On aftermarkets, network effects and dynamic competition for locked in consumers

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Version: April 28, 2009

In this paper, we propose a theoretical model of dynamic competition in primary markets and aftermarkets, explicitly accounting for the network effects resulting from the interplay between the value of complementary goods and services (sold in aftermarkets) and the size of the network in the primary market. Relying on the equilibrium notion of Linear Markov Perfect Equilibrium, we investigate market outcomes at equilibrium. We conclude that equilibrium prices in the primary market tend to be lower when durable goods traded in primary markets are weakly differentiated, competition in aftermarkets is relatively soft, the intrinsic utility of complementary goods and services is high and network effects are considerably intense.

Key Words: Aftermarkets, network effects, differential games, linear Markov perfect equilibrium

1. INTRODUCTION

In several durable-goods markets (primary markets) the value of the durable-good / equipment is essentially derived from using it complementarily with goods and services sold in other markets (aftermarkets): cars and maintenance/repairing services, printers and ink cartridges, coffee machines and coffee capsules, hardware and software,... Moreover, in several cases, the interplay between primary markets and aftermarkets entails network effects: often, the value of the complementary goods and services (CGS) is positively related to the number of consumers who are endowed with a similar equipment in the primary market. For example, the larger the number of consumers buying a specific printer, the easier it is to find a seller of the corresponding ink cartridges. Similarly, the availability of adequate coffee capsules increases with the number

[1]The authors thank Paul Belleflamme, António Brandão, Sofia Castro, Jean Gabszewicz, Filomena Garcia, Ngo Van Long, Paula Sarmento for their helpful comments. Joana Resende acknowledges financial support from CETE, Universidade do Porto and FCT.

of consumers owning a compatible coffee machine. Also, the availability and the quality of PC (*vsus* Mac) assistance/repairing services is positively related with the total number of PC users (*vsus* Mac users)[2].

A number of recent antitrust cases, like *Eastman Kodak Co. v. Image Technical Servs* (copiers and maintenance/reparation services); *Pelikan v. Hewlett-Packard* (printers and ink cartridges), *Canada (Director of Investigation and Research) v. Chrysler Canada, Ltd.* (automobile industry), *Red Lion Medical Safety, Inc. v. Ohmeda, Inc* (medical equipment).... have dealt with antitrust issues arising when equipment producers are also involved in the provision of CGS. Several aftermarket theories have investigated the interplay between primary markets and aftermarkets. However, hitherto, there are no consensual results regarding the rationale for antitrust intervention in aftermarkets. Some of these theories tend to be favorable to antitrust intervention (e.g. Borenstein *et al*, (2000)), claiming that welfare losses entailed by the lack of competition in aftermarkets are not ruled out by competition in primary markets. In contrast, some theories[3] are less prone to support antitrust intervention in aftermarkets, arguing that consumers' injury caused by the lack of competition in aftermarkets tends to be rather small, especially when primary markets are competitive (e.g. Shapiro (1995)). Other theories go even further, pointing that the lack of competition in aftermarkets may be welfare-improving in certain situations (even from consumers' point of view). For example, that can be the case when there are increasing returns in the aftermarkets (Cabral (2008)), when there is uncertainty about equipment' quality (Schwartz and Werden (1996)), or when there are problems due to time inconsistency related with durable-goods (Morita and Waldman (2004), Carlton and Waldman (2001)).

In this paper, we develop a theoretical model of dynamic competition in primary markets and aftermarkets. In line with the previous literature, our model takes into account the following features of aftermarkets: *(i)* goods/ services traded in aftermarkets aim to complement a durable good, whose value considerably depends on CGS consumption; *(ii)* equipment choices precede consumers' purchases of CGS; and *(iii)* consumers are significantly "locked in" to their equipment and therefore, the switch from one equipment' brand to another is highly unlikely. In addition, we formally introduce in our model the network effects resulting from the interplay between the value of CGS and the size of the installed base of equipment owners.

The model used in this paper corresponds to a differential game with a linear-quadratic structure. To investigate agents' optimal decisions and equilibrium market outcomes in the context of this dynamic game, we rely on the equilibrium concept of Linear Markov Perfect Equilibrium (LMPE). This equilibrium concept rests on two assumptions. On the one hand, it is assumed that all the payoff relevant information can be summarized by the state vector (Markov perfection). On the other hand, the concept of LMPE is restricted to equilibrium strategies that

[2] As a matter of fact, quite often, there are learning effects in the provision of CGS. In those circumstances, the expertise of CGS suppliers positively depends on the number of consumers endowed with the corresponding equipment.

[3] For example, Chen and Ross (1993), Shapiro (1995), Schwartz and Werden (1996), Carlton and Waldman (2001), Morita and Waldman (2004), Cabral (2008),...

can constitute affine functions of the state vector. The first assumption is quite common in the literature dealing with problems of dynamic price competition in network industries (see Cabral (2008), Mitchell and Skrzypacz (2006), Laussel *et al* (2004), Doganoglu (2003),...). Concerning the second assumption, our game being linear-quadratic, it is natural to search for the simplest equilibrium strategies that solve linear-quadratic games, namely, the linear strategies.

We provide a necessary and sufficient condition for the existence of a unique LMPE in which both firms have non-negative market shares. When such a unique equilibrium exists, instantaneous equilibrium prices in the primary market depend on *(i)* the degree of differentiation between the available equipment variants; *(ii)* the profitability of the corresponding aftermarkets; and *(iii)* the relative market shares of each equipment variant in the primary market.

As concerns *(i)*, it is not surprising that equipment prices tend to be lower when equipment variants are more similar. Regarding *(ii),* we show that equipment prices are negatively correlated with the profitability of aftermarkets, which depends itself on: the degree of competition in aftermarkets, the intrinsic value of CGS and the intensity of network effects. More precisely, equipment prices are higher in the case of very competitive aftermarkets or when consumers do not value that much CGS (either because CGS have a limited intrinsic utility or because they do not yield a significant network benefit).

Finally, in relation to *(iii),* we show that, when we account for the network effects resulting from the interplay between the value of CGS and the size of the installed base of equipment owners, at each instant of time, the equipment producer with a larger installed base of consumers sells its equipment at a higher price. The intuition for this result is the following. When accounting for network effects, it follows that, *ceteris paribus*, CGS made available to the "dominant" equipment[4] are more valuable than CGS made available to the other equipment (with a lower market share) since the network benefits yielded by the former are larger than the network benefits yielded by the latter. When we concentrate on the unique LMPE in which both firms have non-negative market shares, the producer of the dominant equipment exploits its "qualitative advantage" by setting a higher price than its rival ("fat cat"[5] effect).

Along the LMPE trajectories, the dominant equipment becomes less and less likely to capture new consumers[6] and the asymmetry on the market shares of equipment producers tends to vanish with time (decreasing dominance). Given that we focus on LMPE in which both firms have non-negative market shares, it is not surprising that, in the steady state, firms share the market evenly. However, it is worth noting that there might be a considerable inertia in the evolution of market shares towards this symmetric steady state, especially when products are weakly differentiated, network effects are considerably intense or when the flows of consumers entering and exiting this industry are relatively limited.

From a welfare perspective, we show that the exercise of market power in aftermarkets is always beneficial to equipment producers and detrimental to consumers who were already en-

[4] The dominant equipment corresponds to the equipment that is owned by a larger fraction of consumers.
[5] This concept has been introduced by Fudenberg and Tirole (1983).
[6] Furthermore, this effect is reinforced by the fact that the dominant equipment also registers higher exit flows.

dowed with an equipment. Regarding independent CGS suppliers and consumers who were not initially endowed with an equipment, the effect of restricting competition in aftermarkets is non-monotonic. Adding up all effects, we conclude that the lack of competition in aftermarkets has a detrimental effect on "Marshallian social welfare". In our model, the social damage is caused by the underprovision of CGS due to the lack of competition in aftermarkets[7].

The rest of the paper is organized as follows. Section 2 presents the basic ingredients of the model. Section 3 and Section 4 respectively investigate agents' decisions in aftermarkets and in the primary market. Section 5 analyzes market outcomes from a welfare perspective and, finally, Section 6 concludes.

2. THE MODEL

Consider a duopolistic primary market, where two firms (Firm 1 and Firm 2) produce, at zero marginal cost, infinitely-lived and horizontally differentiated durable goods (Equipment 1 and Equipment 2). Horizontal differentiation is modelled *à la* Hotelling with quadratic transportation costs[8]. The Hotelling line $[0, 1]$ represents the spectrum of all possible equipment variants. Equipment 1 and Equipment 2 are located at the opposite extremes of this line (Equipment 1 is located at point $x_1 = 0$, while Equipment 2 is located at point $x_2 = 1$). Potential equipment buyers are uniformly distributed along the Hotelling line, in accordance with their preferences.

We consider that equipment owners face considerable switching costs[9] (learning costs, habit formation,...) and therefore, the switch from one equipment' brand to another never occurs (full lock in in the primary market). At each instant of time, consumers who are not endowed with an equipment (new consumers) make a lifetime choice between the two available equipment variants. To make such choice, consumers compare *(i)* the intrinsic characteristics of each equipment, *(ii)* their respective prices and *(iii)* the additional value that consumers may extract from their equipment by using it together with the appropriate CGS throughout their lives.

The expected lifetime utility obtained by a consumer located at $x \in [0, 1]$ in the Hotelling line who buys equipment i at instant t, is denoted by $V_i(x, t)$, $i = 1, 2$:

$$V_i(x,t) = \int_t^{\infty} \left[\vartheta - \tau (x - x_i)^2\right] e^{-(r+\mu)(v-t)} dv + \int_t^{\infty} E[u_i(v)] e^{-(r+\mu)(v-t)} dv - p_i(t), \qquad (1)$$

where $\vartheta > 0$ is the instantaneous *intrinsic utility*[10] delivered by the equipment variant corresponding to consumers' most preferred specification; τ is the unit transportation cost; $r+\mu$ is the "adjusted discount rate", with r standing for the conventional discount rate and $\mu < \frac{r}{2}$ standing for the instantaneous probability of exiting the industry; $E[u_i(v)]$ corresponds to the expected utility obtained from CGS consumption at instant $v > t$; and, finally, $p_i(t)$ denotes the price

[7] To be more precise, this welfare loss corresponds to the "deadweightloss" triangle arising in aftermarkets. This result is in line with Shapiro (1995) or Borenstein *et al* (2000).

[8] See d' Aspremont *et al* (1979).

