Rational Laymen versus Over-Confident Experts: 

Who Survives in the Long Run?

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JEL: G12, G13
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1 Introduction and Motivation

Information processing by market participants is one of the key issues in theoretical and empirical financial economics. Equilibrium prices reflect the demand for securities by investors, who determine the quantities they want to trade on the basis of the available information. When investors cannot directly observe a parameter or a state variable, their ability to learn from observed data becomes crucially important. Learning in this context is mostly interpreted in a Bayesian sense, i.e. investors start with some prior about the variable of interest and update their beliefs according to observable quantities. Whereas in a world with perfectly rational investors this available information is processed efficiently without error or bias, more recent papers focus on differences in beliefs between investors, on errors in the learning process, and on the impact of these differences and/or errors on asset allocations and equilibrium prices.

For example, Buraschi and Jiltsov (2003) study an economy with two groups of investors, which have different priors for the drift of the dividend process. The authors are mainly interested in the trading volume in derivative markets which is generated by this heterogeneity and test their model empirically using data from options markets. Dumas, Kurshev, and Uppal (2005) analyze the case in which two groups of investors receive a signal about the stochastic drift of the underlying dividend process, but interpret it differently. Whereas investors in Group B correctly extract the information from the signal, investors in Group A are in some sense overconfident by attributing too much informational content to the signal. In particular, they wrongly assume that innovations to the signal are correlated with innovations to the dividend drift, although in the true model the two Brownian motions are independent. The authors then show that Group B containing the ‘more intelligent’ investors dominates the economy, although, according to their numerical analyses, it would take very long until the group of overconfident investors would disappear from the market. However, the result that investors interpreting signals correctly have an informational advantage over their competitors is intuitively clear.

Our paper adds to this strand of the literature by studying an economy where ‘nobody is perfect’. Each investor has some disadvantage when it comes to learning about the non-observable drift, either because his or her signal is of a lower quality than the signal received by some other investor, or because he or she makes an error in interpreting the signal, e.g. of the kind explained above. This setup allows us to compare the consequences of these deficiencies and to assess their respective impact on equilibrium allocations. We are especially interested in trade-offs between the two types of shortcomings the investors exhibit in our model. Similar to Dumas, Kurshev, and Uppal (2005), we assume that there is a Group A of overconfident investors assigning too much informational value to the innovations in the signal about the dividend drift. However, in our model investors in Group B do not receive a perfect signal either. The investors in this group correctly assume that innovations in the signal are uncorrelated with innovations in the drift of the dividend process, but the signal that they receive has a higher volatility than the signal of the other group.
and is therefore of a lower quality. For example, if investors lack the competence to pre-process a piece of news or an analyst’s opinion fully correctly to generate a signal about the dividend drift, this will not necessarily create a biased signal, but may result in additional noise compared to the fundamental randomness in the news themselves. So these investors will ultimately use less information than the investors generating or receiving the more precise signal. The key question we are asking in our paper is: Given a certain level of overconfidence on the one hand and a certain level of additional noise in the signal on the other, which group of investors is better off in terms of (long-run) consumption and wealth?

To get some intuition for the setup in our model, take an executive in the life sciences sector (especially in the late nineties of the previous century) as a typical representative for the investors in Group A. Due to professional expertise this person may have well been able to properly extract the dividend signal out of, e.g., a research report, but may at the same time have been overconfident concerning his ability to infer the future prospects of his or her firm or of the industry as a whole. In contrast to this overconfident expert, a ‘normal’ investor could have been aware of the fact that noise in the signal was not informative about the dividend process. However, he or she was probably not able to interpret the report perfectly to generate a signal with the highest possible precision, may be due to a lack of special knowledge about the respective industry. For example, it seems fair to assume that the normal investor would be less competent than an industry representative in interpreting the economic consequences of rather technical research reports from a biotech company about its recent advances in trying to develop new medication. In the following, we will call this normal investor a rational layman.

The decisive parameters in our model are the amount of additional noise in the signal received by the less competent rational laymen, or, alternatively, the degree of correlation between the signal and the dividend process assumed by the overconfident experts, i.e. the level of overconfidence. In a sort of sensitivity analysis we investigate the critical degree of overconfidence for the experts in Group A. Below this level, these investors still have an advantage, i.e. obtain on average a larger share of consumption and wealth in the long run. Alternatively, one can infer a critical level of additional noise for the rational laymen in Group B, below which they will be better off than the typical representative from Group A.

Note that there are two limiting cases of our model. First, when type B investors are completely ignorant about the content of some piece of information, their signal will have infinite noise. In our above example this would mean that the investor not associated with the biotech industry would not be able to infer any information at all from a research report, so that the only source of information for type B investors would be the realization of the dividend process. Second, when we set the additional noise in the signal received by type B investors equal to zero, so that these investors have the same competence as the experts in interpreting the contents of the research report mentioned above, without becoming overconfident. Then, only the overconfident experts have a disadvantage, and we are back in the setup studies in Dumas, Kurshev, and Uppal (2005).
The rest of the paper is organized as follows. In Section 2 we will present the model setup, before the equilibrium outcome in general will be discussed in Section 3. Since the special focus of our paper is on long-run consumption shares, Section 4 is devoted to this topic. Section 5 contains a few concluding remarks.

2 Model

2.1 Basic setup

We consider a continuous-time Lucas (1978) tree economy with an infinite time horizon. In this pure-exchange economy there is a single consumption good that is used as the numeraire. In what follows, we assume that all processes and expectations introduced are well-defined, without explicitly stating the regularity conditions.

The exogenous dividend, or aggregate supply, process in the economy $D_t$ follows the stochastic differential equation

$$
\frac{dD_t}{D_t} = \mu_t \, dt + \sigma_D \, dW_{D,t},
$$

(1)

where $W_{D,t}$ is a one-dimensional standard Brownian motion under the true probability measure. $\sigma_D$ is the volatility of the dividend growth rate, which is assumed to be constant. As in Scheinkman and Xiong (2003) and Dumas, Kurshev, and Uppal (2005), the drift $\mu_t$ of the dividend is stochastic and follows an Ornstein-Uhlenbeck process

$$
d\mu_t = -\lambda(\mu_t - \bar{\mu}) \, dt + \sigma_\mu \, dW_{\mu,t},
$$

where $W_{\mu,t}$ is also a standard Brownian motion under the true probability measure. $\sigma_\mu$ is the constant volatility, $\bar{\mu}$ is the long-run mean, and $\lambda \geq 0$ is the speed of mean reversion.

The process of the conditional expected dividend growth rate $\mu_t$ is unobservable. Investors can learn about it from observing the dividend process, but they cannot infer it perfectly. Furthermore, there are two signals $s^A_t$ and $s^B_t$ with dynamics

$$
\begin{align*}
    ds^A_t &= \mu_t \, dt + \sigma_s \, dW_{s,t}, \\
    ds^B_t &= \mu_t \, dt + \sigma_n \, dW_{n,t},
\end{align*}
$$

where $W_{s,t}$ and $W_{n,t}$ are standard Brownian motions. The volatilities $\sigma_s$ and $\sigma_n$ are constant. Both signals have the same drift as the dividend process and thus provide information about the unobserved $\mu_t$, as does the dividend itself. We assume $\sigma_n \geq \sigma_s$, i.e. the amount of noise contained in the signal $s^B_t$ is greater than that contained in $s^A_t$, so that $s^A_t$ is the more precise signal.

For simplicity we assume that the above Brownian motions $W_{D}$, $W_{\mu}$, $W_s$, and $W_n$ are mutually independent. Furthermore, we assume that, although investors do
not observe the dividend drift $\mu_t$, they know all parameters governing the dynamics of the dividend, the dividend drift and the signals.

