big|_{a_0=1}$. The general solution will then take the form $C(\xi) = AC_h(\xi)$.

The remaining terms of the power series are obtained by comparing the coefficients of the different powers of $\xi$ in (8) for $n \geq 0$ and $c=1$. We obtain:

\[
\frac{1}{2} a_{n+1} (n+2)(n+1) + \left( \mu(n+1) - r \right) a_n = 0, \quad n \geq 0
\]  

(10)

Solving (10) and changing the counter from $n+1$ to $n$, we obtain:

\[
a_n = \frac{2[r - n\mu]}{(n+1)n} a_{n-1}, \quad n \geq 1
\]  

(11)

This equation can be used to obtain a solution by determining iteratively as many of the coefficients in the power series of the solution as is needed for acceptable accuracy (from $a_0$ we can find $a_1$, then from $a_1$ we can find $a_2$, etc)\(^\text{10}\).

Besides the boundary conditions at $Q(=\xi) = 0$ the ordinary differential equation is subject to the usual two American boundary conditions,

\[^{10}\text{Notice that the radius of convergence of our power series is given by: } \rho = \lim_{n \to \infty} \left( \frac{n+2(n+1)}{2 \left| r - (n+1)\mu \right|} \right) = \infty. \text{ Consequently, the series converges for all } \xi.\]

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respectively value matching between the exercised and unexercised option (where $K$ denotes the investment cost):

\begin{equation}
C(Q^*) = \frac{Q^*}{r - \mu} - K
\end{equation}

\begin{equation}
C(\xi^*) = \frac{\sigma^2 \xi^*}{r - \mu} - K
\end{equation}

and smooth pasting between their slopes:

\begin{equation}
\frac{dC(Q^*)}{dQ} = \frac{1}{r - \mu} \quad \text{or} \quad \frac{dC(\xi^*)}{d\xi} = \frac{\sigma^2}{r - \mu}
\end{equation}

where we have given each boundary condition both for $Q$ and for the rescaled variable $\xi$.

After designing this series solution near the origin, our final stage is to design a computer program that determines the general numerical solution $C(\xi) = AC_k(\xi)$ subject to the value matching and smooth pasting conditions. This process determines the value of $\xi^*$ that satisfies these two boundary conditions, which is the trigger value at which the monopolist invests\(^\text{11}\).

The program begins by calculating $C_k(\xi)$. As described above this is done by using the Frobenius power series to calculate the function at a point near to $\xi = 0$. The program then uses a numerical integration routine (the Runge-Kutta\(^\text{12}\) scheme) to continue the integration further. In the sample

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\(^\text{11}\) The computer code, written in Mathematica version 5.2, can be obtained from the authors by request.

\(^\text{12}\) See e.g. Edwards and Penney (1985) for a detailed explanation of the Runge-Kutta method.
program we have (arbitrarily) integrated numerically as far as \( \xi = 400 \), implying \( Q = 400\sigma^2 \).

We then use the numerical solution to calculate the following quantities for all values of \( \xi \) on the integration range\(^{13} \):

\[
A_1 = \frac{1}{C_h(\xi)} \left[ \frac{\sigma^2 \xi}{r - \mu} - K \right] \\
A_2 = \frac{1}{dC_h(\xi)/d\xi} \left[ \frac{\sigma^2}{r - \mu} \right]
\]

At \( \xi = \xi^* \) both these functions should have the same value (since then \( A = A_1 = A_2 \), and (12) and (13) are satisfied). Consequently the final part of the solution is determined by comparing \( A_1 \) with \( A_2 \) for values of \( \xi \) increasing from zero. At the value of \( \xi \) where \( A_1 - A_2 \) changes sign we know that \( \xi^* \) lies between that value of \( \xi \) and the previous one, so we use linear interpolation between the two to estimate the actual value of \( \xi^* \).

The evaluation of \( \xi^* \) is placed in a loop that does the calculation over a range of values for \( \sigma \) (using a given fixed value for \( K \)). These values are recorded. Then, from \( \xi^* \), the optimal exercise point \( Q^* \) can be calculated.