[9] On switching costs see, for example, Klemperer (1987) or Farrell and Shapiro (1988).

[10] By intrinsic utiliy, we mean the equipment value that does not depend on CGS consumption.

of equipment i at instant t. The constant ϑ is considered to be large enough for all potential equipment buyers to find an equipment for which $V_i(x,t)$ is positive at equilibrium (full market coverage). We focus on forward-looking agents, assuming that agents are able to perfectly anticipate the future benefits generated by future consumption of CGS (i.e. $E[u_i(v)] \equiv u_i(v)$).

In the aftermarket, two distinct categories of perishable CGS[11] are available. Both categories of CGS are produced at zero marginal cost. We consider that each type of CGS has been specifically designed for a particular equipment variant, being fully incompatible with the other equipment. Hence, also in the aftermarket, consumers are fully locked in to their equipment: equipment owners only benefit from CGS consumption when combining their equipment with the appropriate category of CGS. Equipment 1 must be combined with CGS_1, while Equipment 2 must be combined with CGS_2. Accordingly, the aftermarket can be decomposed in two mutually exclusive market segments: aftermarket of CGS_1 (where the owners of equipment 1 buy CGS throughout their lives) and aftermarket of CGS_2 (where the owners of equipment 2 buy CGS throughout their lives).

In each segment of the aftermarket, the instantaneous utility obtained by a consumer endowed with equipment i who, at instant t, buys a level g_i of CGS_i is equal to $u_i(g_i(t), D_i(t), D_j(t)) \equiv u_i(t)$:

$$u_i(t) = \Upsilon + \alpha g_i(t) - \frac{1}{2} g_i(t)^2 + \gamma [D_i(t) + \phi D_j(t)] - v_i(t) g_i(t), \tag{2}$$

where $\Upsilon > 0$ is a constant that measures the fixed intrinsic utility of CGS_i to the owners of equipment i; $\alpha > 0$ is a constant that measures the marginal benefit of CGS consumption; $g_i(t)$ denotes the level of CGS_i consumption by the owners of equipment i; $\gamma > 0$ is a constant that measures the intensity of network effects; $\phi > 0$ is a constant that measures the degree of compatibility between rival networks; $D_i(t)$ and $D_j(t) = 1 - D_i(t)$ respectively denote the fraction of consumers endowed with equipment i and equipment j at instant t; and, finally, $v_i(t)$ denotes the unit price of CGS_i at instant t.

The specification of $u_i(t)$ in equation (2) is suitable for aftermarkets exhibiting network effects associated with the size of equipment' installed base of (locked in) customers in the primary market. The larger the installed base of consumers endowed with equipment i, the more valuable CGS_i become. We allow for partial compatibility between networks, considering that the number of consumers endowed with the rival equipment has a positive, albeit smaller, impact on the value of CGS[12]. The degree of compatibility between rival networks is measured by the parameter ϕ. Note that network effects are incorporated in $u_i(t)$ linearly, and consequently, they do not influence the marginal benefit of CGS consumption.

Concerning the production of CGS, we introduce the possibility of competition in each segment of the aftermarket. More precisely, we consider that, in each aftermarket segment, the

[11] Examples of perishable CGS include repairing services, maintenance services, and so on.

[12] For example, in the case of learning effects, partial compatibility refers to the fact that CGS suppliers in a given segment of the aftermarket may (partially) benefit from the learning economies generated in another segment of the aftermarket, due to knowledge diffusion/transmission.

equipment producer i competes *à la* Cournot with other $N-1$ similar (independent) suppliers of CGS[13], with $N \geq 1$. We rule out any commitment device in the aftermarket[14].

In this context, we investigate agents' optimal strategies in primary markets and aftermarkets, considering a dynamic game with the following structure. At each instant of time t, there is an inflow of new consumers (arriving in the industry at a rate of μ) and an outflow of old consumers (exiting the industry at the same rate μ). In the primary market, at each instant of time t, Firm 1 and Firm 2 respectively set $p_1(t)$ and $p_2(t)$, accounting for their impact on firms' current and future profits. Newborn consumers observe equipment prices and make a lifetime choice of equipment, taking into consideration equipment prices, their tastes and their rational expectations on the discounted utilities obtained from CGS consumption throughout their lifetime. Finally, in each segment of the aftermarket, at each instant of time t, each equipment producer competes *à la* Cournot with other independent $N-1$ suppliers, establishing the unit-price of CGS_i, $v_i(t)$. Conditional on the instantaneous price of CGS_i, the owners of equipment i make their choices regarding the consumption level of CGS_i $(g_i(t))$.

Note that, under the assumptions that *(i)* consumers are fully locked in to their equipment, and *(ii)* CGS suppliers cannot commit to future output levels, firms' and consumers' decisions in the aftermarket at instant t do not affect future outcomes. Thus, instantaneous Cournot equilibrium outcomes in each segment of the aftermarket[15] only depend on the outcomes in the primary market *via* $D_i(t)$. Accordingly, the vector of market shares $(D_i(t), 1-D_i(t))$ conveys all the payoff-relevant information (state vector), making it possible to define agents' expectations on future discounted aftermarket profits and future discounted utilities conditional only on $D_i(t)$.

To solve the dynamic game previously described, we start by investigating equilibrium outcomes in aftermarkets conditional on $D_i(t)$. Then, we study the decisions of newborn consumers in the primary market (under the assumption of Linear Markov Expectations[16]) and we obtain equipment demands in the primary market conditional on $D_i(t)$. Finally, we study the equilibrium path of equipment prices, solving the dynamic problem faced by equipment producers in the primary market, under the assumption of Linear Markov Price Strategies and Linear Markov Expectations[17].

3. AFTERMARKET

In the segment i of the aftermarket, consumers endowed with equipment i choose the amount of CGS_i they want to consume (by assumption, owners of equipment i do not have any incentives

[13] Note that these $N-1$ firms only produce CGS, they do not interact with equipment producers in the primary market.

[14] This assumption is in line with the aftermarkets literature associated with the "theory of lack of commitment" (see Borenstein *et al* (2000))

[15] Namely consumers' equilibrium instantaneous utilities and firms' equilibrium instantaneous profits in the aftermarket.

[16] This concept is formally introduced in Section 4.

[17] These concepts are formally introduced in Section 4.

to consume CGS_j as their equipment does not deliver any additional value when combined with these CGS). Formally, at each instant of time t, the problem of consumers endowed with equipment i can be formulated as follows:

$$\max_{g_i(t)} u_i\left(t\right), \tag{3}$$

where $u_i\left(t\right)$ is given by (2). The solution[18] to problem (3) corresponds to the consumption level $g_i\left(v_i\left(t\right)\right)$ for which the marginal benefit entailed by the consumption of CGS_i, $\alpha - g_i\left(t\right)$, is perfectly balanced by its marginal cost, $v_i\left(t\right)$. At each instant of time t, the individual demand for CGS_i is given by:

$$g_i\left(v_i\left(t\right)\right) = \alpha - v_i\left(t\right)$$

and the market demand for CGS_i is equal to[19] $Q_i\left(v_i\left(t\right)\right) = g_i\left(v_i\left(t\right)\right) D_i\left(t\right)$. The inverse market demand for CGS_i writes as follows:

$$v_i\left(t\right) = \frac{\alpha D_i\left(t\right) - Q_i\left(t\right)}{D_i\left(t\right)},$$

with $\frac{\partial v_i(D_i(t),Q_i(t))}{\partial Q_i(t)} < 0$.

In aftermarket i, the producer of equipment i competes *à la* Cournot with $N-1$ symmetric firms. Given the inverse demand for CGS_i, at instant t, the aftermarket profits obtained by each firm $k = 1, ..., N$ participating in aftermarket i (including equipment producer i, itself) are equal to:

$$\pi^a_{k,i}\left(q_k\left(t\right), D_i\left(t\right)\right) = \left(\frac{\alpha D_i\left(t\right) - q_k\left(t\right) - \sum_{j=1}^{N-1} q_j\left(t\right)}{D_i\left(t\right)}\right) q_i\left(t\right),$$

where $q_k\left(t\right)$ corresponds to the output of the $k-$supplier of CGS_i, and $\sum_{j=1}^{N-1} q_j\left(t\right)$ represents the aggregate production of CGS_i by the remaining $N-1$ CGS suppliers participating in aftermarket i, with $j \neq k$. Given the symmetry of CGS suppliers participating in each segment of the aftermarket, at equilibrium ,$q_k\left(D_i\left(t\right)\right) = q\left(D_i\left(t\right)\right)$ $\forall$ $k = 1, ..., N$; and

$$q\left(D_i\left(t\right)\right) = \frac{\alpha}{N+1} D_i\left(t\right).$$

The total production of CGS_i is equal to $Q_i\left(t\right) = \frac{\alpha N}{N+1} D_i\left(t\right)$ and the equilibrium price of CGS_i is given by:

$$v_i = \frac{\alpha}{N+1},$$

being constant over time[20].

[18] The solution to problem (3) is obtained for $u_i'\left(g_i\left(t\right)\right) = 0$ and $u_i''\left(g_i\left(t\right)\right) < 0$.

[19] To obtain the market demand for CGS_i note that *(i)* all equipment i owners have the same individual demand for CGS_i : $g_i\left(v_i\left(t\right)\right)$; and *(ii)* only equipment i owners buy CGS_i . Thus market demand is simply $g_i\left(t\right) D_i\left(t\right)$.

[20] This result is similar to Borenstein *et al* (2000).

The equilibrium profits of CGS suppliers participating in this segment of the aftermarket (including the producer of equipment i) are equal to

$$\pi^a(D_i(t)) = \frac{\alpha^2}{(N+1)^2} D_i(t). \tag{4}$$

On consumers' side, the individual equilibrium consumption of CGS_i by consumers endowed with equipment i is constant over time, being equal to:

$$g_i = N\frac{\alpha}{N+1},$$

and yielding an instantaneous utility of

$$u(D_i(t)) = \Upsilon + \frac{1}{2}\left(\frac{N\alpha}{N+1}\right)^2 + \gamma\left[D_i(t) + \phi(1 - D_i(t))\right]. \tag{5}$$

According to equations (4) and (5), in each segment of the aftermarket, equilibrium aftermarket profits $\pi^a(D_i(t))$ and equilibrium instantaneous utilities $u(D_i(t))$ only depend on market outcomes in the primary market through the vector of instantaneous market shares $(D_i(t), 1 - D_i(t))$. In the following section, we use the fact that the vector $(D_i(t), 1 - D_i(t))$ conveys all the payoff relevant information (both to firms and consumers) to investigate the equilibrium trajectories of equipment prices.