In the economy there are two types, called A and B, of infinitely-lived investors. The investors have identical preferences and endowments, and the observations they make are also the same, but they differ with respect to their learning behavior. This simplifying set of assumptions allows us to focus on the main point of the paper, namely the effect of differences in beliefs on the equilibrium wealth distribution. In particular, we assume that all investors have the same CRRA utility function

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

with constant relative risk aversion $\alpha$, and the same time preference parameter $\rho$. With $c_i^t$ as the consumption of type $i \in \{A, B\}$ at time $t$ the lifetime utility is given by

$$\int_0^\infty e^{-\rho t} \frac{1}{1-\alpha} (c_i^t)^{1-\alpha} dt.$$ 

The drift of the dividend growth rate $\mu_t$ is unobservable, and investors have to estimate it based on the available information. The estimates of the dividend drift are

$$\mu_i^t := \mathbb{E}_i[\mu_t | \mathcal{F}_i^t] \quad (i \in \{A, B\}),$$

where $\mathcal{F}_i^t$ represents the information set of type $i$, and $\mathbb{E}_i$ denotes expectation calculated under the type $i$ subjective probability measure. The conditional variance of the estimation error is analogously given by

$$\gamma_i^t := \mathbb{E}_i[(\mu_t - \mu_i^t)^2 | \mathcal{F}_i^t] \quad (i \in \{A, B\}).$$

Before discussing the details of the filtering process by which investors form their estimates of the dividend drift, we take a closer look at the information structure in our model. In our model, both investor types observe the dividend process and the respective signal. Furthermore, we assume that each investor type can observe the estimate of the dividend produced by the other group, i.e. type A investors know what type B investors think the drift currently is, and vice versa. For example, we could think of an economy, in which all market participants always announce their estimates for the dividend drift, but not the signal from which this estimate was derived. The information sets are thus given by

$$\mathcal{F}_i^t = \sigma(D_u, s_i^u, \mu_i^t, 0 \leq u \leq t)$$

and

$$\mathcal{F}_i^t = \sigma(D_u, s_i^u, \mu_i^t, 0 \leq u \leq t).$$

---

1For the joint effect of preferences and beliefs on equilibrium distribution and their trade-off relationship, see Kogan, Ross, Wang, and Westerfield (2006) and Yan (2005).

2This assumption allows us to work in a complete markets framework later on and so to abstract from a possible impact of market incompleteness on the equilibrium outcome in our economy.
We assume that type A and type B investors 'agree to disagree': Although they know the other group’s estimate, and although this estimate differs from their own, they still believe in their own model and simply think that their competitors are wrong.

Type A investors observe the dividend process $D_t$ as well as the inference $\mu_t^B$ made by type B, and their learning is based on the more precise signal $s_t^A$. However, they commit two errors in their own inference of the dividend drift. First, they incorrectly assume that the innovation in the signal process $s_t^A$ is correlated with the innovation in the mean reversion process for $\mu_t$, although the two Brownian motions are actually independent. Second, they assume that the type B investors’ estimate does not contain any useful information about the dividend drift. The ‘learning model’ used by investors in Group A is therefore given by the following set of stochastic differential equations:

$$
\begin{align*}
  dD_t &= \mu_t \, dt + \sigma_D \, dZ_{D,t} \\
  d\mu_t &= -\lambda(\mu_t - \bar{\mu}) \, dt + \sigma_\mu \, dZ_{\mu,t} \\
  ds_t^A &= \mu_t \, dt + \sigma_s \left\{ \phi dZ_{\mu,t} + \sqrt{1-\phi^2} dZ_{s,t} \right\} \\
  d\mu_t^B &= -\lambda(\mu_t^B - \bar{\mu}) \, dt + \frac{\gamma_B}{\sigma_D} \left( \frac{dD_t}{D_t} - \mu_t^B \, dt \right) + \frac{\gamma_B}{\sigma_n} \, dZ_{n,t}.
\end{align*}
$$

$Z_D, Z_s, Z_\mu, Z_n$ are mutually independent standard Brownian motions. $\phi \geq 0$ is the assumed correlation between shocks in the dividend drift and the signal. The process for $\mu_t^B$ as assumed by type A is (with hindsight at this point in time) chosen such that it is line with the true process for $\mu_t^B$, which is the result of the filtering process by type B investors. We will comment on the role of $\mu_t^B$ for investors in Group A at a later point. At the moment, it is just important to note that the estimation of $\mu_t^A$ is not influenced by the inclusion or omission of the $\mu^B$ process. Thus, type A investors (improperly) attribute zero informational content to this process.

Type A investors can ultimately base their decisions on a better signal, but they use a wrong model. We can think of them as investors with special knowledge in a certain industry who are able to generate a signal with lower volatility from some source of news, e.g. from analyst recommendations or research reports, due to their professional competence. However, at the same time these investors sometimes tend to be over-confident and think that their signal contains more information than it actually does. For these reasons we call investors in Group A ‘over-confident experts’.

We will now describe the information setup for investors in Group B. They observe the dividend process $D_t$ and the inference $\mu_t^A$ made by type A investors. Furthermore, they can only generate the noisier signal $s_t^B$. In contrast to type A investors, they use the correct model for the joint dynamics of the dividend drift and their signal. However, just like type A investors ignore the information contained in the estimate $\mu_t^B$, type B investors do not assign any informational value to observing
\( \mu_t^A \). The learning model used by Group B then is

\[
\frac{dD_t}{D_t} = \mu_t \, dt + \sigma_D dZ_{D,t},
\]

\[
d\mu_t = -\lambda (\mu_t - \bar{\mu}) \, dt + \sigma_\mu dZ_{\mu,t},
\]

\[
ds_t^B = \mu_t \, dt + \sigma_n dZ_{n,t},
\]

\[
d\mu_t^A = -\lambda (\mu_t^A - \bar{\mu}) \, dt + \frac{\gamma_A}{\sigma_D} \left( \frac{dD_t}{D_t} - \mu_t^A \, dt \right) + \frac{\gamma_A + \phi \sigma_\mu^2}{\sigma_s} \, dZ_{s,t}.
\]

We call investors of type B ‘rational laymen’. They are not as apt in interpreting public signals, like research reports, because of a lack of special competence in the given field. However, they are rational and correctly specify the joint dynamics of dividend drift and signal. Like the experts, the rational laymen do not attribute any additional information to the inference made by their competitors either, beyond what they already know. They just interpret the additional information in \( \mu_t^A \) as random error in the learning process of Group A.

### 2.2 Filtering

Both the over-confident experts and the rational laymen have to make their consumption and investment decisions without observing the state variable \( \mu_t \). Similar problems have been studied in Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986). In these papers the authors show that this kind of problem may be separated into a part related to filtering, i.e. to the fact that investors have to estimate the current value of the state variable, and a part related to the investment decision, in which they decide on their portfolios, based on the estimates from the first step.

In the filtering problem, both groups of investors base their inference on the observed dividend, their own signal, and the estimate of the other group. They use the model they assume to be correct, and they update their beliefs in a Bayesian way. We assume that the priors of both groups are Gaussian with means \( \mu_0^A(\mu_0^B) \) and variances \( \gamma_0^A(\gamma_0^B) \).

First we look at the filtering problem of Group B, the group of rational laymen. From standard filtering theory discussed in Liptser and Shiryaev (2001), we know that the conditional distribution of the unobservable drift of the dividend process is also Gaussian, with conditional mean \( \mu_t^B \) and conditional variance \( \gamma_t^B \). Group B’s estimate of the dividend drift \( \mu_t^B \) follows the stochastic differential equation

\[
d\mu_t^B = -\lambda (\mu_t^B - \bar{\mu}) \, dt + \frac{\gamma_B}{\sigma_D} \left( \frac{dD_t}{D_t} - \mu_t^B \, dt \right) + \frac{\gamma_B}{\sigma_n^2} \, (d\mu_t^B - \mu_t^B \, dt),
\]

where \( \gamma_B \) is the steady state variance given by

\[
\gamma_B = \sqrt{\frac{\lambda^2 + \sigma_\mu^2 (\sigma_D^2 + \sigma_n^2)}{\sigma_D^2 + \sigma_n^2} - \lambda}.
\]
To justify the steady state assumption, one can think of an infinitely long history of past observations, so that the variance has already converged to its long run value.

Putting the results together, we can write the model of Group B, the rational laymen, after learning as

\[ \frac{dD_t}{D_t} = \mu^B_t \, dt + \sigma_D \, dW^B_{D,t} \]
\[ ds^B_t = \mu^B_t \, dt + \sigma_n \, dW^B_{n,t} \]
\[ d\mu^B_t = -\lambda(\mu^B_t - \bar{\mu}) \, dt + \frac{\gamma^B}{\sigma_D} \, dW^B_{D,t} + \frac{\gamma^B}{\sigma_n} \, dW^B_{n,t} \]
\[ d\mu^A_t = -\lambda(\mu^A_t - \bar{\mu}) \, dt + \frac{\gamma^A}{\sigma^2_D} \left( \frac{dD_t}{D_t} - \mu^A_t \, dt \right) + \frac{\gamma^A + \phi \sigma_D \sigma_s}{\sigma_s} \, dW^B_{s,t} \]

where \( W^B_{D,t}, W^B_{n,t}, \) and \( W^B_{s,t} \) are mutually independent standard Brownian motions under the subjective probability measure of Group B. Note that the dynamics of the estimate \( \mu^B \) have indeed the same structure as assumed by Group A, which justifies the specification used above.

