Multistage auction with endogenous entry: a rational for B2B auction practices

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Abstract

One of the most puzzling aspects of online B2B auction is the increasing use of auction split in multi stages. One of the options proposed in online auction is to conduct a two-stage auction where the number of possible suppliers is narrowed in the first stage of the auction and the supply contract is awarded to the winner of the second stage via first-price sealed-bid auction. This paper derives the strategic properties of such a mechanism and explain why the use of a multistage mechanism rather than traditional direct mechanisms can be justified from the point of view of a revenue maximizing auctioneer.

Assuming risk aversion and that bidders participating in a first-price auction incur a computational cost which may discourage entry, we show why the dilemma of choosing between English (which favors participation) and sealed-bid auction (which favors aggressiveness) can be solved by the use of multistage auction.

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1 Introduction

While the business-to-consumer (B2C) has been the most popular category of online auctions, business-to-business (B2B) online auctions are emerging as a prominent business model. In fact, B2B online auctions totaled $109 billion worth of transactions in 1999 alone, and that number is expected to grow to $2.7 trillion by 2005. Every day, millions of transactions are realized through online business-to-business auctions. Simultaneously, new companies, such as Portum, Synerdeal... propose new solutions to companies, from e-sourcing to e-procurement, but are also developing an important variety of auction design. The rapid growth of online auctions underscores the need for research in this domain. These new mechanisms appear to be far more complex than the ones traditionally used. Why so many different mechanisms? Is it just a mean of closing the market by not letting, in differentiating services, any market gap for a potential competitor? Or do these specific mechanisms have strategic properties still ignored by theorists?

One of the most puzzling aspects of online B2B auction is the increasing use of auctions split in multi stages. For example, Covisint, a marketplace which gathers many car manufacturers, allows buyers some flexibility in designing the auction mechanism. The buyer selects the category for its auction and the type of auction it wishes to conduct (either English auction or first price sealed bid auction). One of the options to buyers in Covisint is to conduct a two-stage auction. In the standard one-stage auction, bidders submit their bids once and the lowest price bid wins the first stage of the auction. In the two-stage auction, the number of possible suppliers is narrowed in the first stage of the auction and the supply contract is awarded to the winner of the second stage (see Tombak and Wang 2005). More precisely, only the most competitive bidders of the first stage will have the opportunity to take part to the second stage, which will select the winner via a first price sealed bid auction (the minimum acceptable bid for this second stage being the one of the last competitor eliminated in the first stage). Similarly, alternative mechanism with a second stage auction with ascending price, but with fixed ending time so that many bidders bid only in the last few seconds is essentially sealed-bid style.

What are the strategic properties of such a mechanism? Can the use of a multistage mechanism rather than traditional direct mechanisms be justified from the point of view of a revenue maximizing auctioneer? This is the main questions we address in this paper.

Klemperer has first pointed out the potential benefits of using a multistage auction for the the sale of the British 3G Telecom licences. Following Klemperer (2001), sealed-bid auctions do better
at promoting entry with asymmetric bidders because they give entrants a better chance of winning against strong incumbents whereas ascending auctions gather informations, useful in common value setting, by enabling bidders observing who is staying in and who is getting out as the price rises. So, he has proposed to run an anglo-dutch auction. The auctioneer begins by running an English auction in which the price is raised continuously until all but two bidders have dropped out. The two remaining bidders are then required to make a final sealed-bid offer that is no lower than the current asking price, and the winner pays the bid. The first stage of the auction (English) means less loss of efficiency than might result from pure sealed-bid auction whereas sealed-bid at the final stage induces some uncertainty about which of the finalists will win, and entrants are attracted by the chance to make it to the final.

Similarly, Tombak and Wang (2005) examines when a buyer would opt for a two-stage auction where in the first round some sellers are eliminated. Looking at some ex-post arguments, the authors find a rationale for such a multistage auction is that the buyer can eliminate outliers in the cost distribution, thereby reducing the variance among bidders in the second round.

In this paper, we depart from these analysis, assuming symmetry of bidders, independent private value and looking why ex-ante a rational profit maximizing auctioneer should use a multistage auction. Revenue equivalence theorem and revelation principle clearly state conditions for direct mechanisms to yield the maximum expected revenue. So, we depart from this framework taking both risk aversion and participation decision into account. Actually, according to Lucking-Reiley (1999, 2000) and Robert (2001), tendering in a first price sealed bid auction generates a cost of computation of the strategic bid, which does not exist in the ascending auction where the dominant strategy consists in bidding up to her own reservation value. The strategic simplicity of ascending auction was one of the argument commonly used for the choice of the FCC spectrum auction during the end of the 90’s. As Cramton (1998) notes, "this strategic simplicity not only promotes efficiency, but also encourages participation, since fewer resources are needed to figure out how to bid".