4. PRIMARY MARKET

At each instant of time, only newborn consumers (who arrive in this industry at a rate of μ) have to choose between the available equipments. Given his/her preferences, each newborn consumer makes a lifetime choice of equipment, comparing the expected lifetime utility delivered by each equipment: $V_1(x,t)$ *versus* $V_2(x,t)$. The expected lifetime utility $V_i(x,t)$ depends on the characteristics and the price of equipment i, $\int_t^\infty \left[\vartheta - \tau(x - x_i)^2\right] e^{-(r+\mu)(v-t)}dv - p_i(t)$, as well as on the expected lifetime benefit yielded by future consumption of the appropriate CGS: $\int_t^\infty E[u_i(v)]e^{-(r+\mu)(v-t)}dv$.

Since we focus on forward-looking consumers, newborn consumers perfectly anticipate equilibrium outcomes in the aftermarket, i.e., they anticipate the equilibrium instantaneous utilities $u(D_i(t), D_j(t))$ in expression (5). Accordingly, the expected lifetime utility delivered by each equipment, $V_i(x,t)$, can be re-written as follows:

$$V_i(x,t) = \frac{\vartheta - \tau(x - x_i)^2 + \Upsilon + \frac{1}{2}\left(\frac{N\alpha}{N+1}\right)^2}{r+\mu} + \gamma\omega_i(t) - p_i(t), \tag{6}$$

where

$$\omega_i(t) = E\left[\int_t^\infty \left[D_i(v) + \phi(1 - D_i(v))\right] e^{-(r+\mu)(v-t)}dv\right]$$

stands for consumers' expected lifetime utility (at instant t) entailed by network effects (i.e. the component of consumers' expected lifetime utility that depends on the trajectory of the equipment' installed base in the primary market). $\omega_i\,(t)$ must verify two properties. First

$$\omega_i(t) \in \left[\frac{\phi}{r+\mu}, \frac{1}{r+\mu}\right] \tag{7}$$

since $D_i(t) \in [0,1]$. Second,

$$\omega_1(t) + \omega_2(t) = \frac{1+\phi}{r+\mu} \tag{8}$$

as a consequence of the full market coverage assumption.

At instant t, for given $p_1\,(t)$, $p_2\,(t)$, $\omega_1(t)$ and $\omega_2(t)$, it is possible to identify the position $\widetilde{x}\,(t) = \widetilde{x}(p_1(t),\, p_2(t),\, \omega_1(t), \omega_2(t))$ of the newborn consumer who is indifferent between buying equipment 1 or equipment 2 : $V_1\,(\widetilde{x}, t) = V_2\,(\widetilde{x}, t)$, with $\widetilde{x}(t)$ defined by:

$$\widetilde{x}(t) = \frac{1}{2} + \frac{(r+\mu)\left[\gamma\omega_1(t) - \gamma\omega_2\,(t) + p_2\,(t) - p_1\,(t)\right]}{2\tau}. \tag{9}$$

When $\widetilde{x}(t) \in \,]0,1[$, newborn consumers located at the left of the indifferent consumer (i.e. $x\,(t) < \widetilde{x}\,(t)$) prefer equipment 1 over equipment 2, while newborn consumers located at the right of the indifferent consumer (i.e. $x\,(t) > \widetilde{x}\,(t)$), prefer to buy equipment 2. When $\widetilde{x}(t) < 0$ (resp. $\widetilde{x}(t) > 1$), at instant t, the entire set of newborn consumers prefer equipment 2 to equipment 1 (resp. equipment 1 to equipment 2). In this paper, we focus on duopolistic equilibrium outcomes, in which both firms keep non-negative market shares. Thereby, we rule out eviction cases, assuming that $-\tau < (r+\mu)\left[\gamma\omega_1(t) - \gamma\omega_2\,(t) + p_2\,(t) - p_1\,(t)\right] < \tau$.

Under the last condition, the instantaneous demands for equipment 1 and 2 are respectively given by:

$$d_1(t) = \mu\widetilde{x}(t); \text{ and } d_2(t) = \mu\left[1 - \widetilde{x}(t)\right], \tag{10}$$

where $\widetilde{x}(t)$ is given by (9).

Firms' instantaneous profits in the primary market are equal to $\pi_i^{PM}\,(t) = d_i\,(t)\,p_i\,(t)$,where $d_i\,(t)$ is given by (10) and $p_i\,(t)$ corresponds to a strategic decision of firm i.

At equilibrium, both equipment producers optimally choose equipment price strategies $p_i\,(t)$, corresponding to the sequence of equipment prices that, at each instant t, maximize the total discounted profits of equipment producer i

$$\int_0^\infty \left(\frac{\alpha^2}{(N+1)^2} D_i\,(t) + d_i\,(t)\,p_i\,(t)\right) e^{-rt} dt, \tag{11}$$

given rival's equipment price strategy as well as consumers' expectation rules. Simultaneously, at equilibrium, equipment choices of newborn consumers must be optimal since consumers are forward-looking agents whose equipment choices are based on expectations that turn out to be true.

Without any further assumption, this dynamic problem has no closed-form solution and it is not possible to provide general descriptions on firms' optimal price strategies or consumers' equilibrium expectation rules. In this context, we focus our analysis on a more restrictive equilibrium notion: the Linear Markov Perfect Equilibrium (LMPE).

This equilibrium concept introduces two further assumptions in our model. First, we assume that expectation rules and price strategies follow Markovian processes. In the context of our model, this constitutes a reasonable assumption since all the payoff relevant information is conveyed by the state vector $(D_i(t), 1 - D_i(t))$. Second, both Markovian processes are assumed to be linear. As far as concerns this assumption, note that our dynamic game corresponds to a linear-quadratic differential game[21], since: *(i)* the instantaneous demand for equipment is linear in the control variables $(p_i(t), p_j(t))$; *(ii)* the equation of motion of $D_i(t)$ is linear in the state variable; and *(iii)* firms' total profits are quadratic in control variables and linear in the state variable (see equation (11)). Our differential game being linear-quadratic, it is natural to search for the simplest strategies that might-solve this category of differential games: the linear strategies. Furthermore, in the context of linear-quadratic games, the best reply to linear strategies is also linear.

Formally, linear Markov expectation rules and linear Markov price strategies are defined as follows:

DEFINITION 1. **Linear Markov expectation rules**
Linear Markov expectation rules correspond to a *pair* of linear functions $(F_1(\cdot), F_2(\cdot))$ that maps any observed vector $(D_1(t), D_2(t))$ on the unit simplex

$$\Delta = \{(D_1(t), D_2(t)) \mid 0 \leq D_i(t) \leq 1 \ , \ D_1(t) + D_2(t) = 1\}$$

to a vector $(\omega_1(t), \omega_2(t))$ such that

$$F_i(\cdot) = \omega_i(t) = \delta_i + b_i D_i(t);$$

with $i = 1, 2$.

DEFINITION 2. **Linear Markov price strategies**
Linear Markov price strategies correspond to a pair of functions $(P_1(\cdot), P_2(\cdot))$ that maps linearly any observed point $(D_1(t), D_2(t))$ on the unit simplex Δ to a vector of prices $(p_1(t), p_2(t))$ such that

$$P_i(D_1(t), D_2(t)) = \widetilde{p}_i(t) = \eta_i + s_i D_i(t),$$

with $i = 1, 2$.

Since $D_1(t) + D_2(t) = 1$, from Definition 2, it follows that

$$P_j(D_i, D_j) = P_j(1 - D_j, D_j) \equiv \widetilde{p}_j(D_j) \equiv \widetilde{p}_j(1 - D_i).$$

[21] See Dockner *et al* (2000) and Long, N.V. and Leonard, D. (1992).

A linear Markovian price strategy, $\widetilde{p}_i^*(D_i)$, is said to be the *best reply* of firm i to both the competitor's price strategy, $P_j(\cdot)$, and consumers' expectation rules, $(F_1(\cdot), F_2(\cdot))$, if the price strategy $\widetilde{p}_i^*(D_i)$ maximizes Firm i's total discounted profits, given $P_j(\cdot)$ and $(F_1(\cdot), F_2(\cdot))$.

Definition 3. The LMPE of the dynamic game is defined by the sequences of price strategies and consumers' expectations rules

$$\{\widetilde{p}_1^*(t), \widetilde{p}_2^*(t), F_1(t), F_2(t)\}$$

such that:
(i) $\widetilde{p}_i^*(t)$ is firm i's best reply to both the price strategy $\widetilde{p}_j(t)$ and the expectation rule $(F_i(t), F_j(.))$, for $i, j = 1, 2, i \neq j$, and
(ii) expectations are rational in the sense that, for $i = 1, 2$,

$$F_i(t) = \omega_i(t) = \int_t^\infty [D_i(v) + \phi(1 - D_i(v))] e^{-(r+\mu)(v-t)} dv.$$

Considering the previous definitions, it follows that firm i's LMPE price strategy $(\widetilde{p}_i^*(D_i(t)))$ corresponds to the solution of the following dynamic optimization problem:[22]

$$\max_{\widetilde{p}_i(t)} \int_0^\infty e^{-rt} \Pi_i(\widetilde{p}_i(t), \widetilde{p}_j(t), D_i(t)) dt \tag{12a}$$

s.t.

$$\frac{dD_i(t)}{dt} = d_i(t) - \mu D_i(t) \tag{12b}$$

$$d_i(t) = \mu \frac{\tau - \gamma(1+\phi)}{2\tau} + \mu(r+\mu) \frac{2\gamma\delta_i + (2\gamma b_i - s_j) D_i(t) + \eta_j + s_j - p_i(t)}{2\tau} \tag{12c}$$

$$F_i(t) = \omega_i(t) = \int_t^\infty [D_i(v) + \phi(1 - D_i(t))] e^{-(r+\mu)(v-t)} dv, \tag{12d}$$

$$\Pi_i(t) \geq 0 \text{ and } 0 < D_i(t) < 1 \tag{12e}$$

where $p_i(t)$ is the control variable, $D_i(t)$ is the state variable and

$$\Pi_i(\widetilde{p}_i(t), \widetilde{p}_j(t), D_i(t)) = \frac{\alpha^2}{(N+1)^2} D_i(t) + d_i(t) p_i(t).$$

Under the assumptions of linear Markov expectation rules and linear Markov price strategies, the problem of equipment producers in the primary market corresponds to an infinite horizon optimal control problem with a linear-quadratic structure. Hence, the closed-form solution of

[22] To obtain (12c), introduce (9) in (10), accounting for (8). Then, introduce linear Markov expectation rules and take into account firm j's linear Markov price strategy.

this problem[23] can be obtained and it is possible to describe the properties of equilibrium price strategies.