For the filtering problem of Group A, the over-confident experts, a similar calculation yields

\[ \frac{dD_t}{D_t} = \mu^A_t \, dt + \sigma_D \, dW^A_{D,t} \]
\[ ds^A_t = \mu^A_t \, dt + \sigma_s \, dW^A_{s,t} \]
\[ d\mu^A_t = -\lambda(\mu^A_t - \bar{\mu}) \, dt + \frac{\gamma^A}{\sigma^2_D} \, dW^A_{D,t} + \frac{\gamma^A + \phi \sigma_D \sigma_s}{\sigma_s} \, dW^A_{s,t} \]
\[ d\mu^B_t = -\lambda(\mu^B_t - \bar{\mu}) \, dt + \frac{\gamma^B}{\sigma^2_D} \left( \frac{dD_t}{D_t} - \mu^B_t \, dt \right) + \frac{\gamma^B}{\sigma_n} \, dW^A_{n,t} \]

Again we assume that the steady state has been reached, and the steady state variance is given by

\[ \gamma^A = \sqrt{\frac{(\lambda + \frac{\sigma^2_D}{\sigma_s} \phi) + \frac{2}{\sigma^2_D} (1 - \phi^2) (\sigma_D^{-2} + \sigma_s^{-2}) - (\lambda + \frac{\sigma^2_D}{\sigma_s} \phi)}{\sigma^2_D + \sigma_s^{-2}}} \]

Analogously to Group B, \( W^A_{D,t}, W^A_{n,t}, \) and \( W^A_{s,t} \) are mutually independent standard Brownian motions under the subjective probability measure of Group A. Again, \( \mu^A \) has the same dynamics as assumed by Group B. The impact of the improper assumption \( \phi > 0 \) can be assessed by looking at the dynamics of the estimated drift. \( \phi \) influences the weights of the observed changes in the dividend and in the signal when updating the estimate for the drift. For \( \phi > 0 \), the investor puts too much weight on the signal and not enough weight on the dividend innovation compared to the case \( \phi = 0 \).

In our model we have three equivalent probability measures: the true measure \( \mathbb{P} \), and two subjective measures \( \mathbb{P}^A \) and \( \mathbb{P}^B \). To get the relation between these measures, first note that the investors observe the dividend and their signal. Equating
the dynamics of these variables under $\mathbb{P}$ and $\mathbb{P}^A$ yields

\[
\begin{align*}
    dW^A_{D,t} &= dW_{D,t} - \frac{\mu^A_t - \mu_t}{\sigma_D} dt \\
    dW^A_{s,t} &= dW_{s,t} - \frac{\mu^A_t - \mu_t}{\sigma_s} dt.
\end{align*}
\]

In the same way we can conclude that

\[
\begin{align*}
    dW^B_{D,t} &= dW_{D,t} - \frac{\mu^B_t - \mu_t}{\sigma_D} dt \\
    dW^B_{n,t} &= dW_{n,t} - \frac{\mu^B_t - \mu_t}{\sigma_n} dt.
\end{align*}
\]

Furthermore, note that the investors observe the estimated drift of the other group, which gives

\[
\begin{align*}
    dW^A_{s,t} &= dW^B_{s,t} \\
    dW^A_{n,t} &= dW^B_{n,t}.
\end{align*}
\]

This last set of equation implies that the type B investor knows the innovation $dW^A_{s,t}$ in the signal $s^A_t$ received by group A. To see this, first note that

\[
ds^A_t = \mu^A_t dt + \sigma_s dW^A_{s,t}. \tag{2}
\]

With the knowledge of $\mu^A_t$ and $dW^A_{s,t}$, the investor from group B can infer the good signal received by A. However, he or she does not know (or refuses to know) that the true process for this signal is

\[
ds^A_t = \mu_t dt + \sigma_s dW_{s,t},
\]

which indeed contains information about the true drift of the dividend. Instead, the type B investor believes that Equation (2) describes the true behavior of the signal.

The triples $(W^A_D, W^A_s, W^A_n)$ and $(W^B_D, W^B_s, W^B_n)$ are three-dimensional Brownian motions under the subjective measures of Group A and Group B, respectively. From the equations above we obtain the relation between these two subjective measures $\mathbb{P}^A$ and $\mathbb{P}^B$ as

\[
\begin{align*}
    dW^A_{D,t} &= dW^B_{D,t} + \frac{\mu^B_t - \mu^A_t}{\sigma_D} dt \\
    dW^A_{s,t} &= dW^B_{s,t} \\
    dW^A_{n,t} &= dW^B_{n,t}.
\end{align*}
\]

Whenever the investors disagree on the expected dividend growth rate, i.e. whenever $\mu^B_t \neq \mu^A_t$, the investors also disagree on the innovations in the dividend process. On the other hand, they always agree on the innovations in the signal processes. Each
group observes only one signal. The investors form their opinion about their signal process, and just adopt other investors’ opinion about the other signal without realizing that this signal would provide useful information.

Let the Radon-Nikodym derivative between the subjective measures of A and B be denoted by $\eta_t$, i.e.

$$
\eta_t := \mathbb{E}^A \left[ \frac{dP^B}{dP^A} \big| \mathcal{F}^A_t \right].
$$

The dynamics of $\eta$ under Group A’s measure follow from a simple application of Girsanov’s theorem:

$$
\frac{d\eta_t}{\eta_t} = \frac{\mu^B_t - \mu^A_t}{\sigma_D} dW^A_{D,t}.
$$

(3)

This equation shows how the current disagreement $\mu^B_t - \mu^A_t$ gets reflected in investors’ beliefs about future events.

3 Equilibrium

3.1 Homogeneous economy

First we analyze the simple case when there are only investors of type $i \in \{A, B\}$ in the economy. In this homogeneous economy investors observe only the dividends and their own signals. To solve for the equilibrium outcome we assume that the market is complete, which allows us to focus on the impact of differences in beliefs, similar to Basak (2005). This completeness assumption may seem restrictive at first sight. However, it will turn out to be innocuous, as we will show below.

Since the market is complete, there is a unique stochastic discount factor $\xi^i_t$ given by

$$
\frac{d\xi^A_t}{\xi^A_t} = -r^A_t dt - \theta^A_{D,t} dW^A_{D,t} - \theta^A_{s,t} dW^A_{s,t}
$$

in a type A economy, and by

$$
\frac{d\xi^B_t}{\xi^B_t} = -r^B_t dt - \theta^B_{D,t} dW^B_{D,t} - \theta^B_{n,t} dW^B_{n,t}
$$

in a type B economy. $r^i_t$ is the respective instantaneous interest rate in a type $i$ economy, $\theta^A_{D,t}$ and $\theta^B_{n,t}$ are the market prices of risk for dividend shocks and signal shocks, respectively. Since we have only one group of investors there are only two sources of risk, namely $W_D$ and either $W_s$ or $W_n$.

As shown by Cox and Huang (1989), in a complete market we can use martingale methods to solve investors’ utility maximization problem subject to their static
budget constraint:

\[ \sup_c \mathbb{E}^i \left[ \int_0^\infty e^{-\rho t} \frac{1}{1-\alpha} (c'_t)^{1-\alpha} dt \right] \]

s.t. \[ \mathbb{E}^i \left[ \int_0^\infty \xi_t c'_t dt \right] = \mathbb{E}^i \left[ \int_0^\infty \xi_t D_t dt \right]. \]  

In equilibrium the representative investor chooses a consumption policy that maximizes his or her expected lifetime utility (under the subjective measure), given all security prices. Market clearing implies that he or she has to consume all the aggregate dividend. The first order condition for consumption and the market clearing condition then give the interest rate and the market prices of risk in a homogeneous economy.

**Proposition 1** The equilibrium the instantaneous interest rate in a type \( i \) economy \((i \in \{A, B\})\) is given by

\[ r^i_t = \rho + \alpha \mu^i_t - 0.5 \alpha (1+\alpha) \sigma^2_D, \]

the market price of risk for dividend shocks is \( \theta^i_{D,t} = \alpha \sigma_D \), and the market prices of risk for signal shocks are \( \theta^A_{s,t} = \theta^B_{n,t} = 0 \).