Our paper is related to numerous works in auction theory. It first borrows to some traditional results of the literature in this field (Vickrey (1961), Milgrom and Robert (1982), Riley and Sammelson (1981)) but also on more recent works introducing the possibility of an endogenous entry. The first models considering the possibility of an endogenous number of bidders are those of Samuelson (1985), Engelbrecht-Wiggans (1987), and McAfee and Mc-Millan (1987). In these works, the bidders undergo a participation cost (either a cost related to the acquisition of information on their evaluation of the object, or related to the effort of resolution of a strategic offer). The existence of this cost leads
some bidders not to take part. Levin and Smith (1994) show that if the entry is random, the seller’s expected surplus is decreasing according to the potential number of bidders. The explanation is that the variance of the effective number of participants is increasing in the potential number of bidders, and that this variance is expensive. These models are based on the assumption that the bidders decide or not to enter before knowing their reservation value.

Menezes and Monteiro (2000) give up this assumption and consider that the bidders decide entry once knowing their reservation value. We will make the same assumption here. We consider however that this participation cost is closely related to the complexity of the bidding strategy, that is to the agent’s optimal response to the other bidders strategy. Thus, this cost only appears in the situations where each competitor must determine her bid according to the reaction of the other competitors. This strategic computation requires many external informations, can be very complex and thus raises an high level of expertise. Clearly, such a computation cost does not appear in the situation where the bid simply consists in submitting its own reservation value, as it is the case in the second price sealed bid auction or in the japanese ascending auction.

Within the standard independent and private values framework, with risk-averse bidders, we derive conditions under which a multistage mechanism can yield higher expected revenue than a first-price auction in one stage or an ascending auction. In the one hand, the computational cost of first-price sealed mechanisms can discourage the bidders’ participation and thus reduce competition in sealed bid auction. Therefore, the ascending auction tends to yields higher expected revenue. However, in the second hand, when the bidders are risk-averse, with an equal number of competitors, the first price auction dominates the ascending auction. Actually, the bidding strategy is more aggressive than the one adopted by risk neutral bidders. Besides, in an ascending auction, the bidders’ strategy is not modified by risk-aversion. The dilemma of choosing between English and sealed-bid auction can so be partially solved by the use of multistage auction, which consists in an ascending auction followed by a first price sealed-bid auction. It can then aggregate the advantages of both auctions: to attract more competitors in the first stage and to incite with more aggressive bids in the second stage when the bidders are risk-averse. Strategic non-participation during the second stage of the multistage auction is reduced compared to the non-participation in a first price sealed bid auction in one stage: competing with fewer opponents in the second stage makes entry more profitable to bidders than supporting the computation cost before large competition in a single stage sealed auction.

The paper is organized as follows. In section 2, we define the general framework of the model. Section 3 compares the mechanisms from the point of view of the seller’s expected revenue in the
absence of participation cost. Section 4 shows the interest of the multistage auction in the presence of a strategic participation cost when bidders are risk-averse. Section 5 concludes.

2 Assumptions and notations

Let us consider \( N \) bidders \( i = 1, ..., n \) competing for a good. Each bidder \( i \) has a private valuation for the object noted \( v_i \). These valuations are identically and independently drawn from a cumulative function \( F(.) \) over \([0, 1]\), with \( f(.) \) the strictly positive associated density.

All the bidders are risk averse, with a CRRA utility function given by:

\[
u(x) = x^\theta \quad \forall x \geq 0, 1 > \theta > 0\]

where \( 1 - \theta \) is the CRRA parameter.

In the following, we’ll consider three alternative mechanisms: the standard first-price sealed bid auction (hereafter FPSA), the ascending auction (hereafter AA) and the multistage auction (hereafter MSA).

During the MSA, an ascending auction (Japanese one) first takes place. The auctioneer continuously increases the price until only \( k \) bidders remain active\(^1\). These \( k \) remaining bidders can participate to the second stage and the object is allocated by the means of a first price sealed bid auction with reserve price sets to the ending price of the first stage ascending auction. Remark that \( k = 2 \) corresponds to the Klemperer’s (2001) Anglo-Dutch Auction.

Participating to a first-price sealed bid auction (in the SBA or in the second stage of the MSA) involves a cost \( c \), whereas participating to an ascending auction is assumed costless. Such assumption implicitly reflects the ”computational” or ”strategic” cost associated with deriving optimal strategy in first price sealed bid auction. Facing such a cost, bidders only know the number of potential participants. After seeing their values for the object, potential participants decide whether or not to enter the auction. They may not want to enter the auction since they have to pay participation costs. Finally, it is worth noting the assumption that ascending auction is costless is consistent with numerous comments of the ”strategic simplicity” of such a mechanism (see e.g. McMillan 1994).

\(^1\)This mechanism may be implemented by a proxy bidding system
3 Bidding strategies and preliminary results

Before turning to consider the entry decisions when bidding may be costly, let us characterize equilibrium bidding strategies for both standard FPSA, AA and MSA when participation is exogenous, that is assuming \( c = 0 \). It will allow us to compare the expected revenues generated by each mechanism when all the \( N \) potential acquirers participate to the auctions. It will shed light on some interesting preliminary results.