LEMMA 1. ***Properties of expectation rules and price strategies***
In the duopolistic LMPE of the game:
(i) Linear Markov expectation rules are perfectly symmetric, with:

$$b_1 = b_2 = b \text{ and } \delta_1 = \delta_2 = \delta;$$

(ii) Linear Markov price strategies are also perfectly symmetric, with

$$s_1 = s_2 = s \text{ and } \eta_1 = \eta_2 = \eta.$$

Proof. see the Appendix. ∎

According to Lemma 1, at equilibrium, linear Markov expectation rules and price strategies are symmetric. In this context, at equilibrium, instantaneous equipment prices $p_1(t)$ and $p_2(t)$ only differ when there is an asymmetry regarding the size of equipment producers' "installed base of customers" $D_i(t)$. The same applies, *mutatis mutandis,* to the linear Markov expectation rules.

The symmetry properties identified in Lemma 1 are brought about by (i) the symmetry of the primitives of our model and *(ii)* the exclusive focus on duopolistic outcomes. Concerning the symmetry of the primitives of the model, note that, equipment are considered to be symmetric with respect to every characteristic except their initial installed base of consumers. Would one of the equipment producers have an exogenous advantage over its rival (e.g. a quality advantage[24] or a "better location" on the Hotelling line, being closer to the preferences of a larger fraction of consumers), asymmetric price strategies and expectation rules would be expected at the LMPE. Concerning the focus on duopolistic outcomes, this assumption rules out the case where the market tips and therefore, our analysis rules out situations in which the firm with a larger installed base of equipment owners (dominant equipment producer) makes use of this exogenous advantage to evict the rival firm.

It is worth noting that, even in the context of our model, the symmetric LMPE may not exist. Proposition 1 identifies the necessary and sufficient conditions for the existence of a unique LMPE.

PROPOSITION 1. ***Existence and uniqueness of the LMPE***
A unique LMPE exists if and only if the intensity of network effects,γ *is sufficiently small in*

[23] For theorems stating necessary and sufficient conditions, see, for example, Long, N.V. and Leonard,D. (1992), Chapter 9.
[24] See Argenziano (2008) for a model of static price competition in a duopoly with vertical and horizontal differentiation and network effects.

relation to the degree of product differentiation, τ.

$$\gamma \leq \overline{\gamma} \equiv \frac{\tau (3r + 2\mu)}{(1 - \phi)(r + 2\mu)}. \tag{13}$$

Proof. see the Appendix. ∎

According to Proposition 1, a unique symmetric LMPE exists only when the intensity of network effects, γ, is not too strong in relation to τ. When network effects are very strong, firms may have incentives to move away from the symmetric price strategies, in order to amplify as much as possible the network effects created by their installed base of consumers.

The likelihood of fulfilling condition $\gamma \leq \overline{\gamma}$ depends on the remaining parameters of the model (τ, ϕ, r, μ). When equipments are considerably differentiated (τ is large) or the degree of compatibility between network effects is large (ϕ is large), the existence and uniqueness condition (13) is more likely satisfied. This is also the case when firms discount more future profits (r is large). In contrast, in a scenario of rapid turnaround in the composition of the population of equipment owners (μ is large), it becomes more difficult to fulfill the existence and uniqueness condition.

In what follows, we assume that condition (13) is met and we investigate further properties of equilibrium price strategies.

4.1. LMPE price strategies

Proposition 2. *Along the LMPE price trajectories, the equipment with a larger instantaneous installed base of consumers is more expensive than the rival equipment:*

$$p_i^* (t) - p_j^* (t) = s^* [D_i (t) - D_j (t)]$$

with $s^* > 0$.

Proof. See the Appendix. ∎

Proposition 2 puts forward the existence of a *"network premium"* for equipment producers enjoying a larger installed base of customers (dominant equipment). When aftermarkets exhibit network effects, the larger the installed base of consumers endowed with equipment i, the more valuable CGS_i become. Hence, from the perspective of newborn consumers, the dominant equipment has a qualitative advantage over its rival[25]. However, when condition (13) is fulfilled, the producer of the dominant equipment is not interested in adopting aggressive price-cutting strategies in the primary market. Under (13), network effects tend to be relatively weak in relation to equipment differentiation and, therefore, it is highly unlikely that newborn consumers "located distantly" from the dominant equipment choose to buy it. The dominant equipment producer

[25] When taking their equipment decisions, newborn consumers account for this qualitative advantage but they also consider the location of equipment on the Hotelling line.

acts like a "*fat cat*"[26], accommodating itself to the rival equipment producer by charging a higher price for its "superior" equipment[27].

The fat cat effect just described tends to be more significant in a scenario of stronger network effects (γ is larger) or weaker product differentiation (τ is smaller), leading to a greater price differential between the dominant equipment and its rival. In fact from (40) in Appendix, it follows also that the derivative $\frac{\partial s(b^*)}{\partial b}$ is positive when $r > 2\mu$. In these circumstances, the gap in the instantaneous equipment prices $p_i^*(t) - p_j^*(t)$ depends positively on b, which is turn is determined by the parameters $\gamma, \tau, r\,, \mu$ and ϕ. In particular, for $r > 2\mu$, it follows that b^* is an increasing function of $\frac{\gamma}{\tau}$ for (γ, τ) such that condition (13) holds.

Finally, it is worth noting that s^* does not depend on N and therefore, the degree of competition in aftermarkets does not affect the equilibrium differential of instantaneous equipment prices.

Proposition 3. *In the LMPE, $\eta(b^*)$ is positively related to the degree of differentiation between equipment (τ) and negatively related with the degree of aftermarkets' profitability, which is determined by the degree of competition in aftermarkets (N), the intrinsic value of CGS (α) and the relative intensity of network effects $\left(\frac{\gamma}{\tau}\right)$. Depending on the interplay of these elements, at equilibrium $\eta(b^*) \lesseqgtr 0$.*

Proof. See the Appendix. ∎

In the context of linear Markov price strategies, η corresponds to the component of price that does not depend on equipment' installed base of customers. At equilibrium:

$$\eta(b^*) = \underbrace{\frac{\tau}{(r+\mu)}}_{(a)} + \underbrace{\tau\frac{(1-\phi-b^*(r+2\mu))}{b^{*2}\mu(r+\mu)^2}(2(1-\phi)-b^*(r+\mu))}_{(b)} - \underbrace{\frac{\alpha^2}{(r+\mu)(N+1)^2}}_{(c)} \qquad (14)$$

From (14), it follows that, in line with Proposition 2, $\eta(b^*)$ is the sum of three terms: (a) the "Hotelling price" $\left(\frac{\tau}{r+\mu}\right)$, (b) a "lock in discount", and (c) a discount related to the lack of competition in aftermarkets. The Hotelling price corresponds to the price that would be charged by equipment producers if they were not involved in the production of CGS. When equipment producers simultaneously participate in the primary market and the aftermarket, both equipment producers compete more fiercely for new (lock in) consumers to avoid loosing profits in the aftermarket[28]. The more profitable are aftermarkets in comparison with primary markets ($\frac{\gamma}{\tau}$ is larger), the larger is discount (b) in equation (14). In contrast, in the absence of network effects $(\gamma = 0)$, there is no lock in discount since component (b) in equation (14) is equal to zero.

[26] We borrow this terminology from Fudenberg and Tirole (1983).

[27] The superiority of the equipment is caused by network effects. The fact that the dominant equipment benefits from a larger installed basis of consumers makes future consumption of the compatible CGS more valuable.

[28] Remind that the profits of each CGS supplier (including equipment producers) are positively correlated with the number of consumers endowed with the corresponding equipment.

In addition, $\eta(b^*)$ depends positively on the intensity of competition in the aftermarket[29]. When competition in aftermarkets is tougher (N is larger), $\eta(b^*)$ is higher and, therefore, both equipment are more expensive (regardless of their installed base of consumers). At the limit, when aftermarkets are perfectly competitive, the discount (c) in equation (14) is null. Notice also that, the increase in equipment price triggered by tougher competition in aftermarkets is greater the higher the marginal benefit of CGS consumption (α) and the lower the effective discount rate $(r+\mu)$.

Corollary 1. *In the LMPE:*

$$-\frac{\alpha^2}{4(r+\mu)} \leq \eta^* \leq \frac{\tau}{r+\mu}.$$

Proof. See the Appendix. ∎

Corollary 1 defines the set of possible values of $\eta(b^*)$. When there are no network effects and aftermarkets are perfectly competitive ($\gamma = 0$ and $N \to \infty$), there is no "lock in discount" nor "lack of competition" discount, and $\eta(b^*) = \frac{\tau}{r+\mu}$. When the intensity of network effects reaches its upper limit (γ fulfills condition (13) in equality), the "lock in discount" is maximum and it is equal to $\frac{\tau}{r+\mu}$, yielding $\eta^* = -\frac{\alpha^2}{(N+1)^2(r+\mu)} < 0$. Accordingly, η^* reaches its minimum value when γ fulfills condition (13) in equality and, simultaneously, equipment producers have the monopoly of CGS provision ($N = 1$), yielding $\eta^* = -\frac{\alpha^2}{4(r+\mu)}$.

When aftermarkets are very competitive and/or network effects are relatively strong in relation to differentiation between equipment, $\eta(b^*)$ can be negative, eventually leading to negative equipment prices, especially in the case of the dominated equipment producer ("lean and hungry" effect[30]).

4.2. LMPE trajectories and Steady state

This subsection analyzes the process of convergence to the steady state LMPE, investigating the LMPE trajectories of equipment' instantaneous market shares $D_i(t)$ and equipment' prices $p_i(t)$.

By definition, in the steady state equilibrium $d_i(t) = \mu D_i(t)$, or equivalently

$$\frac{dD_i(t)}{dt} = 0, \tag{15}$$

yielding[31] $\overline{D}_i = \frac{1}{2}$,where $\overline{D}_i$ stands for the steady state LMPE market share of equipment

[29] Section 5 shows that any additional profits on aftermarkets translate into lower profits in the primary market, since equipment producers compete more fiercely to attract newborn consumers.

[30] We borrow this terminology from Fudenberg and Tirole, 1983.