**Proof:** See Appendix A.

The interest rate and the market price for dividend risk are equal to those in a model with known dividend drift. However, the most important part of the proposition is that signal risk is not priced. To get the intuition, note that the aggregate dividend does not depend on the signal process. Since the representative investor has to consume aggregate dividends, this is also true for his consumption. Thus, the signal shock has no impact on his marginal utility, and is therefore not priced.

The market prices of risk in Proposition 1 are calculated for the pricing kernel of investor \( i \in \{A, B\} \), i.e. they specify the change from investor’s subjective measure to the risk-neutral measure. Relative to the true probability measure, the risk premia have to be adjusted by the change between the true and the subjective measures. The market price of dividend risk \( \theta^i_{D,t} - \frac{\mu^i_t - \mu_t}{\sigma_D} \) is then equal to \( \alpha \sigma_D - \frac{\mu^i_t - \mu_t}{\sigma_D} \).

For the market price of signal risk, we have to distinguish between investor A and investor B. If there is only investor A in the economy, then the premium \( \theta^A_{s,t} - \frac{\mu^A_s - \mu_s}{\sigma_s} \) is equal to \( -\frac{\mu^A_s - \mu_s}{\sigma_s} \), while in a type B economy we obtain in an analogous fashion a premium of \( -\frac{\mu^B_n - \mu_n}{\sigma_n} \).

The risk premia relative to the true measure are actually part of the so-called ‘empirical kernel’, which is the Radon-Nikodym derivative of the risk-neutral measure with respect to the true measure. As shown in Bondarenko (2003), the empirical kernel can be decomposed into the pricing kernel and the belief kernel, which is equal to the Radon-Nikodym derivative of the subjective measure with respect to the true
measure. An outside empiricist can only observe the empirical kernel. However, the same empirical kernel can result from many different combinations of a pricing kernel, which depends on investors’ preferences, and a belief kernel, which depends on their beliefs. Therefore, risk aversion and incorrect beliefs sometimes have the same implications for securities prices.

3.2 Heterogeneous economy

Now we turn to the economy where the over-confident experts (Group A) and rational laymen (Group B) coexist, and analyze the effect of differences in beliefs and learning behaviors on asset prices and consumption shares in equilibrium.

Again we assume a complete market. This implies the existence of a unique SDF $\xi^i$ $(i \in \{A, B\})$ for each investor group with dynamics

$$
\frac{d\xi^A_t}{\xi^A_t} = -r_t dt - \theta^A_{D,t} dW^A_{D,t} - \theta^A_{s,t} dW^A_{s,t} - \theta^A_{n,t} dW^A_{n,t},
$$

(6)

$$
\frac{d\xi^B_t}{\xi^B_t} = -r_t dt - \theta^B_{D,t} dW^B_{D,t} - \theta^B_{s,t} dW^B_{s,t} - \theta^B_{n,t} dW^B_{n,t},
$$

(7)

where $r_t$ is the instantaneous interest rate in this heterogeneous economy, and $\theta^A_{D,t}$, $\theta^A_{s,t}$, and $\theta^A_{n,t}$ are the market prices of the dividend shock and the signal shocks, respectively, as perceived by type $i$ investors.

In the economy, investors have heterogeneous beliefs, so the market price of risk for each group may differ from each other. However, since we have assumed a complete market, all claims must be priced equally by Group A and B. The belief kernels and pricing kernels for both groups may be different, but the empirical kernel in this economy is unique. This implies that the Radon-Nikodym derivative $\eta_t$ of B’s measure with respect to A’s measure can be written as

$$
\eta_t = \frac{\xi^A_t}{\xi^B_t}.
$$

(8)

A general equilibrium in this economy is defined as a price system and a pair of consumption-portfolio processes such that all investors maximize their expected utility given the perceived price processes, and markets clear. To solve for the equilibrium we have to solve the optimization problem for each investor group separately. Since we assume a complete market, we can use the martingale approach again. The utility maximization problem faced by Group $i$ is

$$
\sup \mathbb{E}^i \left[ \int_0^\infty e^{-\rho t} \frac{1}{1 - \alpha} (c^i_t)^{1-\alpha} dt \right]
$$

s.t. $\mathbb{E}^i \left[ \int_0^\infty \xi^i_t c^i_t dt \right] = \beta^i \mathbb{E}^i \left[ \int_0^\infty \xi^i_t D_t dt \right],
$$

where $\beta^i$ is the initial share of aggregate endowment owned by investor $i$. 

11
The first order condition for optimal consumption equates marginal utility to \( y_i \xi^i_t \), where \( y_i \) is the (constant) Lagrange multiplier of the budget constraint. The optimal consumption for type \( i \) is given by

\[
c^i_t = (y_i \xi^i_t e^{\rho t})^{-\frac{1}{\alpha}}
\]

The market clearing condition for the commodity market requires that the total dividend has to be consumed by A and B investors together, i.e.

\[
c^A_t + c^B_t = (y_A \xi^A_t e^{\rho t})^{-\frac{1}{\alpha}} + (y_B \xi^B_t e^{\rho t})^{-\frac{1}{\alpha}} \equiv D_t. \tag{9}
\]

From equations (8) and (9) we obtain the optimal consumption policy for each group in equilibrium.

**Proposition 2** In a heterogeneous economy the optimal consumption policies for type A and B are given by

\[
c^A_t = D_t \frac{y_A^{-\frac{1}{\alpha}}}{y_A^{-\frac{1}{\alpha}} + (\frac{\eta_B}{\eta_A})^{-\frac{1}{\alpha}}}
\]

\[
c^B_t = D_t \frac{\frac{\eta_A}{\eta_B}}{y_A^{-\frac{1}{\alpha}} + (\frac{\eta_B}{\eta_A})^{-\frac{1}{\alpha}}}
\]

**Proof:** See Appendix B.

The consumption sharing rule in equilibrium is linear in the dividend \( D_t \). This is a consequence of our simplifying assumption that both groups have the same risk aversion, and is in line with results obtained in Basak (2005), Buraschi and Jiltsov (2003), and Dumas, Kurshev, and Uppal (2005). The fraction of the total dividend allocated to each group depends on \( \eta \), which describes the difference in beliefs of both groups. Since \( \eta \) is stochastic, the consumption shares are also stochastic.

The interest rate and the market prices of risk in equilibrium follow from Equations (6), (8) and (9) and an application of Ito’s lemma:

**Proposition 3** In an equilibrium with heterogeneous investors, the instantaneous interest rate is

\[
r_t = \frac{c^A_t}{D_t} r^A_t + \frac{c^B_t}{D_t} r^B_t - \frac{1 - \alpha}{2\alpha} \frac{(\mu^A_t - \mu^B_t)^2}{\sigma^2_D} \frac{c^A_t}{D_t} \frac{c^B_t}{D_t}, \tag{10}
\]

where \( r^A_t \) and \( r^B_t \) are the equilibrium interest rates in homogeneous economies of type A and B, respectively.

**Proof:** See Appendix B.
The equilibrium interest rate in the heterogeneous economy is thus equal to the weighted average of the equilibrium interest rates in the two homogeneous economies with only type A and type B investors, respectively, plus an additional term reflecting the interaction between the two groups. The weight for the respective homogeneous interest rate is the consumption share of the group. The additional term can be interpreted as another ‘precautionary savings’ term, now with respect to the difference in beliefs $\mu^A_t - \mu^B_t$ and is thus similar in spirit to the terms introduced into the short rate equation by other forms of investor heterogeneity, as shown, e.g., in Dieckmann and Gallmeyer (2005). It disappears for several special cases, namely when investors in both groups agree on the expected dividend growth rate, i.e. when $\mu^A_t = \mu^B_t$, or when the consumption share of one group is equal to 0, which represents a homogeneous economy, or when investors have log utility ($\alpha = 1$), since myopic investors do not care about changes in the state variables.

The next proposition deals with the market prices of dividend risk in the two groups.