3.1 Strategies and expected revenue in traditional mechanisms.

In an ascending auction, it is well known that bidding up to her actual value is a dominant strategy. Risk-aversion has no effect on a bidder’s optimal strategy.

Thus, the seller’s expected revenue corresponds to the second highest bidder’s expected value:

\[
ER_A = \int_0^1 vn(1 - F(v))F(v)^{n-1} f(v) dv
\]  

In the first-price sealed bid auction (FPSA), each bidder’s optimal strategy consists in bidding according to the following symmetrical increasing bid function \( b_1(.) \), solution of:

\[
\max_{b_1} (v_i - b_1(v_i))^{\theta} F(b_1^{-1}(b_1(v_i)))^{n-1}
\]

And we easily obtain: (cf. appendix A):

\[
b_1(v) = v - \int_0^v F(s) \frac{(n-1)}{(n-s)} ds \quad \forall v \in [0,1]
\]  

This optimal strategy corresponds to bidding the true value less a strategic mark-up. This latter depends on belief relative to the opponent values. Risk aversion tends to increase the bidders’ aggressiveness, reducing the mark-up. The seller’s expected revenue corresponds to the expected highest bidder’s strategy:

\[
ER_{FPSA} = \int_0^1 \left( v - \int_0^v F(s) \frac{(n-1)}{F(v)} ds \right) nF(v)^{n-1} f(v) dv
\]  

and we have:

\[
ER_{FPSA} \geq ER_A
\]

with equality for \( \theta = 1 \), i.e. when bidders are risk-neutral.
3.2 The multistage auction

The multistage auction is divided into two stages. The first one consists in a button ascending auction which stops when they only remain \( k \) bidders who are selected for the second stage. This one is a first-price sealed bid auction where the highest bidder wins the item. The reserve price at the second stage is the highest one reached at the first stage. Then,

**Lemma 1** At the first stage, the dominant strategy consists in bidding up to her own reservation value. The starting price at the second stage corresponds to the reservation value of the \((k+1)\) highest bidder, \( v_{k+1} \).

**Proof.** During the first stage, bidders have no incentive to overbid their reservation value as the starting price at the second stage is the highest price reached at the first stage. Thus, using similar arguments as in a traditional button ascending auction, the dominant strategy consists in bidding up until her own reservation value. Therefore, the highest price reached corresponds to the reservation value of the \((k+1)\) highest bidder, \( v_{k+1} \), since the auction stops when only \( k \) bidders remain active.

At the second stage, each bidder bids according to the symmetric bidding strategy derived in the preceding section, given that it is now common knowledge that they are only \( k \) bidders whose valuation \( v_i \) is higher than \( v_{k+1} \). Then, *bidders’ beliefs concerning the distribution of competitors’ valuations are updated* and it is now commonly known that the \( k \) bidders’ valuations are independently drawn from the same cumulative function, \( G(.) \), with a positive and continuously differentiable density function \( g(.) \) on \([v_{k+1}, 1]\) such as:

\[
G(v) = \frac{F(v) - F(v_{k+1})}{1 - F(v_k)} \quad \forall v \in [v_{k+1}, 1]
\]

Each bidder’s optimal strategy consists in bidding according to the following symmetric increasing bid function \( b_2(.) \):

\[
b_2(v) = v - \int_{v_{k+1}}^v \left( \frac{F(s) - F(v_{k+1})}{1 - F(v_k)} \right)^{\frac{k-1}{k}} ds = v - \int_{v_{k+1}}^v \left( \frac{F(v) - F(v_{k+1})}{1 - F(v_k)} \right)^{\frac{k-1}{k}} ds
\]

(4)

And the seller’s expected revenue corresponds to the expected highest bid among the \( k \) bidders:

\[
\int_{v_{k+1}}^1 \left( v - \frac{\int_{v_{k+1}}^v \left( \frac{F(s) - F(v_{k+1})}{1 - F(v_k)} \right)^{\frac{k-1}{k}} ds}{F(v) - F(v_{k+1})} \right) \left( \frac{f(v)}{1 - F(v_{k+1})} \right)^{k-1} dv
\]
in expectation over \( v_{k+1} \), the \( k + 1 \) highest valuation among the \( N \) bidders. Then, we obtain:

\[
ER_{MSA} = \int_0^1 \left( \int_{v_{k+1}}^1 \left( F\left(\frac{n}{(n-k-1)\theta}F\left(v_{k+1}\right)^{n-k-1}\left(1-F\left(v_{k+1}\right)\right)^k f\left(v_{k+1}\right)\right) \frac{n}{(n-k-1)\theta} dv \right) k \left( \frac{f\left(v\right)}{1-F\left(v_{k+1}\right)} \right)^{k-1} dv \right) dv_{k+1}
\]

with \( k \left( \frac{f\left(v\right)}{1-F\left(v_{k+1}\right)} \right) \left( \frac{F\left(v\right)-F\left(v_{k+1}\right)}{1-F\left(v_{k+1}\right)} \right)^{k-1} \) the density of the first rank order statistics among the \( k \) highest bidders and \( \frac{n}{(n-k-1)\theta}F\left(v_{k+1}\right)^{n-k-1}\left(1-F\left(v_{k+1}\right)\right)^k f\left(v_{k+1}\right) \) the \( k+1 \)st density of order statistics among the \( N \) bidders.