[31] To obtain the steady state market share, in (15), substitute $d_i(t)$ by the expression (12c). Then, introduce $\delta = \frac{1+\phi-b(r+\mu)}{2(r+\mu)}$ (see (20)) and account for the linear Markov price strategies of firm i. Finally, solve equation (15), obtaining $\overline{D}_i = \frac{1}{2}$.

producer $i, i = 1, 2$. Not surprisingly, in the steady state LMPE, equipment producers have symmetric market shares in the primary market. This result is a direct consequence of the fact that we focus exclusively on duopolistic equilibrium outcomes within a model where firms are symmetric with respect to every characteristic but initial market shares in the primary market.

The producer of the dominant equipment assumes a "fat cat" behavior, charging higher equipment prices and, accordingly, it becomes increasingly less attractive to new consumers until the symmetric steady state is reached. Whether this process of market shares convergence is slower or faster depends on the characteristics of the markets.

PROPOSITION 4. *In the LMPE, market shares converge to the symmetric steady state equilibrium at a rate equal to*

$$\psi = \frac{1-\phi}{b^*} - (r + \mu).$$

Proof. See the Appendix. ∎

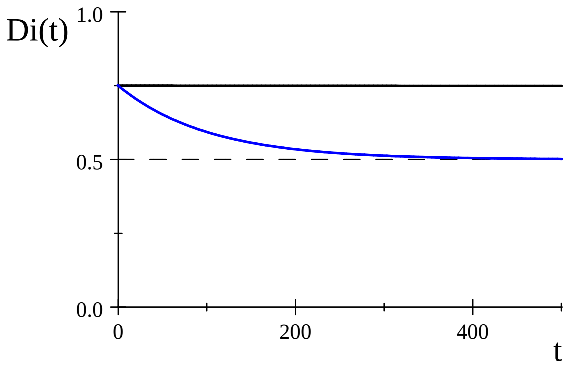
COROLLARY 2. *In the LMPE, the speed of convergence to the symmetric steady state is such that* $0 \leq \psi \leq \mu$.

Proof. Follows directly from Proposition 4 and the fact that $\frac{1-\phi}{r+2\mu} \leq b^* \leq \frac{1-\phi}{r+\mu}$. ∎

Proposition 4 states that the speed of convergence to the symmetric steady state depends on the effective discount rate $(r + \mu)$, on the degree of compatibility between network effects and on the equilibrium value of b^*, which is determined by the parameters γ, τ, ϕ, r and μ. Convergence to the symmetric steady state is slower in industries characterized by weaker product differentiation (τ is smaller) and stronger network effects (larger γ). In these circumstances, the rational expectations on the future market shares are more sensitive to equipment' instantaneous base of consumers (b^* is larger) and the "fat cat" effect tends to be more limited. Accordingly the equilibrium gap $|p_i(t) - p_j(t)|$ is also smaller and the asymmetry between firms' market shares prevails longer. This result is in line with Doganoglu (2003) and Laussel *et al* (2004).

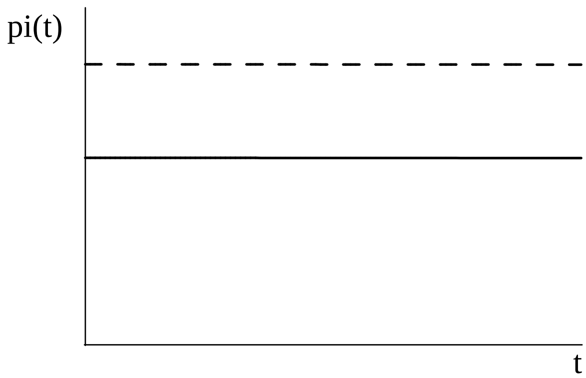
It is also worth noting that, as illustrated by Corollary 2, there might be a considerable degree of inertia in the evolution of equipment market shares. This is more likely to occur when network effects are stronger (larger γ) or product differentiation is weaker. In any case, when the turnaround in the population of equipment owners is relatively limited (μ is small), the convergence to the steady state LMPE tends to be slower, independently of the preferences of equipment owners (γ, ϕ, τ).

The following graph shows the LMPE trajectories of $D_i(t)$ in two different industries. The black line represents an industry with stronger network effects and weaker product differentiation ($\tau = 1,\ \phi = 0.05,\ \gamma = 2.22$) in comparison with the industry corresponding to the blue line ($\tau = 1.5,\ \phi = 0.5,\ \gamma = 1$). In both cases, we consider $D_i(0) = 0.75$, $\mu = 0.01$ and $r = 0.025$.



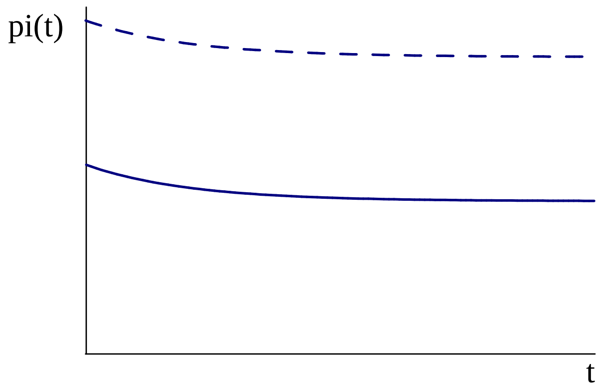
LMPE trajectories of $D_i(t)$, with $D_i(0) = 0.75, r = 0.02, \mu = 0.01$

The LMPE price trajectories mimic the evolution of firms' market shares in the primary market. As illustrated in the following figure, when there is a lot of inertia in the evolution of market shares, the volatility in equipment prices tends to be very low. The figure illustrates as well the interdependence between equipment prices and the degree of competition in the aftermarkets, with both equipment variants being more expensive when competition in aftermarkets is stronger. We assume $\tau = 1$, $\phi = 0.05$, $\gamma = 2.22$, $D_i(0) = 0.75$, $\mu = 0.01$, $r = 0.025$ and $\alpha = 1$. The solid line considers monopolist aftermarkets ($N = 1$), while the dashed line considers perfect competition in the aftermarket ($N \to \infty$).



LMPE price path.

When convergence to the steady state is faster, equipment prices evolve more rapidly to their steady state level. The speed of convergence is independent of the degree of competition in the aftermarkets, but price levels are not: the tougher the competition in aftermarkets, the higher equipment prices are. The following figure illustrates this point, considering $\tau = 1.5$, $\phi = 0.5$, $\gamma = 1$, $D_i(0) = 0.75$, $\mu = 0.01$, $r = 0.025$ and $\alpha = 1$.



LMPE price path.

LEMMA 2. *In the steady state LMPE, firms charge identical equipment prices* $p_i = p_j = \overline{p}$, *with:*

$$\frac{\tau}{r+\mu}\left(\frac{r}{r+2\mu}\right) - \frac{\alpha^2}{4\,(r+\mu)} \leq \overline{p} \leq \frac{\tau}{r+\mu}.$$

Proof. See the Appendix. ∎

LMPE steady state equipment prices are symmetric. Equilibrium prices depend on the degree of product differentiation as well as on the profitability of aftermarkets. When there are no network effects and aftermarkets are perfectly competitive, steady state equipment prices correspond to the "Hotelling price", $\frac{\tau}{r+\mu}$. As aftermarkets become more profitable, competition in the primary market becomes tougher and steady state LMPE equipment prices are below the "Hotelling level". In particular, when aftermarkets are not so competitive (N is small) and consumers significantly value CGS consumption (α is large in relation to τ), steady state LMPE prices may be negative.

5. WELFARE

In the context of antitrust cases on aftermarkets, the most widely debated question is associated with the potential welfare-loss caused by the lack of competition in aftermarkets. In this section, we investigate to what extent, in our model, the "lack of competition" in aftermarkets (measured by N) might be detrimental to each of the participants in the primary market and the corresponding aftermarkets: *(i)* consumers, *(ii)* independent suppliers of CGS, and *(iii)* equipment producers.

In the case of consumers, a further distinction is considered: "old consumers", who were already endowed with an equipment at $t = 0$, and "new consumers" who arrived later in this industry.

PROPOSITION 5. *When aftermarkets become more competitive, the aggregated equilibrium lifetime surplus of "old consumers"* (CS^*_{old}) *always increases. In contrast, the relation between* *the degree of competition in aftermarkets* (N) *and the aggregated equilibrium lifetime surplus of "new consumers"* (CS^*_{new}) *is non-monotonic. More precisely:*

$$\begin{aligned} \frac{\partial CS^*_{old}}{\partial N} &= \frac{1}{r+\mu}\frac{\alpha^2 N}{(N+1)^3} > 0; \\ \frac{\partial CS^*_{new}}{\partial N} &= \frac{\mu}{(r+\mu)\,r}\frac{\alpha^2 (N-2)}{(N+1)^3}. \end{aligned} \tag{16}$$

Proof. See the Appendix. ∎

Proposition 5 shows that the lack of competition in aftermarkets is always detrimental to old consumers. These consumers were already endowed with an equipment and therefore they are only interested in purchasing CGS, being penalized by the lack of competition in aftermarkets. In contrast, new consumers are not endowed with an equipment and, before purchasing CGS, they have to buy the equipment itself. This entails a non-monotonic relation between the degree of competition in aftermarkets and CS^*_{new} : when aftermarkets become more competitive, new consumers benefit from cheaper CGS but both equipment become more expensive (see equation (14)).

Proposition 6. *When aftermarkets become more competitive, the aggregated equilibrium lifetime profits of independent CGS suppliers* (Π^*_{IS}) *decreases for* $N > 3$, *with:*

$$\frac{\partial \Pi^*_{IS}}{\partial N} = -\alpha^2 \frac{N-3}{r(N+1)^3}. \tag{17}$$

Proof. See the Appendix. ∎

According to Proposition 6, when competition in aftermarkets becomes very tough, the aggregated equilibrium lifetime profits of independent CGS suppliers are lower, due to the simultaneous decline of the unit price of CGS and the individual production of each CGS supplier.