**Proposition 4** In equilibrium with heterogeneous investors, the market prices of dividend risk for the two investor groups are given by

\[
\theta^A_{D,t} = \alpha \sigma_D + \frac{\mu^A_t - \mu^B_t}{\sigma_D} \frac{c^B_t}{D_t}
\]

\[
\theta^B_{D,t} = \alpha \sigma_D - \frac{\mu^A_t - \mu^B_t}{\sigma_D} \frac{c^A_t}{D_t}.
\]

**Proof:** See Appendix B.

Note that the market prices of risk are given under each group’s subjective probability measure, and they also depend on the difference in beliefs $\mu^A_t - \mu^B_t$. It is intuitive that if at a certain point in time investors, for example, type A investors are more optimistic, i.e. $\mu^A_t > \mu^B_t$, then they perceive a higher risk premium for dividend shocks. This creates a higher incentive for them to take on more dividend risk, and to sell insurance to their counterpart.

A simple calculation shows that the average of the two market prices of risk, with weights given by the consumption shares, is equal to the market price of risk in the homogeneous economy:

\[
\frac{c^A_t}{D_t} \theta^A_{D,t} + \frac{c^B_t}{D_t} \theta^B_{D,t} = \alpha \sigma_D.
\]

The risk premia perceived by each investor group are different whenever the investors disagree on the expected dividend growth rate. But since we have a unique empirical kernel, the risk premium with respect to the true measure must be unique. Indeed, from either of the subjective risk premia, we can obtain the market price of dividend.
shock relative to the true measure, which is given by
\[
\theta_{D,t} = \alpha \sigma_D + \mu_t - \left( \frac{c^A_t}{D_t} \mu^A_{t} + \frac{c^B_t}{D_t} \mu^B_{t} \right)
\]
\[
= \frac{c^A_t}{D_t} \theta^A_{D,t} + \frac{c^B_t}{D_t} \theta^B_{D,t} + \mu_t - \left( \frac{c^A_t}{D_t} \mu^A_{t} + \frac{c^B_t}{D_t} \mu^B_{t} \right).
\]

It is equal by the weighted average of the market prices of risk of the two investor groups relative to their subjective measure, plus an additional term which depends on the difference between the true drift of the dividend and the weighted average of the estimated dividend drifts.

**Proposition 5** In an equilibrium with heterogeneous investors, the signal shocks are not priced for either investor type:

\[
\theta^A_{s,t} = \theta^B_{s,t} = 0
\]
\[
\theta^A_{n,t} = \theta^B_{n,t} = 0
\]

**Proof:** See Appendix B.

This result is very similar to that obtained for the homogeneous economy in Proposition 1, but different from the result in Dumas, Kurshev, and Uppal (2005). They find that the market price of risk for the signal shock is different from zero for both groups, and that only the weighted average of the subjective risk premia is zero. The reason is that in their model, both investors observe the same signal, but have different beliefs about the properties of the signal shock. The investors then trade with each other conditional on these differences in beliefs. As a consequence, their individual consumption depends on the signal shocks even if aggregate consumption does not, and signal risk is priced in equilibrium.

In our setup, the investors observe their own signal and the estimate of the other group. As argued above, this implies that both groups of investors can observe both signals. In contrast to Dumas, Kurshev, and Uppal (2005), however, the investors do not disagree on the signal shocks. Remember that each investor group learns from only one of the signals, and just adopts the opinion of the other group for the second signal. Thus, there is no way for the two groups to disagree on the shocks in the signal, which implies that they are not going to trade with each other conditional on such a disagreement. Furthermore, since the aggregate dividend does not depend on the signal shocks, and since there is no difference in beliefs concerning the signal shocks, individual consumption does not depend on the signal shocks either. As a consequence, signal shocks are not priced. In addition, there is no trading in signal shocks for the purpose of achieving optimal consumption patterns. In summary, there is zero trading volume in signal shock risk, and thus our initial assumption of market completeness does not seem overly restrictive, since for the implementation of the equilibrium, only the stock and the money market account are needed.
This indicates that an analysis like ours could also be performed on an incomplete market with just these two assets, since the completeness restriction is not binding in the framework of our model. In this context it is important to note that we are analyzing the case of a heterogeneous economy, so that even a zero aggregate supply condition for derivatives written on signal risk would not automatically imply zero trading volume.

Again, the market prices of signal shocks in Proposition 5 are calculated from the subjective perspective of the investors. From the perspective of the true model, we have

\[
\theta_{s, t} = -\frac{\mu_t^A - \mu_t}{\sigma_s}, \\
\theta_{n, t} = -\frac{\mu_t^B - \mu_t}{\sigma_n}.
\]

Now we have identified the interest rate and market prices of all risks in equilibrium, and thus each investor’s SDF. Since we have assumed a complete financial market, we can now determine any security price in the economy. In particular, for a general claim \( P \) with dynamics

\[
\frac{dP_t}{P_t} = \mu_P(P_t, t)dt + \sigma_{P,D}(P_t, t)dW_{D,t} + \sigma_{P,s}(P_t, t)dW_{s,t} + \sigma_{P,n}(P_t, t)dW_{n,t}.
\]

the following drift restriction must hold:

\[
\mu_P(P_t, t) = r_t + \sigma_{P,D}(P_t, t)\alpha\sigma_D + \sigma_{P,D}(P_t, t)\mu_t - \left( \frac{c^A_t \mu_t^A}{\sigma_s} + \frac{c^B_t \mu_t^B}{\sigma_n} \right), \\
+ \sigma_{P,s}(P_t, t) \left( \frac{\mu_t - \mu_t^A}{\sigma_s} \right) + \sigma_{P,n}(P_t, t) \left( \frac{\mu_t - \mu_t^B}{\sigma_n} \right).
\]

Here we can see that the expected excess return in an economy with unobserved state variables and heterogeneous beliefs is different from the expected excess return in a homogeneous economy with complete information. First, \( \mu_P(P_t, t) \) contains a component representing the additional compensation for dividend risk depending on the difference between the (average) estimated drift and the true drift. The more pessimistic the investors are, the higher this extra risk premium will be. Second, the drift contains risk premia for the signal shocks. These premia depend on the difference between the true drift and the drift estimated by investors who observe the respective signal directly. When investors are pessimistic, i.e. when they underestimate the drift, the premia will be positive.

4 Consumption Sharing

In this section we investigate the distribution of wealth and consumption between the two groups of investors in equilibrium. For different choices of the key parameters
(the amount of additional noise in the signal received by the laymen and the over-confidence level of the experts) we analyze which group of investors has an advantage over their counterparts and therefore survives in the long run. The issue about survival of irrational traders is addressed in several papers like Dumas, Kurshev, and Uppal (2005), Kogan, Ross, Wang, and Westerfield (2006), and Yan (2005). In these papers the authors always give one group of investors an obvious disadvantage, e.g. irrational learning behavior or incorrect beliefs. The question is then whether these investors can survive in the long run, and if they cannot, how long it will take until they have lost most of their wealth. In our model, however, all investors exhibit certain deficiencies, and we focus on the evolution of their consumption shares. As shown in the previous section, consumption shares are not only natural indicators for the long run survival but also represent the weights of investor groups in determining almost all important quantities in the heterogeneous economy.

4.1 Dynamics of consumption shares

Proposition 2 gives the optimal consumption policy of both investor groups in equilibrium. Let $\delta^A_t$ denote the fraction of the total dividend consumed by type A at time $t$ given by

$$
\delta^A_t \equiv \frac{c^A_t}{D_t} = \frac{y_A^{-\frac{1}{\alpha}}}{y_A^{-\frac{1}{\alpha}} + \left(\frac{y_B}{\eta_t}\right)^{-\frac{1}{\alpha}}}
$$

$y_A$ and $y_B$ are the constant Lagrange multipliers in the maximization problems of type A and B. Therefore, the process of $\delta^A_t$ solely depends on the dynamics of $\eta_t$, the Radon-Nikodym derivative between A’s and B’s subjective measures. The stochastic differential equation for $\eta_t$ under A’s subjective measure has already been derived in Equation (3) as $d\eta_t = \eta_t \left(\frac{\mu^B_t - \mu^A_t}{\sigma^2_D}\right) dW^A_{D,t}$. An application of Ito’s lemma gives the dynamics of A’s consumption share:

$$
d\delta^A_t = \frac{\delta^A_t \delta^B_t \left(\mu^A_t - \mu^B_t\right)^2}{\sigma^2_D} \left(1 + \frac{1}{\alpha} - \delta^A_t\right) dt + \frac{\delta^A_t \delta^B_t \left(\mu^B_t - \mu^A_t\right)}{\sigma^2_D} dW^A_{D,t}. \quad (11)
$$

Note that this SDE is given under the subjective measure of type A investors. The consumption shares $\delta^A_t$ and $\delta^B_t$ always lie between 0 and 1. Thus, an overconfident expert with $\alpha > 1$ thinks that his or her consumption share always has a positive drift. Even if his relative risk aversion is below 1, the dynamics of his consumption share show mean reversion with a ’long run mean’ equal to $\frac{1}{2} + \frac{1}{2 \alpha}$, which is greater than or equal to $\frac{1}{2}$.