### 3.3 Seller’s revenues comparisons

In the multistage auction, all the \( k \) highest bidders remain active during the second stage and the highest bidders amongst the \( k \) wins the second stage. Hence, as the ascending auction or the FPSA, the multistage auction is an efficient mechanism, allocating the object to the bidder with the highest valuation. Thus, with risk-neutrality, independent private values, the revenue equivalence theorem applies and the three auction mechanisms yield the same expected revenue.

**Remark 1** When bidders are risk-neutral and have no strategic cost, the multistage auction is equivalent to the two other mechanisms, whatever \( k \), the number of competitors in the second stage.

Of course, risk-aversion modifies this result and we can easily show that:

**Lemma 2** When bidders are risk-averse, the FPSA yields higher expected revenue for the seller than the multistage auction which yields higher expected revenue than the ascending auction.

**Proof.** see appendix B. ■

This result is rather intuitive. Let us first compare the two traditional mechanisms. In a FPSA, risk-aversion induces the bidders to bid more aggressively but risk-aversion has no effect on the bidder’s optimal strategy in an ascending auction. Therefore, the FPSA yields a higher expected revenue for the seller. Basically, risk-aversion impacts the strategic mark-up of the bidders; with our utility function specification, each bidder bids as if she was competing against \( \frac{(n-1)}{\theta} > (n-1) \) opponents for \( \theta < 1 \). (cf equation 2).

Of course, this "aggressiveness effect" also applies in the second stage of the multistage auction where (cf. equation 4) the "virtual" competition is now of \( \frac{(k-1)}{\theta} \) competitors. But this aggressiveness
effect is lower than in the FPSA. Implicitly, in the FPSA, bidder $i$ expects to pay the expected highest of the $\frac{n-1}{\theta}$ bidders’ values, conditional on her own value being the highest. In the multistage auction, the $k$ highest bidders expect to pay the expected highest of the $\left(\frac{k-1}{\theta}\right)$ opponent’s values conditional on her own value being highest and conditional on the reserve price $v_{k+1}$, the $k+1$ highest valuation among the $N$ bidders. But ex ante (in expectation over $v_{k+1}$), the aggressiveness effect of the risk-aversion do not modify this value.

Under such assumptions, with an equal number of competitors participating in all the three alternative mechanisms, there is no reason why a seller may decide to use a multistage auction: a revenue maximizing seller will ever prefer a FPSA when facing risk averse bidders.

Nevertheless, looking on how the seller’s expected revenues varies with the level of participation helps us to highlight some trade-off we will more deeply analyze in the following section, where the participation will be supposed to be endogenous. Therefore, we will suppose thereafter, in order to provide an explicit formulation of the expected revenues, that $F(\cdot)$ is a uniform cumulative function. In this case, we obtain for respectively $n$, $n'$ and $n''$ competitors:

\[
\begin{align*}
ER_{FPSA} &= \frac{n(n-1)}{(n+1)(n-1+\theta)} \\
ER_A &= \frac{n'-1}{n'+1} \\
ER_{MSA} &= \frac{n''(k+\theta-1)-k\theta}{(n''+1)(k+\theta-1)}
\end{align*}
\]

Hence, of course, with different participation levels, the preceding ranking of auction formats does not hold anymore; any increase in competition increases the seller’s expected revenue generated by each mechanism. Intuitively, the multistage auction will dominate the first price sealed-bid auction if the increase in competition more than offsets the lack of aggressiveness of the $n-k$ first bidders not bidding during the second stage. Following the same argument, the ascending auction will dominate the multistage auction as soon as the participation increase more than offsets the lack of aggressiveness of the $k$ bidders remaining in the second stage of the multistage auction.

The use of a multistage auction can so be justified only if the first of the two effects is checked. In any alternative situations, the seller will either prefer the ascending auction if it ensures a very high level of participation, or the FPSA in order to benefit from the bidders’ risk-aversion.