Proposition 7. *When aftermarkets become more competitive, the aggregated equilibrium lifetime profits of equipment producers* (Π^*_{EP}) *always decreases:*

$$\frac{\partial \Pi^*_{EP}}{\partial N} = \frac{-2\alpha^2}{(N+1)^3 (r+\mu)} < 0. \tag{18}$$

The previous effect coincides with the effect of N *on the aggregated equilibrium lifetime profits that equipment producers extract from "old consumers".*

Proof. See the Appendix. ∎

Proposition 7 shows that the degree of competition in aftermarkets (N) does not affect the equilibrium lifetime profits yielded by the interactions of equipment producers with the successive cohorts of new consumers. Since new consumers are not endowed with an equipment, when aftermarkets become more competitive, equipment producers are able to increase equipment prices to compensate profit losses in the respective aftermarket. When aftermarkets become more competitive, equipment become more expensive (the price discount $-\frac{\alpha^2}{(r+\mu)(N+1)^2}$ in equation (14) becomes smaller), exactly compensating the lifetime profit-losses (per new consumer) that equipment producers suffer in the aftermarket as a consequence of increasing competition.

From the preceding propositions, it follows that the "lack of competition" in aftermarkets affects differently the agents involved in this industry. The following Proposition investigates the effect of N on equilibrium total social welfare ("Marshallian social welfare"):

$$W^* = CS^*_{old} + CS^*_{new} + \Pi^*_{IS} + \Pi^*_{EP}.$$

PROPOSITION 8. *The lack of competition in aftermarkets has a detrimental effect on total social welfare, with:*

$$\frac{\partial W^*}{\partial N} = \frac{\alpha^2}{r\,(N+1)^3} > 0.$$

Proof. Follows directly from (16), (17) and (18). ∎

According to Proposition 8, the lack of competition in aftermarkets has a detrimental effect on total social welfare.

The magnitude of the welfare loss is determined by the characteristics of the aftermarket, namely the marginal benefit entailed by CGS consumption $\left(\frac{\partial W^*}{\partial N}\text{ depends positively on } \alpha\right)$ and the discount factor $\left(\frac{\partial W^*}{\partial N}\text{ depends negatively on } r\right)$.

In line with other models dealing with aftermarket competition (see, for example Shapiro (1995) and Borenstein *et al* (2000)), in the context of our model, total social damage corresponds to the sum of the discounted value of the deadweightloss triangles in each segment of the aftermarket. The lack of competition in aftermarkets increases the price of CGS, reducing their consumption below the socially desirable level. In the case of the primary market, there is no welfare detrimental effect, since the price distortions created by the lack of competition in aftermarkets[32] do not affect equipment choices of newborn consumers.

6. CONCLUSION

In this paper, we propose a theoretical model of dynamic competition in primary markets and aftermarkets. In line with the previous literature, our model encompasses the key elements to an aftermarket: *(i)* the complementarity between durable goods and CGS; *(ii)* the existence of a time lag between equipment purchases and CGS consumption; and *(iii)* consumers' lock in. In addition, our model takes into consideration the network effects resulting from the interplay between the value of CGS and the market shares of equipment producers.

To investigate strategic interaction between firms we develop a linear-quadratic differential

[32] Remind that equipment prices are negatively affected by the degree of competition in aftermarkets.

game. In the context of this game, we search for the unique LMPE in which both firms have non-negative market shares and we provide a necessary and sufficient condition for the existence of such equilibrium. When the existence and uniqueness condition is fulfilled, we show that, at equilibrium, instantaneous equipment prices are determined by the interplay of the degree of differentiation between equipment variants and the profitability of aftermarkets, which depends itself on the on the degree of competition in aftermarkets, on the intrinsic value of CGS and on the intensity of network effects. Equipment prices tend to be lower when *(i)* equipment variants are weakly differentiated, *(ii)* competition in aftermarkets is relatively soft, *(iii)* CGS have a high intrinsic utility, and *(iv)* network effects are rather intense. Moreover, when effects *(i)*, *(ii)* or *(iii)* are very significant, instantaneous equipment prices may be negative at equilibrium.

In line with the literature on dynamic price competition in network industries, we found that, at equilibrium, the equipment producer with a larger installed base of consumers quotes a higher equipment price that reflects the greater network benefits generated in the corresponding aftermarket (fat cat behavior). The dominant equipment becomes less and less attractive to newborn consumers and, along the LMPE trajectories, market shares converge to the symmetric steady state (decreasing dominance). However, when network effects are rather intense in relation to differentiation between equipment variants, there might be a considerable degree of inertia in the evolution of market shares. Furthermore, the speed of convergence never exceeds the rate of consumers' turnaround (entry/exit rate).

From a welfare perspective, we conclude that the social damages entailed by the exercise of market power in aftermarkets are exclusively related to the underprovision of CGS in aftermarkets. Any additional profits on aftermarkets translate into tougher competition in the primary market, yielding lower equipment prices.

In our future research, we intend to investigate to which extent network effects resulting from the interplay between primary markets and aftermarkets may lead to eviction outcomes: when network effects are very strong in relation to product differentiation, equipment producers may have incentives to adopt aggressive pricing policies in the initial periods, in order to the evict the rival equipment producer. In our future research, we intend to study in which circumstances eviction outcomes may arise at equilibrium, investigating also the corresponding welfare implications.

We are also interested in extending our model to more complex types of network effects, evaluating in which circumstances network effects may lead to the "increasing dominance" phenomenon described by Cabral (2008). In particular, we intend to study the case of "multiplicative network effects", occurring when the size of the network positively effects the marginal benefit of CGS consumption. Finally, we also intend to consider asymmetric competition in aftermarkets, investigating how the intensity of competition in one aftermarket segment may affect market outcomes in the other aftermarket segment.

Appendix

Proof of Lemma 1

The proof is organized as follows. First, we prove that, when consumers are forward-looking, the assumption of linear Markov expectation rules implies $b_1 = b_2 = b$. Then, we show that $s_1 = s_2 = s$ for equilibrium price strategies to be compatible with the assumptions of linear Markov expectation rules and linear Markov price strategies. Finally, at the end of the proof, we show that $\eta_1 = \eta_2 = \eta$ and $\delta_1 = \delta_2 = \delta$.

Under the assumption of linear Markov expectation rules (Definition 1), we have that

$$F_i(\cdot) = \delta_i + b_i D_i(t) = \omega_i(t). \tag{19}$$

Considering (8), condition (19) implies $\delta_2 + b_2 D_2(t) = \frac{1+\phi}{r+\mu} - [\delta_1 + b_1 D_1(t)]$, or equivalently

$$\Leftrightarrow \delta_2 + b_2 D_2(t) = \frac{1+\phi}{r+\mu} - \delta_1 - b_1 + b_1 D_2(t). \tag{20}$$

From (20), it follows that $b_1 = b_2 = b$.

To show that $s_1 = s_2 = s$, we introduce the current-value co-state variable $\lambda_i(t)$ and we define the current-value Hamiltonian for firm i as follows:

$$H_i(t) = [p_i(t) + \lambda_i(t)]\, d_i(t) + \alpha^2 \frac{D_i(t)}{(N+1)^2} - \mu\lambda_i(t) D_i(t), \tag{21}$$

where, under linear Markovian expectation rules and linear Markovian price strategies, $d_i(t)$ is given by (12c).

The necessary conditions to guarantee optimality of firm i's equipment price strategies include:

$$\frac{\partial H_i(t)}{\partial p_i(t)} = 0 \tag{22}$$

and

$$\frac{d\lambda_i(t)}{dt} = r\lambda_i(t) - \frac{\partial H_i(t)}{\partial D_i(t)}, \tag{23}$$

with $i = 1, 2$.

From the first order conditions (22) and (23), we obtain:

$$\lambda_i(t) = \frac{\tau - \gamma(1+\phi)}{r+\mu} + \eta_j + s_j[1 - D_i(t)] - 2p_i(t) + 2\gamma[\delta_i + b_i D_i(t)], \tag{24}$$

and

$$\frac{d\lambda_i(t)}{dt} = (r+\mu)\lambda_i(t) - \frac{\mu(r+\mu)(2\gamma b_i - s_j)}{2\tau}[p_i(t) + \lambda_i(t)] - \frac{\alpha^2}{(N+1)^2}, \tag{25}$$

with $i = 1, 2$.

In condition (25), replace $\lambda_i(t)$ for the RHS of condition (24) and introduce the linear Markov price strategy of firm i, $p_i = \eta_i + s_i D_i(t)$, obtaining $\frac{d\lambda_i(t)}{dt} = A_i + B_i D_i(t)$, where the polynomials A_i and B_i depend on the parameters of the model as well as on the values of b, δ_i, δ_j, η_i, η_j, s_i

and s_j. Then, introduce firm i's linear Markov price strategy in condition (24) and differentiate it with respect to time, obtaining

$$\frac{d\lambda_i(t)}{dt} = (2\gamma b_i - s_j - 2s_i)\,\frac{dD_i(t)}{dt}, \tag{26}$$

where $\frac{dD_i(t)}{dt}$ is determined by the motion equation. Accounting for the linear Markov price strategy of firm i, $\frac{dD_i(t)}{dt}$ is equal to

$$\mu\left(\tfrac{\tau-\gamma(1+\phi)+(r+\mu)\left[2\gamma\delta_i+(2\gamma b_i-s_j-s_i)D_i(t)+\eta_j+s_j-\eta_i\right]}{2\tau} - D_i\left(t\right)\right).$$

Introducing this expression in (26) and putting $D_i\left(t\right)$ on evidence, one obtains $\frac{d\lambda_i(t)}{dt} = F_i + G_i D_i(t)$,where again F_i and G_i depend on the parameters of the model as well as on the values of b, δ_i, δ_j, η_i, η_j, s_i and s_j.

In the LMPE of the game, it must be the case that, at each instant $t \geq 0$, and for $i = 1, 2$, $A_i + B_i\ D_i(t) = F_i + G_i\ D_i(t)\forall D_i\left(t\right)$. Thus, it follows that $A_i = F_i$ and $B_i = G_i$, for $i = 1, 2$.

The condition $B_i = G_i$ implies that:

$$\left(r+\mu\right)\left(\frac{\mu}{\tau}\left(-s_i - s_j + 2b\gamma\right)^2\right) + \left(2s_i + s_j - 2b\gamma\right)\left(r + 2\mu\right) = 0, \tag{27}$$

for $i = 1, 2$. Subtracting the second from the first, we obtain $\left(s_1 - s_2\right)\left(r + 2\mu\right) = 0$, which is consistent with $s_1 = s_2 = s$.