We also can derive the dynamic of Group B’s consumption share $\delta^B_t$ under B’s subjective measure:

$$
d\delta^B_t = \frac{\delta^A_t \delta^B_t \left(\mu^A_t - \mu^B_t\right)^2}{\sigma^2_D} \left(1 + \frac{1}{\alpha} - \delta^B_t\right) dt + \frac{\delta^A_t \delta^B_t \left(\mu^B_t - \mu^A_t\right)}{\sigma^2_D} dW^B_{D,t}.
$$
It has the same form as Equation (11), and investor B also thinks, that he or she will be the only one surviving in the long run or at least end up with a larger average consumption share than investors of group A.

These results should not be very surprising. In our economy, both types of investors determine their optimal consumption and investment policy based on the information available to them. Given their posterior beliefs, the consumption shares in equilibrium are part of their utility maximizing strategies, which are, in their opinion, optimal. They have the same preferences, and all differences in their portfolios are due to differences in beliefs. The investors are aware of these differences and their implications. And since the investors agree to disagree, it is quite intuitive that each investor thinks that he has an (informational) advantage over his competitors, so that he will be better off and (on average) consume more than his competitors.

However, in order to answer the question which group is really better off, we have to look at the dynamics of $\delta_t^A$ and $\delta_t^B$ under the true probability measure. Under the true measure the expression for $\delta_t^A$ is more involved than the one under A’s subjective measure in Equation (11):

$$d\delta_t^A = \frac{\delta_t^A \delta_t^B \mu_t^A - \mu_t^B}{\sigma_D^2} \left[ \left( \frac{1}{2} - \frac{1}{2} \alpha - \delta_t^A \right) (\mu_t^A - \mu_t) + \left( \frac{1}{2} - \frac{1}{2} \alpha - \delta_t^B \right) (\mu_t^B - \mu_t) \right] dt$$

$$+ \frac{\delta_t^A \delta_t^B \mu_t^A - \mu_t^B}{\alpha \sigma_D} dW_{D,t}.$$  

Now the drift term depends on additional state variables, e.g. the estimation errors made by Group A, $(\mu_t^A - \mu_t)$, and by Group B, $(\mu_t^B - \mu_t)$.

### 4.2 Expected consumption shares

As argued before, in order to analyze whether type A or type B investors survive in the long run, we need to compute the expected consumption shares for the two types under the true probability measure. To do so, we need to know the distribution of $\eta$, conditional on the state variables $\eta$, $\mu_A$, $\mu_B$ and $\mu$ at time $t$. Although the dynamics of all these variables are given explicitly, the conditional distribution of $\eta$ is not easy to obtain. However, since this problem has a structure very similar to the one solved in Dumas, Kurshev, and Uppal (2005), we can follow their approach to first obtain the characteristic function of $\eta$, and then to compute its distribution function via Fourier inversion.

Let $g_i^t$ denote the estimation bias made by group $i$, where $i \in \{A, B\}$, i.e. $g_i^t \equiv \mu_i^t - \mu_t$ ($i \in \{A, B\}$). The $g^i$ serve as state variables, and the characteristic function $F(\eta, g^A, g^B, t; \tau; \chi)$ of $\eta$, for $\chi \in \mathbb{C}$ is given as

$$F(\eta, g^A, g^B, t; \tau; \chi) \equiv \mathbb{E}_{t,\eta;g^A,g^B} [\eta^\chi].$$

$F$ is the solution of a partial differential equation with an initial condition described in Appendix C. As shown in Dumas, Kurshev, and Uppal (2005), the coefficients of
the differential equation have a special functional form, and therefore the solution for $F$ can be represented as

$$F(\eta, g^A, g^B, t; \tau, \chi) = \eta^\chi f(g^A, g^B, t; \tau; \chi),$$

where the function $f(g^A, g^B, t; \tau; \chi)$ is again defined in Appendix C.

Once we have obtained the characteristic function of $\eta\tau$, we can compute its density by Fourier inversion. Then, the expectation of $\delta^A_\tau$ under the true measure can be obtained by (numerical) integration:

$$E_{\eta, g^A, g^B}[\delta^A_\tau] = \int_0^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(g^A, g^B, t; \tau; i\chi)}{\eta \chi} \right] d\eta d\tau$$

4.3 Numerical examples

Although the characteristic function of $\eta$ and thus the expected consumption shares can be obtained in nearly closed form, the effect of over-confidence and limited information processing capability on investors’ long run survival cannot be determined analytically. We look at some examples to illustrate the effect of these two key variables in our model. We would like to choose reasonable parameter values and keep comparability with the results in Dumas, Kurshev, and Uppal (2005). The choice of parameter values in the dividend process is based on the estimation of models similar to ours undertaken in Brennan and Xia (2001). Values of other parameters and state variables are similar to those taken in Dumas, Kurshev, and Uppal (2005). Table 1 lists the values we specify for the parameters and the state variables in order to compute the expected consumption shares.

First we study the case where type A investors are over-confident, but have no better signal than type B investors. This means we set $\sigma_s = \sigma_n = 0.1$ and give only Group A a handicap. Figure 1 illustrates the evolution of the expected consumption share of over-confident experts starting out with a share of 50% of total wealth. We plot this variable for different levels of over-confidence represented by parameter values $\phi = 0, 0.5, 0.9, 0.99$. $\phi = 0$ corresponds to the special case when the experts are also rational. In this case, the two groups are symmetric, no one has an advantage over the other. Therefore the average consumption share of each group at any point of time should be 50%, as we can see from the curve corresponding to $\phi = 0$.

For positive values of $\phi$, we have a situation very close to the one described in Dumas, Kurshev, and Uppal (2005). Figure 1 shows that, not surprisingly, the expected consumption share of the over-confident experts decreases over time. It is also plausible that, the higher their level of over-confidence, the faster their consumption share will fall. Although the irrational investors lose wealth permanently, this process is excessively slow. With an initial consumption share of 50%, even extremely irrational investors ($\phi = 0.99$) still consume on average more than 42% of
the total dividend after 200 years. This is in line with the results in Dumas, Kurshev, and Uppal (2005) and Yan (2005) who find that ultimately irrational agents become extinct, but it takes on average several hundreds of years for them to lose half of their initial wealth.

Figure 2 illustrates another special case, where type A investors are fully rational, i.e. \( \phi = 0 \), and type B investors, the laymen, have a noisier signal. We plot the expected consumption share for different amounts of noise \( \sigma_n \) contained in \( s^B \) (\( \sigma_n = 0.1, 0.2, 0.5, 5 \)). In the case \( \sigma_n = 0.1 \), the signal received by the laymen \( s^B \) has the same quality as \( s^A \). For \( \sigma_n > 0.1 \), Group A has an advantage, and therefore its expected consumption share increases permanently, although, like in the opposite case, at a very slow rate. It is also not surprising that for an increasing amount of noise in \( s^B \), the consumption share for type A grows faster. An interesting observation concerns the effect of an increase in \( \sigma_n \) relative to the current value of this parameter. Increasing \( \sigma_n \) from 0.1 to 0.2 generates a much stronger effect on the wealth distribution than an increase from 0.5 to 5. We also computed the expectation of \( \delta^A \) for even larger values of \( \sigma_n \) like 20 or 50, but the pictures hardly changed compared to \( \sigma_n = 5 \). This may be interpreted as some sort of convergence.

However, we are mostly interested in the issue of a trade-off between over-confidence and lower information quality, and which disadvantage is more severe for investors. In our setting over-confidence is modeled as a non-zero correlation coefficient assumed by the investors in Group A between two processes whereas these processes are actually independent. Since a correlation coefficient can never exceed one in absolute value, the mistake made by the over-confident experts seems to be bounded, whereas the informational disadvantage of the rational laymen, represented by the noise size in their signal, can basically become infinitely large. Intuitively, one might conclude that for every level of over-confidence there would be a critical noise size, such that if the noise contained in the rational laymen’s signal exceeded this level, the over-confident experts would still be better off.