But, the number of additional bidders which enables the multistage auction to dominate the FPSA may be lower than that necessary to the ascending auction to dominate the multistage auction.
Formally, the three procedures are revenue-equivalent when \( n \) participates to the FPSA, \( n' \) to the multistage auction and \( n'' \) to the ascending auction with:

\[
\begin{align*}
n' &= n + \frac{(k - n)(n + 1)(\theta - 1)\theta}{(k + \theta - 1)(n - 1 + \theta + n\theta)} \\
n'' &= n' + \frac{(1 - k)(1 + n)(-1 + \theta)(-1 + n + \theta)}{(k + \theta - 1)(n - 1 + \theta + n\theta)}
\end{align*}
\]

We verify easily that:

\[
n' - n > n'' - n' \iff \frac{(n^2 - 1)(1 - \theta)}{n - 1 + \theta + n\theta} > 0
\]

The additional participation necessary to make the ascending auction’s expected revenue higher than the multistage auction’s one is quite higher than the additional participation which makes the latter higher than the FPSA’s expected revenue.

In a similar way, we can determine the minimal number \( k \) of bidders at the end of the first stage of the multistage auction so that the entry of only one additional competitor during the first stage is enough to ensure an higher expected revenue than the FPSA. This value of \( k \) (by neglecting the constraints of integer values) is given by:

\[
k > \frac{(1 - \theta)(n - 1 + (1 + n)^2\theta)}{-\theta^2 + n(2 - \theta)\theta + 1}
\]

The following graph illustrates this value threshold for \( n = 3, ..., 100 \) according to the degree of risk-aversion.
4 Multistage auction with endogenous entry

The preceding section has shed light on the value, for the seller, of increased competition and showed why entry of an additional bidder may be more profitable in the multistage auction than in an ascending auction, while entry was implicitly assumed exogenous. We depart now from this standard hypothesis and examine auctions where potential bidders, after seeing their values for the object, decide whether or not to enter the auction. They may not want to enter the first price sealed bid auction or the second stage of the multistage auction since they have to pay a participation cost, \( c \). This computational cost is sunk once the strategic bid is computed, and a bidder can not recover this cost if, once the bidding strategy determined, she decides not to enter the auction.

We characterize equilibrium bidding strategies and entry decisions for both first-price sealed-bid and multistage auctions when participation is endogenous. Following Menezes and Monteiro (2000), we show, for each mechanism, that there is a pure strategies entry equilibrium where only bidders with values greater than a certain cut-off point actually bid in these auctions. This cut-off point is determined such that each bidder is indifferent between participating or not. Because the bidding strategy is computationally simpler in an ascending auction, where remaining active as long as the price is above the valuation is a dominant strategy, we assume that the ascending auction is less costly for the bidder and, in order to alleviate notations and without loss of generality, we normalize this cost to zero.

So let us in the following assume that computing bids generates a cost \( c \) for the \( N \) potential bidders of the first price sealed bid auction or alternatively for the \( k \) potential bidders remaining after the first stage of the multistage auction whereas all the \( N \) potential bidders actually participate to the ascending auction.

4.1 Bidding and entry strategies

Given their values, each bidder decides whether or not to submit a bid (and pay \( c \)) without knowing how many bidders will submit bids. In what follows, we characterize the individual participation decision and derive the optimal bidding strategy for both first-price sealed-bid auctions and multistage auction.

Let us first consider the first-price sealed-bid auction. The timing is as follows: each bidder privately learns her own value. Given this value and her prior belief, she decides whether or not to enter the auction. If she decides to enter, she determines her bidding strategy and support cost \( c \). The
seller reveals the number of actual bidders and bidders transmit their offers. Intuitively, entry occurs if and only if the expected utility for a bidder is greater than zero. Following Menezes and Monteiro (2000), to derive the optimal bidding strategy, let us suppose a cutoff value $v_s$ is given and that bidder $i$ participates if and only if $v_i \geq v_s$.

The cut-off point is such that if a participant has a value $v_i = v_s$ then she is indifferent between entering and not entering the auction. In a nutshell, assuming bidding strategies increasing with the bidder’s value, a participant with value just equal to the cut-off $v_s$ wins only if she is the sole participant. Bidders with values below $v_s$ do not enter the auction. Thus, the cutoff is defined as a solution of:

$$U(v_s - c)F^{n-1}(v_s) + U(-c)(1 - F^{n-1}(v_s)) = 0$$

The first term of the equation reflects the expected utility of a bidder with value $v_s$ winning the auction. This latter can only occur if the bidder is the sole participant, and the successful bidder supports cost $c$. The second term of the equation reflects the cost generated by computing the bid while the bidder is unsuccessful which occurs as soon as at least one other bidder enters the auction. Collecting terms enables us to derive the equation determining implicitly the cut-off value $v_s$, solution of: \(^2\)

$$(U(v_s - c) - U(-c))F^{n-1}(v_s) = -U(-c)$$

Remark that in order to decide whether or not to enter the auction, the agents is not induced to compute the equilibrium bid so that entry decision is computationally costless. The entry decision is only based on the fact that a bidder indifferent between entering or not will only win if no other potential bidders enter the auction. Namely, in an efficient auction, a bidder wins only if she has the highest value (with continuous distribution of value, the probability of two agents having the same value equals zero). With value just equal to the cut-off value, having the highest value can only happen when no one of the potential $n - 1$ opponents enter.