To demonstrate that $\eta_1 = \eta_2 = \eta$, note that equating A_i and F_i for $i = 1, 2$, yields two equations in η_1 and η_2. Subtracting the second from the first, one obtains

$$\frac{(r+\mu)}{\tau}\left(\left(\eta_1 - \eta_2\right)\left(3\tau + 4s\mu - 4b\gamma\mu\right) + 2\gamma\left(\delta_1 - \delta_2\right)\left(-\tau - 2s\mu + 2b\gamma\mu\right)\right) = 0. \tag{28}$$

In addition, given linear Markov expectation rules we have that $\delta_i + bD_i = \omega_i\left(t\right)$, where $\omega_i\left(t\right)$ is given by (12d) since consumers are forward-looking agents. Differentiating both sides of this equality with respect to time, we obtain

$$b\frac{dD_i\left(t\right)}{dt} = \left(r+\mu\right)\left[\delta_i + bD_i(t)\right] - \left[D_i(t) + \phi(1 - D_i(t))\right] \tag{29}$$

Replacing in (29) the law of motion $\frac{dD_i(t)}{dt}$ by the expression derived above and considering $s_i = s_j = s$, it follows the condition $MD_i\left(t\right) + N_i = 0$, where M is a function of b, s and the parameters of the model, while N_i is a function of the parameters of the model as well as on the values of b, s, η_i, η_j, δ_i and δ_j. Since condition $MD_i\left(t\right) + N_i = 0$ must hold for all values of $D_i \in [0, 1]$, it follows that $M = 0$, or equivalently

$$\frac{1}{2\tau}\left(2\gamma b^2\mu\left(r+\mu\right) - 2b\left(s\mu\left(r+\mu\right) + \tau\left(r + 2\mu\right)\right) + 2\tau\left(1 - \phi\right)\right) = 0 \tag{30}$$

It also follows that $N_i = 0$, yielding

$$\delta_i = \frac{2b\mu(r+\mu)(\eta_i - \eta_j) + 2\gamma b\mu(1+\phi) - 2b\mu\tau - 4\tau\phi - 2bs\mu(r+\mu)}{2(r+\mu)(2\gamma b\mu - 2\tau)}, \tag{31}$$

for $i = 1, 2$. Subtracting δ_2 from δ_1, we obtain:

$$\delta_1 - \delta_2 = b\mu \frac{\eta_1 - \eta_2}{b\gamma\mu - \tau}. \tag{32}$$

Replacing condition (32) in equation (28), we obtain $(r+\mu)(\eta_1 - \eta_2)\frac{5b\gamma\mu - 3\tau - 4s\mu}{b\gamma\mu - \tau} = 0$, which is consistent with $\eta_1 = \eta_2 = \eta$.

Finally, given that $\eta_1 = \eta_2 = \eta$, equation (32) yields $\delta_1 = \delta_2 = \delta$.■

Proof of Proposition 1

The proof is organized as follows. First, we use equilibrium conditions (27) and (30) to express the $b - value$ at the LMPE equilibrium as a function of the parameters of the model. Then we investigate the existence of further constraints on the equilibrium $b - value$ imposed by the hypothesis of rational expectations together with the assumptions of linear Markov expectations.

To start, solve equation (30) with respect to s, obtaining $s(b)$ equal to:

$$s(b) = \frac{2\gamma b^2\mu(r+\mu) + 2\tau(1 - \phi - b(r+2\mu))}{2b\mu(r+\mu)}. \tag{33}$$

Then, in condition (27) consider $s_1 = s_2 = s$ and replace s for its value in equation (33), obtaining the LMPE equilibrium condition that implicitly expresses b^* as a function of the parameters of the model:

$$\frac{\gamma}{\tau}\mu(r+\mu)(r+2\mu)b^{*3} + (r+2\mu)^2 b^{*2} - 5(1-\phi)(r+2\mu)b^* + 4(\phi-1)^2 = 0, \tag{34}$$

However, note that, not all the values of b that solve (41) constitute a LMPE. In particular, rational expectations, together with the assumptions of linear Markov expectation rules impose the following additional restrictions:

$$0 < b^* \leq \frac{1-\phi}{r+\mu}. \tag{35}$$

To obtain that $b^* > 0$, notice that equation (29) corresponds to a differential equation. Upon integration, we get:

$$D_i(v) = K + (D_i(t) - K)\, e^{\frac{(r+\mu)b - (1-\phi)}{b}(v-t)}, \tag{36}$$

with $K = \frac{\delta(r+\mu)-\phi}{(1-\phi)-b(r+\mu)}$

Finally, substituting (36) in the RHS of (12d), we obtain:

$$\omega_i(t) = \tfrac{\delta(1-\phi)-b\phi}{(1-\phi)-b(r+\mu)} + (1-\phi)(D_i(t) - K)\textstyle\int_t^\infty e^{-\frac{(1-\phi)}{b}(v-t)}dv.$$

The previous expression can only be equal to $F(\cdot) = \delta + bD_i(t)$ if b is strictly positive.

Regarding the condition $b \leq \frac{1-\phi}{r+\mu}$, notice that condition (7) requires:

$$\frac{\phi}{r+\mu} \leq \delta + bD_i(t) \leq \frac{1}{r+\mu} \ \forall\ D_i(t) \in [0,1]. \tag{37}$$

In particular, for $D_i = 0$, condition (37) implies $\frac{\phi}{r+\mu} \leq \delta \leq \frac{1}{r+\mu}$. Similarly, for $D_i = 1$, condition (37) implies $\frac{\phi}{r+\mu} - b \leq \delta \leq \frac{1}{r+\mu} - b$. Therefore

$$\frac{\phi}{r+\mu} \leq \delta \leq \frac{1}{r+\mu} - b. \tag{38}$$

To guarantee that the set of $\delta-$values in (38) is non-empty, b must be small enough. More precisely $b \leq \frac{1-\phi}{r+\mu}$.

In this context, the LMPE value of b^* correspond to the root(s) of the cubic polynomial in (34) that are included in the interval $0 < b^* \leq \frac{1-\phi}{r+\mu}$.

To conclude the proof, we investigate under which conditions the polynomial in condition (34), hereafter denoted by $X(b)$ has only one root such that $b^* \in \left(0, \frac{1-\phi}{r+\mu}\right]$. In this case, the LMPE exists and it is unique.

To start, notice that $X(b)$ is a third-degree polynomial and it has, at most, three roots. Given that $X(-\infty) = -\infty$ and $X(0) = 4\tau(\phi-1)^2 > 0$; one of the roots of $X(b)$ is necessarily negative, violating condition (35). Hence, at most two roots verify condition (35). It is worth noting that

$$X\left(\tfrac{1-\phi}{r+\mu}\right) = \tfrac{\mu(\phi-1)^2(-\tau(3r+2\mu)-\gamma(\phi-1)(r+2\mu))}{(r+\mu)^2} \lesseqgtr 0,$$

and $X(+\infty) = +\infty$. Therefore, a sufficient and necessary condition for the existence of a unique root of $X(b)$ satisfying condition (35) is $X\left(\frac{1-\phi}{r+\mu}\right) \leq 0$, or equivalently $\gamma \leq \frac{\tau(3r+2\mu)}{(1-\phi)(r+2\mu)}$. When $X\left(\frac{1-\phi}{r+\mu}\right) > 0$, either the polynomial $X(b)$ has two roots in the interval $\left(0, \frac{1-\phi}{r+\mu}\right]$, or it has no root in this interval. Thus, condition (13) is a necessary and sufficient condition for the existence of a unique LMPE.■

Proof of Proposition 2

The equilibrium value of s is jointly determined by equilibrium conditions (27) and (30). First, we solve condition (30) with respect to s, obtaining $s(b)$ in equation (33). Then, we solve the equilibrium condition (34) for γ, obtaining

$$\gamma = \tau\left(4(1-\phi) - b^*(r+2\mu)\right)\frac{b^*(r+2\mu) - (1-\phi)}{b^{*3}\mu(r+\mu)(r+2\mu)}, \tag{39}$$

where b^* denotes the equilibrium value of b. Introducing (39) in (33), we obtain the equilibrium value of s conditional on b^* :

$$s\left(b^*\right) = 2\tau\left(\left(1-\phi\right) - b^*\left(r+2\mu\right)\right)\frac{b^*\left(r+2\mu\right) - 2\left(1-\phi\right)}{b^{*2}\mu\left(r+\mu\right)\left(r+2\mu\right)} \tag{40}$$

Concerning the sign of $s\left(b^*\right)$, it follows that $s\left(b^*\right) > 0$. To show that, at equilibrium $s\left(b^*\right)$ must be strictly positive, note that $b^* \leq \frac{1-\phi}{r+\mu}$ and therefore $\frac{b^*(r+2\mu)-2(1-\phi)}{b^{*2}\mu(r+\mu)(r+2\mu)} < 0$. Furthermore, at equilibrium it must be the case that $b^* > \frac{1-\phi}{r+2\mu}$ and therefore $2\tau\left(\left(1-\phi\right) - b^*\left(r+2\mu\right)\right) < 0$. To verify that $b^* > \frac{1-\phi}{r+2\mu}$ note that $X\left(\frac{1-\phi}{r+2\mu}\right) = \gamma\mu\left(1-\phi\right)^3 \frac{r+\mu}{(r+2\mu)^2} > 0$. Therefore when the existence and uniqueness condition (13) is met, it must be the case that the unique LMPE is obtained for $\frac{1-\phi}{r+2\mu} < b^* < \frac{1-\phi}{r+\mu}$.■

Proof of Proposition 3

In the Proof of Lemma 1, from (24) and (25) it has been derived the equilibrium condition $A_i = F_i$, $i = 2$. Solving this condition for η, after introducing in condition $A_i = F_i$ the equilibrium requirement of symmetry regarding expectation rules and price strategies, we obtain

$$\eta\left(b,\delta,s\right) = \left(\frac{\tau - \gamma(1+\phi)}{(r+\mu)} + s + 2\gamma\delta\right)\left(1 + 2\frac{\mu}{\tau}\left(s - b\gamma\right)\right) - \frac{\alpha^2}{\left(r+\mu\right)\left(N+1\right)^2}. \tag{41}$$

Furthermore, analyzing condition (20) in the Proof of Lemma 1 in the light of the symmetry equilibrium requirement, it follows that $\delta = \frac{1+\phi-b(r+\mu)}{2(r+\mu)}$. Introducing this condition and condition (33) in (41), we get (14).