\(^{13} \text{Notice that until now we have the solution of } C_h(\xi). \text{ } C(\xi) \text{ is found by multiplying } C_h(\xi) \text{ by the arbitrary constant } A. \text{ This is also valid for the boundary conditions (12) and (13) where } C(\xi^*) \text{ and its derivative should be substituted by } C_h(\xi^*) \text{ and its derivative multiplied respectively by the arbitrary constants } A_1 \text{ and } A_2. \)
**IV – Numerical Results**

In Figure 2, we present for the volatility level of $\sigma = 30\%$ the value of the monopolist’s opportunity to invest $C(Q)$ and the net present value of investing immediately, $\frac{Q}{(r - \mu)} - K$, as functions of the potential rate of sales $Q$.

![Graph showing sensitivity of option and net present values to quantity](image)

**Fig. 2** – The parameters are: $r = 3\%$, $\sigma = 30\%$, $\mu = 1\%$ and $K = 100$

The top curve in Figure 2 represents the value of the perpetual option $C(Q)$ to invest “now or later”. The bottom line represents the net present value $\frac{Q}{(r - \mu)} - K$ of investing immediately. The two lines meet at an annualised rate of sales $Q = 6.18$. Thus for $Q < 6.18$ the monopolist will be idle, and the value of his or her opportunity is that of the unexercised option to invest (upper curve). If the sales rate $Q$ exceeds 6.18 the value of his or her opportunity is given by the net present value of having invested immediately (lower line).
Clearly, as expected, an increase in $Q$ increases both the value of the call option $C(Q)$ to invest, and the net present value of investing immediately.

For comparison we present in Figure 3 the values obtained using geometric Brownian motion to model the rate of sale $Q$, using identical parameter values\textsuperscript{14}:

![Graph showing sensitivity of option and net present values to quantity](image)

**Fig. 3** – The parameters are: $r=3\%$, $\sigma=30\%$, $\mu=1\%$ and $K=100$

The upper curve in Figure 3 represents the value of the perpetual call option to invest in a market in which there are no time-constant operating costs, and the potential annualized rate of sales (physical and/or financial) follows a geometric Brownian motion. We are using that model with the minor difference of interpretation that the underlying variable is the stochastic quantity sold at a fixed price, and not the (stochastic) price of a fixed quantity sold. The final value of money sales is the same under either interpretation.

\textsuperscript{14} We used here the model presented on page 184 of Dixit and Pindyck (1994) which determines the optimal price at which to invest and the value of the perpetual call option to invest, in a monopoly market where the price of the underlying follows a geometric Brownian motion. We are using that model with the minor difference of interpretation that the underlying variable is the stochastic quantity sold at a fixed price, and not the (stochastic) price of a fixed quantity sold. The final value of money sales is the same under either interpretation.
geometric Brownian motion. The lower line is the net present value of immediate investment, as before. The trigger sales rate for investing in this market is $Q=8.82$. This trigger is around 1.4 times the value of the trigger obtained above by assuming a birth and death process for $Q$. The Marshallian trigger for the same set of parameters is 2, which is around $\frac{1}{3}$rd of the birth and death trigger, and around $\frac{1}{4}$th of the geometric Brownian motion trigger. In this example the higher volatility of the geometric Brownian motion process generates a higher option value, and therefore a higher trigger value must be reached before it is optimal to kill the option. Such a result is expected in general for a higher volatility process, and especially at high levels of $Q$, since these produce larger differences between $\sqrt{Q}$ (1) and $Q$ in (2).

Conversely, it is interesting to see if there is a volatility at which the geometric Brownian motion process and the birth and death process give similar trigger levels $Q^*$ and therefore similar values of the call at exercise.