In order to alleviate notations and to derive explicit solutions, we will assume in the following $U(x) = x \forall x < 0$. We tackle with risk aversion only if the agents enter the auction. The preceding equation can so be rewritten:

$$((v_s - c)^\theta + c)F^{n-1}(v_s) = c$$

\(^2\)It can be easily shown that with $c < 1$, it always exists a unique $c < v_s < 1$ satisfying the equation.
Only bidders $i$ with value $v_i \geq v_s$ enter the auction. The auctioneer publicly reveals the number $h$ of actual bidders. Each participant can therefore updates her belief, knowing that no one of the $h - 1$ opponents has value lower than $v_s$. Technically, it is now common knowledge that values are independently drawn from the continuous distribution $F_s(v_i) = \frac{F(v_i) - F(v_s)}{1 - F(v_s)}$ with support $[v_s, 1]$. Following similar development to that of the previous section, the optimal symmetric equilibrium bidding strategy for a bidder with value $v_i$ facing $h - 1$ opponents can be written:

$$b_1(v_i, h) = v_i - \int_{v_s}^{v_i} F_s(s) \frac{t^{h-1} ds}{F_s(v)} \forall v \in [v_s, 1]$$

(6)

Remark that bidders adapt their strategies to the level of actual competition, reducing their mark-up as competition increases. Nevertheless, from the auctioneer point of view, the level of actual participation being unknown before the auction takes place, his expected revenue corresponds to:

$$ER_{FPSA} = \sum_{h=2}^{n} \binom{n}{h} h F(v_s)^{n-h} \int_{v_s}^{1} b_1(v, h)(F(v) - F(v_s))^{h-1} f(v) dv$$

(7)

where $\binom{n}{h} h F(v_s)^{n-h}$ corresponds to the probability that $h$ agents enter the auction that is having value greater than $v_s$. Note that if only one bidder enters the game, then he will optimally offer 0. From the previous expression, we can conclude that allowing participation to be endogenous reduces the auctioneer’s revenue given a fixed population of potential bidders.

Let us now turn to consider the multistage auction. Similarly to the previous analysis, the $k$ agents selected after the first stage of the multistage auction can also decide to give up, and not participate to the second stage. Nevertheless, because the first stage of the multistage auction is an ascending mechanism, which induces no cost, all the $N$ potential bidders actually enter the first stage. Each one can, after the first stage and before the second step, if selected, decide to support the computational cost $c$ in order to prepare the second stage bid. Actually, given their values and the price reached during the first stage, but which is unknown ex-ante, each bidder decides whether or not to submit a bid (and pay $c$) without knowing how many bidders amongst $k$ will submit bids.

The timing is as follows: each bidder privately learns her own value. All the $N$ bidders enter the first stage. $k$ bidders are selected. Given the private value and the price reached, if selected, the agent decides whether or not to enter the second stage of the auction. If she decides to enter, she determines her bidding strategy and support cost $c$. The seller reveals the number of actual bidders amongst the $k$ remaining ones and bidders transmit their offers.

Because entering the first stage and bidding up to her own value is a dominant strategy, it is common knowledge that each one of the $k$ remaining bidders has value greater than the price reached.
during the first stage. The cut-off point is now such that if a participant has a value \( v_i = v_{\sigma} \) then she is indifferent between entering and not entering the auction so that the cutoff is defined as a solution of:

\[
(u(v_{\sigma} - c) + c) \left( \frac{F(v_{\tau}) - F(v_{k+1})}{1 - F(v_{k+1})} \right)^{k-1} = c
\]

where \( v_{k+1} \) corresponds to the price reached during the first stage which gives an implicit relationship between \( v_{\tau} \) and \( v_{k+1} \). Let \( v_{\sigma}(v_{k+1}) \) denotes this cut-off value. In this case, if one of the \( k \) remaining bidders has a value just equal to the cut-off value, she can win only if no one the \( k-1 \) opponents enters, each of them having value greater than \( v_{k+1} \).

The optimal symmetric equilibrium bidding strategy for a bidder with value \( v_i \) facing \( h - 1 \) opponents amongst \( k \) pre-selected potential bidders can be written:

\[
b_2(v_i, h) = v_i - \int_0^{v_i} F_\sigma(s) F_{v_i}^{(h-1)} ds \quad \forall v \in [v_{\sigma}, 1]
\]

where \( F_\sigma(s) = \frac{F(s) - F(v_{\tau})}{1 - F(v_{\tau})} \).