The sign of $\eta\left(b^*\right)$ is *a priori* indeterminate. The term $-\frac{\alpha^2}{(r+\mu)(N+1)^2}$ is negative, while the term $\tau\left[b^*\left(r+3\mu\right) - 2\left(1-\phi\right)\right]\frac{b^*(r+\mu)-(1-\phi)}{(b^*)^2\mu(r+\mu)^2}$ is necessarily positive for $b^* < \frac{1-\phi}{r+\mu}$ given the assumption $\mu < \frac{r}{2}$.■

Proof of Corollary 1

From (14) it follows that

$$\frac{d\eta\left(b^*\right)}{db} = \tau\left(1-\phi\right)\frac{b^*\left(3r+5\mu\right) - 4\left(1-\phi\right)}{\left(b^*\right)^3\mu\left(r+\mu\right)^2},$$

which is negative given the assumption $\mu > \frac{r}{2}$ and the result $b^* < \frac{1-\phi}{r+\mu}$.

Furthermore,

$$\frac{d\eta\left(b^*\right)}{dN} = \frac{2\alpha^2}{\left(r+\mu\right)\left(N+1\right)^3} > 0.$$

Accordingly, the highest value of $\eta\left(b^*\right)$ occurs for $b^* \to \frac{1-\phi}{r+2\mu}$ and $N \to \infty$, yielding $\eta^* = \frac{\tau}{r+\mu}$. Conversely, the lowest value of $\eta\left(b^*\right)$ occurs for $b^* \to \frac{1-\phi}{r+\mu}$ and $N = 1$, yielding $\eta\left(b^*\right) = -\frac{\alpha^2}{4(r+\mu)}$.■

Proof of Proposition 4

Consider the motion law $\frac{dD_i(t)}{dt} = d_i\left(t\right) - \mu D_i\left(t\right)$. Then, substitute $d_i\left(t\right)$ by expression (12c), after introducing $\delta = \frac{1+\phi-b(r+\mu)}{2(r+\mu)}$ (see (20)), accounting for the linear Markov price strategies of firm i and substituting s for expression (33). The resulting expression is a first-order differential equation, whose close solution is given by

$$D_i(t) = \frac{1}{2} + \left(D_i(0) - \frac{1}{2}\right) e^{-\left(\frac{1-\phi}{b} - (r+\mu)\right)t}$$

and the speed of convergence is equal to $\left(\frac{1-\phi}{b} - (r+\mu)\right)$.■

Proof of Lemma 2

In the steady state LMPE, firms share the equipment market evenly and, therefore, price equipment is given by $\eta_i + \frac{s_i}{2}$. Considering the equilibrium values of η^* and s^*, one obtains

$$\overline{p} = \frac{\tau}{r+\mu} - 2\tau(1-\phi)\frac{b^*(r+2\mu) - (1-\phi)}{b^{*2}(r+2\mu)(r+\mu)^2} - \frac{\alpha^2}{(r+\mu)(N+1)^2}, \tag{42}$$

which depends positively on N and negatively on b^*. Evaluating (42) for $b^* = \frac{1-\phi}{r+\mu}$ and $N = 1$, we obtain $\overline{p} = \frac{\tau}{r+\mu}\left(\frac{r}{r+2\mu}\right) - \frac{\alpha^2}{4(r+\mu)}$. Evaluating (42) for $b^* = \frac{1-\phi}{r+2\mu}$ and $N \to \infty$, we obtain $\overline{p} = \frac{\tau}{r+\mu}$.■

Proof of Proposition 5

Concerning the aggregated life-time equilibrium surplus of "old consumers" , note that, at equilibrium, the expected lifetime utility delivered by equipment i to an old consumer located at x is equal to

$$\frac{\vartheta - \tau(x-x_i)^2 + \Upsilon + \frac{1}{2}\left(\frac{N\alpha}{N+1}\right)^2}{r+\mu} + \omega_i^*(t),$$

since this consumer is already endowed with an equipment. Given that $\omega_i^*(t)$ does not depend on N, the effect of N on the expected life-utility of any "old consumer" is always equal to $\frac{1}{r+\mu}\frac{N\alpha^2}{(N+1)^3}$ independently of the equipment version owned by this consumer. Since there is a unit mass of "old consumers", the derivative $\frac{\partial CS^*_{old}}{\partial N}$ is equal to $\frac{1}{r+\mu}\frac{N\alpha^2}{(N+1)^3}$, which is always positive.

Concerning new consumers, consider the mass μ of newborn consumers at instant t. At equilibrium, the indifferent newborn consumer is located at

$$\widetilde{x}^*(t) = \frac{1}{2} + \frac{(r+\mu)(2D_1(t)-1)(-s^* + b^*\gamma)}{2\tau}, \tag{43}$$

which is independent of N, given that neither $D_i(t)$; s^* nor b^* depend on N. At equilibrium, the lifetime benefit of consumers arriving in the market at instant t is equal to

$$\mu\left(\int_0^{\widetilde{x}^*(t)} V_1^*(x,t)\,dx + \int_{\widetilde{x}^*(t)}^1 V_2^*(x,t)\,dx\right),$$

with $\widetilde{x}^*(t)$ being given by (43) and $V_i(x,t)$ given by (6) evaluated at equilibrium. Since $\widetilde{x}^*(t)$ does not depend on the degree of competition in aftermarkets, the impact of N on the lifetime surplus of consumers arriving in this industry at instant t is equal to

$$\mu\left(\int_0^{\widetilde{x}^*(t)}\frac{\partial V_1(x,t)}{\partial N}dx+\int_{\widetilde{x}^*(t)}^1\frac{\partial V_2(x,t)}{\partial N}dx\right).$$

From (6), it follows that $\frac{\partial V_i(x,t)}{\partial N}=\frac{\alpha^2(N-2)}{(N+1)^3(r+\mu)}$ and therefore the effect of N on the lifetime surplus of consumers arriving in this industry at instant t is equal to $\mu\frac{\alpha^2(N-2)}{(N+1)^3(r+\mu)}$. Considering that, in our infinite horizon model, there is an inflow of newborn consumers at each instant of time $t\geq 0$, the effect of N on the discounted lifetime surplus of all "new consumers" is equal to

$$\int_0^\infty \mu\frac{\alpha^2(N-2)}{(N+1)^3(r+\mu)}e^{-rt}dt=\frac{\mu}{r}\frac{\alpha^2(N-2)}{(N+1)^3(r+\mu)}, \tag{44}$$

which is non-negative for $N\geq 2$ and negative otherwise.■

Proof of Proposition 6

At equilibrium, aggregated instantaneous profits of independent CGS suppliers participating in aftermarket i are equal to $\frac{(N-1)\alpha^2}{(N+1)^2}D_i(t)$. Aggregating equilibrium instantaneous profits of all independent CGS suppliers (in both aftermarkets), we obtain $\frac{(N-1)\alpha^2}{(N+1)^2}$, given that $D_1(t)+D_2(t)=1$. Therefore, the aggregated equilibrium lifetime profits of independent CGS suppliers (Π_{is}^*) is given by $\int_0^\infty\frac{(N-1)\alpha^2}{(N+1)^2}e^{-rt}dt$, which is equal to $\frac{(N-1)\alpha^2}{r(N+1)^2}$. Accordingly, $\frac{\partial\Pi_{is}^*}{\partial N}=-\alpha^2\frac{N-3}{r(N+1)^3}$.■

Proof of Proposition 7

Aggregated equilibrium lifetime profits of equipment producers (Π_{EP}^*) can be decomposed on two components: *(i)* the aggregated profits that equipment producers extract from old consumers, and *(ii)* the aggregated profits that equipment producers extract from new consumers.

Equilibrium lifetime profits that equipment producer i extracts from old consumers are given by $\int_0^\infty\frac{\alpha^2}{(N+1)^2}D_i(0)e^{-(r+\mu)t}$, that is equivalent to $\frac{D_i(0)}{(r+\mu)}\frac{\alpha^2}{(N+1)^2}$. Accordingly, the aggregated equilibrium lifetime profits that both equipment producers extract from old consumers are equal to $\frac{1}{(r+\mu)}\frac{\alpha^2}{(N+1)^2}$, with $\frac{\partial}{\partial N}\left(\frac{1}{(r+\mu)}\frac{\alpha^2}{(N+1)^2}\right)$ equal to $\frac{-2\alpha^2}{(r+\mu)(N+1)^3}<0$.

Concerning equilibrium lifetime profits that equipment producer i extracts from "new consumers", in addition to profits from CGS, we have to consider profits yielded by equipment sales. The equilibrium lifetime profits that equipment producer i obtains from selling CGS to new consumers are given by

$$\int_0^\infty\frac{\alpha^2}{(N+1)^2}D_i(t)e^{-rt}-\frac{1}{(r+\mu)}\frac{\alpha^2}{(N+1)^2}D_i(0).$$

Differentiating this expression with respect to N, we obtain:

$$\frac{-2\alpha^2}{(N+1)^3}\left(\frac{1-\phi-b^*(r+\mu)}{2r(1-\phi-b^*\mu)}+\frac{b^*(r+2\mu)-(1-\phi)}{(1-\phi-b^*\mu)(r+\mu)}D_i(0)\right)<0. \tag{45}$$

In the primary market, equilibrium instantaneous profits that equipment producer i obtains from selling equipment to the successive cohorts of "new consumers" are equal to $p_i^*(t)\, d_i^*(t)$. Hence, at equilibrium, the lifetime profits that equipment producer i obtains from equipment sales are equal to:

$$\int_0^\infty \left[\mu\left(\eta^* + b^* D_i(t)\right)\left(\frac{1}{2} + \frac{(r+\mu)\left(2D_i(t) - 1\right)\left(-s^* + b^*\gamma\right)}{2\tau}\right)\right] e^{-rt} dt. \tag{46}$$

In the previous equation, only η^* depends on the degree of competition in aftermarkets. Replacing s^* by expression (33), and considering the equilibrium path of $D_i(t)$, it follows that the influence of N on (46) is given by

$$\frac{2\alpha^2}{(N+1)^3}\left(\frac{1-\phi-b^*(r+\mu)}{2r\left(1-\phi-b^*\mu\right)} + \frac{b^*(r+2\mu)-(1-\phi)}{\left(1-\phi-b^*\mu\right)(r+\mu)} D_i(0)\right) > 0,$$

which is symmetric to (45). As a consequence N does not have any effect on the equilibrium lifetime profits that equipment producer i extracts from new consumers. Accordingly, the effect of N on equilibrium lifetime profits of equipment producer i is given by $\frac{-2\alpha^2}{(r+\mu)(N+1)^3} D_i(0)$ and the effect of N on the aggregated equilibrium lifetime profits of both equipment producers is equal to $\frac{-2\alpha^2}{(r+\mu)(N+1)^3}$, which is always negative.$\blacksquare$

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