In Figure 4 we present the Marshallian, the geometric Brownian motion and the birth and death triggers for a range of volatility values, holding all other parameters constant.
Fig. 4 – The parameters are: $r=3\%$, $\mu=1\%$ and $K=100$

For a sufficiently low level of volatility (for the parameter levels used in Figure 4 this is $\sigma=0.03$) the geometric Brownian motion trigger is little different from the birth and death trigger. Nevertheless, it seems that at the higher volatility levels more frequently seen in practice, between 0.2 and 0.3, if real option owners were to assume geometric Brownian motion, in a case where the birth and death assumption would be more appropriate, this could lead to over-optimistic option values, therefore to excessively high triggers, so causing significantly sub-optimal levels of investment. For a competitive firm, delayed investment might also cause a loss of competitive advantage, though our call option model does not attempt to model this. Such effects may well be scale- and parameter- sensitive, so it will be useful to check for them numerically in each problem application.
In this paper we have introduced to the real options literature a new application of birth and death processes. Birth and death processes were modelled for financial options by Cox and Ross (1976), but their dynamics did not find acceptance as being realistic for stock prices, in competition with geometric Brownian motion. Slightly more complex dynamics than birth and death, including mean-reverting dynamics, have been used in the square root model of interest rates of Cox, Ingersoll and Ross (1985) whose empirical realism has been questioned by e.g. Chan, Karolyi, Longstaff and Sanders (1992). Lately Biepke, Klumpes and Tippett (2003) and Klumpes and Tippett (2004) have used variants of the CIR model to represent sales revenue.

In terms of the use of the pure birth and death model to value derivatives, Cox and Ross (1976) value a European Call, and we have valued a perpetual American call. To solve this call, we have introduced a new computation strategy to the financial literature on birth and death processes. The technique is a variant of the method of Frobenius, which constructs an analytic series solution as close as practicably possible to the singular point, and uses the high speed of numerical methods to generate the whole of the remaining solution domain. This method may be useful under many other stochastic specifications, when the value function of an option contains one or more singular points, preventing a closed form solution, and preventing a conventional finite difference solution. This compound of series solution and numerical methods may often be faster than the method suggested for slightly different types of models by Biepke, Klumpes and Tippett (2003) who favour
using repeated applications of series to generate the entire solution domain. The question of the relative speed and accuracy of both methods might be a subject for future research.

There are many practical examples of birth and death processes, where applications of our model may be appropriate. Control of epidemics with the rates of birth (spread) and death (or immunization) may involve irreversible investment-type expenditures such as culling or vaccinations. Often there are local monopolies on the provision of bulky leisure facilities such as swimming pools, where customers come and go, based on population size, quality of the facility, and “referrals” similar to birth and death processes (the latter hopefully limited, during the use of the facilities).

The birth and death specification is potentially of practical use for real options. We have argued that one of the future tasks for real options research is to pay explicit attention to the physical processes: their stochastic dynamics, their payoff functions and their operating constraints. The goal should be eventually to integrate stochastic models of physical processes with (potentially different and differently constrained) stochastic models for selling prices. Richer methods for physical processes in isolation, such as described here, are a contribution to this eventual goal of integrated price-quantity modelling. Based on the findings of Paxson and Pinto (2005) we should expect non-trivial insights to arise only if the processes for price and physical quantity have different constraints.

The numerical properties of our solution are parameter-value-specific, although the general property is well known that greater volatility increases the
value of a call, and delays investment\textsuperscript{15}. Hence, unsurprisingly, for moderate to high values of volatility, the investor will enter the market a lot sooner if he or she has the correct parameter values, but believes the underlying dynamics are birth and death. Prima facie, therefore, an investor who incorrectly believes that the dynamics are geometric Brownian motion will under-invest, both absolutely and relative to better informed investors. One of several topics for future numerical research would be to investigate possible mitigating factors for this bias – for example how wrong will investors be – for a given set of “true” parameter values and ranges of $Q$, if they wrongly interpret an observation of $\sigma Q$ as an observation of $\sigma \sqrt{Q}$? A more general extension of the dynamics themselves would be to develop models for marketing applications, such as models of innovation diffusion and saturation and models of customer churn.

\textit{VI – References}


\textsuperscript{15} There are some exceptions to this rule. In the Margrabe (1978) exchange option the Vegas are positive if the correlation coefficient is negative or zero, but when the correlation is positive, depending on the magnitude and on the relative volatilities of the two assets, the Vega can be negative. Negative Vegas can be seen in many correlation options; this happens because the aggregate volatility may decline as the individual volatilities increase, given certain correlation coefficient values (see Zhang (1997) for details).