Basically, entering the second stage of the multistage auction conveys two informations. First of all, the \( k \) remaining agents have a value greater than \( v_{k+1} \), drawn from the continuous distribution\( F(v) - F(v_{k+1}) \). But only the agents with a value greater than the cut-off \( v_{\sigma} \) entering the second stage, the updated distribution of values becomes

\[
\frac{F(v) - F(v_{k+1})}{1 - F(v_{k+1})} = F(v) - F(v_{\sigma}) = F_\sigma(v).
\]

For a given \( v_{k+1} \), the expected revenue for the seller corresponds to:

\[
R_{MSA} = v_{k+1}(kG(v_{\sigma})^{k-1}(1 - G(v_{\sigma})) + \sum_{h=2}^{k} \frac{k!}{(k-h)!h} F(v_{\tau})^{n-h} \int_{v_{\sigma}}^{1} b_2(v, h)(F(v) - F(v_{\sigma}))^{h-1} f(v) dv)
\]

If no one of the \( k \) selected agents participate to the second stage, the revenue of the auction is equal to zero. If only one agent enters the second stage, then she offers no more than the reserve value \( v_{k+1} \), and it occurs if only one agent has value greater than \( v_{\sigma} \) that is with probability \( kG(v_{\sigma})^{k-1}(1 - G(v_{\sigma})) \). Finally, if more than one bidder enter the second stage (\( h = 2 \) to \( k \) ), then the object is allocated to the highest bidder.

The latter equation enables us to derive the seller’s expected revenue, given that \( v_{k+1} \) the price reached during the first stage corresponds to the value of the \( k+1 \)th higher bidder amongst \( N \):

\[
ER_{MSA} = \int_0^{1} R_{MSA} \frac{n!}{(n-k-1)!k!} (F(v_{k+1})^{n-k-1})(1 - F(v_{k+1}))^{k} f(v_{k+1}) dv_{k+1}
\]
4.2 Mechanisms comparison

As soon as we assume that preparing a bid for a first-price sealed-bid auction involves a sunk cost for participating bidders, entry in the auctions must be assumed to be endogenous. Therefore, there is no longer raison why all the three alternative mechanisms, namely ascending auction, first price auction and multistage auction, should yield the same level of participation. Actually, for each mechanism, we have shown that there is a pure strategies entry equilibrium where only bidders with values greater than a certain cut-off point actually bid in these auctions. This cut-off point is determined such that each bidder is indifferent between participating or not. More precisely, because the bidding in an ascending auction is supposed costless, it is also the only mechanism where all the potential bidders actually enter the auction. Besides, either for the first-price sealed-bid auction or for the multistage auction, the number of actual bidders is reduced.

Nevertheless, the impact of actual competition reduction on the expected revenue of the first-price sealed-bid auction is not the same as the one of the multistage auction. This result holds even if the participation cost is the same for both mechanisms. In the first case, the cut-off value defined by (5) may lead a large number of potential bidders to give up the auction. Such bidders will remain active during the first stage of the multistage auction. But, during the second stage, because the potential competition is reduced to \( k \) potential bidders, the strategic non-participation implicitly defined by (8) may be reduced.

Hence, we obtain the following proposition:

**Proposition 3** With risk averse bidders, and participation cost, the seller’s expected revenue of the multistage auction can be greater than the expected revenue of the ascending auction and greater than the expected revenue of the first-price sealed-bid auction.

The following illustration is a sufficient proof of the preceding proposition.

**Example 1** Assume \( n = 10 \) and \( k = 2 \) and consider \( \theta = \frac{1}{2} \) and \( c = \frac{1}{20} \).

Even with this small participation cost, given (5) only agents with value \( v_i \geq 0.7277 \) participate to the first-price sealed-bid auction. Here, competition reduction is large.

In order to compare the competition reduction between the first-price sealed-bid auction and the multistage auction, consider, say, that at the end of the first stage of the multistage auction, the price reached \( v_k = 0.8 \). In this case, the cut-off value of the second stage \( v_\sigma = 0.8108 \), with \( v_\sigma \) close to \( v_k \), the probability of having all the \( k = 2 \) agents bidding in the second stage remains high. With \( n \) bidders
entering the first stage and \( k \) bidders entering the second stage with high probability, almost all the potential bidders actually participate.

Finally, all the \( n = 10 \) bidders participate to the ascending auction. But, with risk-averse bidders, the ascending auction does not benefit from strategic aggressiveness of sealed-bid mechanisms.

Hence, even if the bidders suffer from the same participation cost in the first-price sealed-bid auction and during the second stage of the multistage auction, the actual competition decrease is reduced. Simultaneously, because the second stage of the multistage auction is a sealed-bid first-price auction, it benefits from the risk aversion of the bidders. Because during the multistage auction, the strategic entry decision is made after a first competition between \( n \) bidders, the remaining potential acquirers are more induced to participate to the second stage. The following table exhibits the seller’s expected revenue for the three alternative mechanisms:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>MSA</th>
<th>FPSA</th>
<th>AA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected revenue</td>
<td>0.8404</td>
<td>0.7124</td>
<td>0.8182</td>
</tr>
</tbody>
</table>