To check this conjecture, we computed combinations of \( \phi \) and \( \sigma_n \), for which the expected consumption shares of both groups are still equal to 50% after 10, 50, and 200 years, respectively. The results are shown in Figure 3. We can see that for low levels of over-confidence, the curves are flat or only moderately upward-sloping. However, as \( \phi \) increases, the curves become very steep, even almost vertical. If the experts are only moderately over-confident, we can find laymen with a sufficiently large informational handicap to offset the negative effect of over-confidence, and the over-confident experts will eventually survive. For higher levels of over-confidence the disadvantage becomes more and more difficult to compensate. As we had seen in Figure 2, an increase in noise size does not have a large impact on the wealth distribution in the economy if the noise size is already large.

The indifference curves in Figure 3 seem to have vertical asymptotes. This would mean that, beyond a certain level of over-confidence, type A investors will not survive in the long run, no matter how noisy the signal for type B investors. To check this conjecture we let the parameter \( \sigma_n \) go to infinity. In this limiting case, the
signal $s^B$ does not contain any information at all about the dividend growth rate. Type B investors will just ignore this signal and make their inference based only on the observed dividend process. Figure 4 shows the evolution of type A’s expected consumption share in this limiting case for different over-confidence levels given by $\phi = 0.1, 0.45, 0.6$. It becomes obvious that if the experts are only moderately over-confident ($\phi = 0.1, 0.45$), they are still better off. However, if they are too over-confident ($\phi = 0.6$), they will lose wealth permanently and cannot survive in the long run.

5 Conclusion

In this paper we have analyzed an economy with heterogeneous investors. The elements of heterogeneity are differences in the fundamental information processing capacity on the one hand and overconfidence on the other. One group of investors is able to generate signals with higher precision than the other, but at the same time these experts are overconfident, i.e. they overestimate the informational value of their signals. The rational investors, who do not suffer from overconfidence, can only generate signals with a lower precision. The key issue we have studied in this paper is, what impact these two types of suboptimal learning behavior have on long-run equilibrium consumption shares.

An important feature of the equilibrium in our economy is that signal risk does not command a premium under the respective subjective probability measures for the two groups. Together with the fact that signal shocks do not influence aggregate dividends, and that furthermore the two investor groups have identical beliefs with respect to signal shocks, this zero premium also implies that there is no need to trade primitive assets written on signal shocks. This finding is important, since it shows that our initial assumption of market completeness is not critical.

The most important results in our paper, however, are those concerning the trade-off between the two types of shortcomings in information processing. Not surprisingly, in the case when only one of the two groups exhibits such a deficiency, the other group will dominate in the long run. However, it is nevertheless noteworthy that also in our model it takes quite a long time before a significant reduction in the consumption share for the group exhibiting the disadvantage becomes observable. When both over-confidence and lower information processing ability are present in the economy, our numerical analyses show that, e.g. for a planning horizon of ten years, a level of over-confidence $\phi$ for type A investors beyond 0.3 can hardly be compensated by higher noise in the signal for Group B. The indifference curve showing the combinations of $\phi$ and $\sigma_n$ which result in an expected consumption share of 50% for both groups becomes so steep that a loss in consumption share for Group A is practically inevitable. For longer horizons this critical value for $\phi$ is larger, but even for a horizon of 200 years, it is below 0.5. We can conclude from this investigation that in a certain sense over-confidence is a more severe problem than a limited information processing capability.
References


A Equilibrium in the Homogeneous Economy

We only show the derivation for the economy with type A investors, since the result for the type B economy can be derived analogously. As shown in Cox and Huang (1989) the optimal consumption policy of type A investors is given by

\[ c_t^A = (y_A \xi_t^A e^{r_t})^{-\frac{1}{\alpha}}, \]

where \( y_A \) denotes the Lagrange multiplier for the budget constraint in the optimization problem (4)–(5). In equilibrium market clearing implies \( c_t^A \equiv D_t \). Given the dynamics of \( \xi_t^A \),

\[ d\xi_t^A = \xi_t^A (r_t dt - \theta_{A,t}^A dW_{A,t}^A - \theta_{s,t}^A dW_{s,t}^A), \]

and applying Ito’s lemma to obtain the dynamics of \( c_t^A \), we can compare the coefficients of \( dc_t \) to the coefficients of \( dD_t \) to obtain the result.

B Equilibrium in the Heterogeneous Economy

The market clearing condition in the heterogeneous economy and the optimal consumption rules of both groups imply

\[ c_t^A + c_t^B = (y_A \xi_t^A e^{r_t})^{-\frac{1}{\alpha}} + (y_B \xi_t^B e^{r_t})^{-\frac{1}{\alpha}} = D_t \]

Plug \( \xi_t^B = \xi_t^A \eta_t \) into the above equation and solve for \( \xi_t^A \) to obtain

\[ \xi_t^A = D_t^{-\alpha} \left( y_A^{-\frac{1}{\alpha}} + \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}} \right)^{-\alpha} e^{-r_t}. \]

Then plug this into the expression for the optimal consumption policy \( c_t^A = (y_A \xi_t^A e^{r_t})^{-\frac{1}{\alpha}} \) to obtain the consumption sharing rule stated in Proposition 2.

The dynamics of the stochastic discount factors for the two groups are given in Equations (6) and (6). Together with the dividend process from Equation (1) and the dynamics of the Radon-Nikodym derivative from Equation (3), an application of Ito’s lemma yields under type A’s subjective measure:

\[ \frac{d \left( y_A^{-\frac{1}{\alpha}} + \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}} \right)^{-\alpha}}{y_A^{-\frac{1}{\alpha}} + \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}}} = \frac{1 - \alpha}{\alpha} \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}} \frac{\left( \mu_t^A - \mu_t^B \right)^2}{\sigma_D^2} dt - \frac{1 - \alpha}{2} \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}} \frac{\sigma_D^2}{\mu_t^A - \mu_t^B} dW_{A,t}^B - \frac{\left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}}}{y_A^{-\frac{1}{\alpha}} + \left( \frac{y_B}{\eta_t} \right)^{-\frac{1}{\alpha}}} \frac{\mu_t^A - \mu_t^B}{\sigma_D} dW_{s,t}^A \]

Using this result we can again apply Ito’s lemma on the right hand side of equation (12) and compare coefficients with the dynamic of \( \xi_t^A \) and do the same for \( \xi_t^B \). This yields the results stated in Propositions 3, 4, and 5.
C Characteristic Function of $\eta_t$

The dynamics of $\eta_t$ and the state variables $g^A_t$ and $g^B_t$ under the true measure are given by

$$\frac{dg^A_t}{\eta_t} = \frac{g^A_t - g^B_t}{\sigma^2_D} \frac{g^A_t}{\sigma_D} dt - \frac{g^A_t - g^B_t}{\sigma_D} dW_{D,t},$$

$$dg^A_t = -g^A_t (\lambda + \frac{\gamma^A}{\sigma^2_D} + \frac{\phi \sigma_s \mu + \gamma^A}{\sigma^2_s}) dt + \frac{\gamma^A}{\sigma_D} dW_{D,t} + \frac{\phi \sigma_s \mu + \gamma^A}{\sigma_s} dW_{s,t} - \mu_d dW_{\mu,t},$$

$$dg^B_t = -g^B_t (\lambda + \frac{\gamma^B}{\sigma^2_D} + \frac{\gamma^B}{\sigma^2_n}) dt + \frac{\gamma^B}{\sigma_D} dW_{D,t} + \frac{\gamma^B}{\sigma_n} dW_{n,t} - \mu_d dW_{\mu,t}.$$

The characteristic function of $\eta F(\eta, g^A, g^B, t; \tau; \chi) := \mathbb{E}^F_{t, \eta, g^A, g^B}[\eta^\chi]$ satisfies the partial differential equation

$$0 = \frac{\partial F}{\partial \eta} g^A - g^B - \frac{\partial F}{\partial g^A} g^A \left( \lambda + \frac{\gamma^A}{\sigma^2_D} + \frac{\phi \sigma_s \mu + \gamma^A}{\sigma^2_s} \right)$$

$$- \frac{\partial F}{\partial g^B} g^B \left( \lambda + \frac{\gamma^B}{\sigma^2_D} + \frac{\gamma^B}{\sigma^2_n} \right) + \frac{\partial F}{\partial t} + 1 \frac{\partial^2 F}{\partial \eta^2} \left( \frac{\gamma^A - g^B}{\sigma_D^2} \right)$$

$$+ \frac{\partial^2 F}{\partial \eta \partial g^A} g^A - g^B + \frac{\partial^2 F}{\partial \eta \partial g^B} \left( \frac{\gamma^A - g^B}{\sigma_D^2} \right).$$

Given the boundary condition $F(\eta, g^A, g^B, t; \tau; \chi) \equiv \eta^\chi$ the solution to this equation has the form

$$F(\eta, g^A, g^B, t; \tau; \chi) = \eta^\chi f(g^A, g^B, t; \tau; \chi),$$

where $f(g^A, g^B, t; \tau; \chi)$ satisfies the partial differential equation

$$0 = \chi \frac{\partial f}{\partial g^A} g^A - g^B - \frac{\partial f}{\partial g^A} g^A \left( \lambda + \frac{\gamma^A}{\sigma^2_D} + \frac{\phi \sigma_s \mu + \gamma^A}{\sigma^2_s} \right)$$

$$- \frac{\partial f}{\partial g^B} g^B \left( \lambda + \frac{\gamma^B}{\sigma^2_D} + \frac{\gamma^B}{\sigma^2_n} \right) + \frac{\partial f}{\partial t} + 1 \frac{\partial^2 f}{\partial \eta^2} \left( \frac{\gamma^A - g^B}{\sigma_D^2} \right)$$

$$+ \frac{\partial^2 f}{\partial \eta \partial g^A} g^A - g^B + \frac{\partial^2 f}{\partial \eta \partial g^B} \left( \frac{\gamma^A - g^B}{\sigma_D^2} \right).$$

with boundary condition $f(g^A, g^B, t; \tau; \chi) = 1.$

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The solution to this equation can be written as
\[
f(g^A, g^B, t; \tau; \chi) = \exp \left\{ A(\tau - t) + B(\tau - t)(g^A)^2 + C(\tau - t)(g^B)^2 + 2D(\tau - t)g^Ag^B \right\},
\]
where
\[
A(\tau - t) = \int_t^\tau \left[ C(v - t) \left( \frac{(\gamma^B)^2}{\sigma_D^2} + \frac{(\gamma^B)^2}{\sigma_n^2} + \sigma_\mu^2 \right) + B(v - t) \left( \frac{(\gamma^A)^2}{\sigma_D^2} + \frac{(\phi\sigma_\epsilon\sigma_\mu + \gamma^B)^2}{\sigma_s^2} + \sigma_\mu^2 \right) + 2D(v - t) \left( \frac{\gamma^A\gamma^B}{\sigma_D^2} + \sigma_\mu^2 \right) \right] dv,
\]
and \(B, C, D\) are defined via
\[
\left( \begin{array}{cc} C & D \\ D & B \end{array} \right) = Y(\tau - t)[X(\tau - t)]^{-1}.
\]

\(X\) and \(Y\) are the solutions to the system of differential equations
\[
\dot{X} = Q^{11} X + Q^{12} Y \\
\dot{Y} = Q^{21} X + Q^{22} Y
\]
with initial conditions \(X(0) = I\) and \(Y(0) = 0\). The matrices \(Q^{11}, Q^{12}, Q^{22}\) are defined as follows:
\[
Q^{21} = \left( \begin{array}{cc} \frac{1}{2} \chi (\chi - 1) & \frac{1}{2} \chi^2 \frac{1}{\sigma_D^2} \\ -\frac{1}{2} \chi^2 \frac{1}{\sigma_D^2} & \frac{1}{2} \chi (\chi + 1) \frac{1}{\sigma_D^2} \end{array} \right),
\]
\[
Q^{11} = -(Q^{22})^T = \left( \begin{array}{cc} \chi + \frac{\gamma_A}{\sigma_D^2} + \frac{\gamma_B}{\sigma_D^2} - \chi \frac{\gamma_A}{\sigma_D^2} & \chi \frac{\gamma_A}{\sigma_D^2} \\ -\chi \frac{\gamma_A}{\sigma_D^2} & \chi + \frac{\gamma_A}{\sigma_D^2} + \frac{\gamma_B}{\sigma_D^2} \end{array} \right),
\]
\[
Q^{12} = \left( \begin{array}{cc} -2\left( \frac{(\gamma_A)^2}{\sigma_D^2} + \frac{(\gamma_B)^2}{\sigma_n^2} + \sigma_\mu^2 \right) & -2\left( \frac{\gamma_A\gamma_B}{\sigma_D^2} + \sigma_\mu^2 \right) \\ -2\left( \frac{\gamma_A\gamma_B}{\sigma_D^2} + \sigma_\mu^2 \right) & -2\left( \frac{(\gamma_A)^2}{\sigma_D^2} + \frac{(\gamma_B)^2}{\sigma_n^2} + \sigma_\mu^2 \right) \end{array} \right).
\]
### Parameters for dividend and signal processes

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility of dividend growth rate</td>
<td>$\sigma_D$ 0.13</td>
</tr>
<tr>
<td>Volatility in the mean reversion process of dividend growth rate</td>
<td>$\sigma_\mu$ 0.015</td>
</tr>
<tr>
<td>Long run mean of dividend growth rate</td>
<td>$\bar{\mu}$ 0.015</td>
</tr>
<tr>
<td>Mean reversion parameter</td>
<td>$\lambda$ 0.2</td>
</tr>
<tr>
<td>Volatility of signal observed by group A (Over-confident experts)</td>
<td>$\sigma_s$ 0.1</td>
</tr>
<tr>
<td>Volatility of signal observed by group B (Rational laymen)</td>
<td>$\sigma_n$ [0.1, $+\infty$)</td>
</tr>
</tbody>
</table>

### Parameters for investors

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative risk aversion of both groups</td>
<td>$\alpha$ 3</td>
</tr>
<tr>
<td>Correlation assumed by group A (Over-confident experts)</td>
<td>$\phi$ [0, 1)</td>
</tr>
<tr>
<td>Ratio between both groups’ initial wealth</td>
<td>$y^B/y^A$ 1</td>
</tr>
</tbody>
</table>

### State variables

<table>
<thead>
<tr>
<th>State variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change between subjective measures at time 0</td>
<td>$\eta_0$ 1</td>
</tr>
<tr>
<td>Estimation error of group A at time 0 $\mu^A_0 - \mu_0$</td>
<td>$\eta^A_0$ 0.005</td>
</tr>
<tr>
<td>Estimation error of group B at time 0 $\mu^B_0 - \mu_0$</td>
<td>$\eta^B_0$ -0.005</td>
</tr>
<tr>
<td>The population’s average belief about the expected growth rate</td>
<td>$\bar{\mu}$</td>
</tr>
</tbody>
</table>

Table 1: Values for Parameters and State Variables

The table shows the values for parameters and state variables used in the numerical analysis of our model. The parameter values for the dividend process are based on the estimation results in Brennan and Xia (2001). The values for the other parameters and state variables are similar to those in Dumas, Kurshev, and Uppal (2005) to facilitate a comparison of the results.
Figure 1: Expected Consumption Share for Group A for the Case of Over-Confidence and Equal Information Quality

The figure shows the expected consumption share of Group A under the true probability measure as a function of time (measured in years), when investors in Group A are over-confident, and investors in Group B have no information disadvantage ($\sigma_n = \sigma_s$). This represents the case analyzed in Dumas, Kurshev, and Uppal (2005). The values for the parameters and the state variables are listed in Table 1.
Figure 2: Expected Consumption Share for Group A for the Case of Full Rationality in Group A and Information Disadvantage for Group B

The figure shows the expected consumption share of Group A under true probability measure as a function of time (measured in years), when investors in Group A are rational ($\phi = 0$) and investors in Group B receive a noisier signal ($\sigma_n > \sigma_s$). The values for the parameters and state variables are listed in Table 1.
Figure 3: Indifference Curves for $\phi$ and $\sigma_n$

The figure shows the combinations of the level of over-confidence $\phi$ and the signal noise $\sigma_n$, which yield equal expected consumption shares for the two groups after 10, 50, and 200 years, respectively. The initial endowment of each group is set to 0.5. The values for the other parameters and state variables are listed in Table 1.
Figure 4: Expected consumption share of Group A when investors in Group B ignore their noisy signal

The figure shows the expected value of the consumption share for Group A under the true probability measure as a function of time measured in years, when type B investors ignore their signal $s^B$. 

φ=0.1
φ=0.45
φ=0.6