5 Conclusion

This paper gives a theoretical support for the use of a multistage auction, within the IPV paradigm. Assuming computational costs for bidding in a first price sealed bid mechanism, participating to the auction must be viewed as a strategic decision, so that entry becomes endogenous. Deriving entry and bidding strategies in ascending, first-price sealed bid and multistage auctions, we have shown that the latter can generate higher expected revenue than the traditional one-stage mechanism. For each mechanism, an increase in the competition level, tends to increase the revenue, while risk aversion tends to favor sealed bid mechanisms. But, because potential competition is reduced \( \text{per se} \) in the second stage of the multistage auction, relative to the first-price sealed bid auction, strategic non-entry is less stringent. This competition effect can more than offset the lack of aggressiveness effect during the first stage of ascending auction with risk averse bidders. Nevertheless, relative to the ascending auction, because the second stage remains a first-price sealed-bid one, even if all the potential bidders participate to the ascending auction, the aggressiveness effect of the second stage can more than offset the lack of competition effect during the second stage.

However this result cannot be generalized. The domination of the multistage auction on both the first price sealed-bid auction and the ascending auction closely depends on the strategic participation cost which in turn condition the strategic entry decision of the agents. If the participation cost is
low, the actual participation in the FPSA will be higher and the multistage auction will be of less interest. In addition, the multistage auction is justified only in the situation where the bidders have risk-aversion: the second stage, which is a first price sealed-bid mechanism, provides incentives to more aggressive bidding. When the bidders are risk-neutral, insofar as the strategic participation cost is equal to zero, the ascending bidding must be preferred.

References


6 Appendix

6.1 Appendix A

The bidder’s expected utility is given by:

\[ EU_1(w) = (v - b(w))^\theta F(w)^{n-1} \]

Then, we have:

\[ \left. \frac{\partial EU_1(w)}{w} \right|_{w=v} = (n-1)F(v)^{n-2}f(v)(v - b(v))^\theta - \theta(v - b(v))^{\theta-1}b'(v)F(w)^{n-1} = 0 \]

which becomes

\[ (n-1)f(v)(v - b(v)) = \theta b'(v)F(v) \]

with the limit condition \( b(0) = 0 \)

\[ b'(v) = -\frac{(n-1)f(v)}{\theta F(v)} b(v) + \frac{(n-1)f(v)v}{\theta F(v)} \]

We then obtain:

\[ b(v) = e^{\int_0^v (-\frac{(n-1)f(s)}{\theta F(s)})ds} \left[ \int_0^v e^{\int_0^s (-\frac{(n-1)f(\xi)}{\theta F(\xi)})d\xi} (n-1)f(s)s \frac{1}{\theta F(s)} ds \right] \]
\[ b(v) = F(0)^{(n-1)} F(v)^{-(n-1)} 0 + F(v)^{-(n-1)} \left( \int_0^v \left( F(s)^{(n-1)} (n-1)f(s)s \right) ds \right) \]

with
\[ (F(s)^{(n-1)})' = \frac{(n-1)f(s)}{\theta F(s)} \]

So:
\[ b(v) = F(0)^{(n-1)} F(v)^{-(n-1)} 0 + F(v)^{-(n-1)} \left( v(F(v)^{(n-1)}) - \int_0^v F(s)^{(n-1)} ds \right) \]

and we have:
\[ b_{FPSA}(v) = v - \frac{\int_0^v F(s)^{(n-1)} ds}{F(v)^{(n-1)}} \]

### 6.2 Appendix B

The ranking of first-price sealed-bid and ascending auction with risk-averse bidders and CRRA utility function is a well known result. Because the second stage of the multistage auction is an ascending one, similar argument applies.

Let us now compare FPSA and the multistage auction. Invoking the revenue equivalence theorem, with risk neutral agents and for the same number \( N \) of bidders, whatever the number \( k \) of bidders selected for the second stage, each multistage auctions is revenue-equivalent.

As for example, ex-ante, the bidding process amongst \( k \) bidders at the second stage with a reserve price of \( v_{k-1} \) is equivalent with a bidding process amongst \( k+1 \) bidders with a reserve price of \( v_k \).

When bidders are risk averse, for the same distribution of \( v_{k-1} \) and \( v_k \), the gap in the competition level is no more \( (k+1) - k \) but rather \( \frac{k+1}{\theta} - \frac{k}{\theta} = \frac{1}{\theta} > 1, \forall \theta < 1 \).

Given that the bidding strategy \( b_2(.,.,) \) is increasing with respect to the number of competitors and given that the gap in the competition level is larger, the two multistage auctions, with respectively \( k \) and \( k+1 \) remaining bidders are no more equivalent. Besides, the multistage auction with \( k+1 \) remaining bidders yields higher expected revenue. Hence, we can conclude that the expected revenue of the multistage auction is increasing with respect to \( k \). But, noting that the case \( k = N \) corresponds to the first-price sealed-bid auction establishes the proof.