Auctions with Financial Externalities*

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Abstract

We study auctions with financial externalities, i.e., auctions in which losers care about how much the winner pays. In the first-price auction, larger financial externalities result in lower expected revenue, while the revenue of the second-price auction may increase if financial externalities increase. Still, the seller cannot gain more revenue by guaranteeing the losers a fraction of the auction revenue. With a reserve price, we find that both auctions may have pooling at the reserve price. This finding suggests that identical bids need not be a signal of collusion, in contrast to what is sometimes argued in anti-trust cases.

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1 Introduction

In this paper, we study sealed-bid auctions with financial externalities. Financial externalities arise when losers benefit directly or indirectly from a high price paid by the winner(s). In auction theory, it is generally assumed that losers are indifferent about how much the winner(s) pay(s). However, in real life auctions, this assumption may be false. In reality, an auction is not an isolated game, as winners and losers also interact after the auction. Paying a high price in the auction may make a winner a weaker competitor later.

The series of UMTS (third generation mobile telecommunication) auctions that took place in Europe offers a concrete example of auctions where losers benefit *indirectly* from a high price paid by the winners. In this context, there are at least three ways how firms that do not acquire a license may benefit from a winning firm paying a high price. First, the share values of winning firms may drop, which makes the winner vulnerable to a hostile take-over by competing firms. For instance, the drop of the share value of the Dutch telecom company KPN with about 95% is partly explained by the huge amount of money the company spent to acquire British, Dutch and German UMTS licences.[1,2] Second, if firms are budget constrained, a high payment in the first auction may give competing firms an advantage in the later auctions. Third, high payments may force the winning firms to cut their budget for investment, which may be favorable for the losers' position in the telecommunications market, as the losing firms are not only competitors of the winning firms in the auction, but in the telecommunications market as well. Indeed, Börgers and Dustmann (2001) argue that financial externalities may (partly) explain seemingly irrational bidding in the British UMTS auction.

Financial externalities occur *directly* when losing bidders get money from the winner(s). For instance, this may happen in the case of bidding rings, in which a member of the ring receives money when she does not win the object (McAfee and McMillan, 1992). Also, partnerships are dissolved using an auction in which losing partners obtain part of

[1] In the UK, KPN bought part of the TIW license after the auction. In Germany, KPN has a majority share in E-plus.

[2] The other part of the drop is probably explained by the changed sentiment in the market.

the winner's bid (Cramton et al., 1987). Furthermore, the owner of a large estate may specify in his last will that after his death, the estate is sold to one of the heirs in an auction, where the auction revenue is divided among the losers (Engelbrecht-Wiggans, 1994). Finally, in the 'Amsterdam auction', the loser receives a premium proportional to the difference between his bid and the minimum price (Goeree and Offerman, 2004).[3]

In Section 2, we present a model of bidding in sealed-bid auctions with financial externalities. The first-price sealed-bid auction (FPSB) or the second-price sealed-bid auction (SPSB) is used to sell an indivisible object. We assume an independent private signals model, with private values and common value models as special cases. Financial externalities are exogenously given and modelled by a parameter φ that is inserted in the bidders' utility functions. This is the simplest extension of the independent private signals model which incorporates financial externalities. Despite its admitted simplicity, this model appears to be sufficiently rich to generate interesting insights.

In Section 3, we derive results for FPSB and SPSB without reserve price. We find a unique symmetric and efficient bid equilibrium for each of the two auction types. Equilibrium bids in FPSB decrease as φ increases. An intuition for this result is that larger financial externalities make losing more attractive for the bidders so that they submit lower bids. The effect of financial externalities on the equilibrium bids in SPSB is ambiguous. A possible explanation is that in SPSB, a bidder is not only inclined to bid less the higher is φ (as she gets positive utility from losing), she also has an incentive to bid higher, because, given that she loses, she is able to influence directly the level of payments made by the winner. We construct an example in which the seller's revenue increase if φ increases. This finding suggests that the seller may gain more revenue by guaranteeing the losers a fraction of the auction revenue. This, however, turns out not to be the case. Finally, we give a revenue comparison between FPSB and SPSB. We find that SPSB results in a higher expected revenue than FPSB.

In Section 4, we characterize equilibrium bid strategies for the case that a reserve price is imposed in FPSB and SPSB. For simplicity, we assume a model with independent private values. In this section, we introduce the concept of a weakly separating Bayesian

[3] More examples can be found in Goeree et al. (2005).

Nash equilibrium, which is an equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function. We find that FPSB has no weakly separating Bayesian Nash equilibrium. However, we derive an equilibrium in which bidders with low signals abstain from bidding, bidders with intermediate signals pool at the reserve price, and bidders with high signals submit a bid according to a strictly increasing bid function. SPSB has a weakly separating Bayesian Nash equilibrium if and only if the reserve price is sufficiently low. Otherwise, the equilibrium involves pooling at the reserve price.

These findings may shed a new light on observations of identical bids in auctions. Theoretical work suggests that several bidders submitting a bid equal to the reserve price is a signal that they are colluding.[4] Indeed, Comanor and Shankerman (1976) report that competition authorities in the US required auctioneers to report instances of identical bids. Moreover, in *U.S. versus Pfizer*, the court stated: "Certainly, uniformity of price may be and has been considered some evidence tending to establish an illegal agreement".[5] However, our finding indicates that several bidders submitting a bid equal to the reserve price may be explained by the existence of financial externalities instead of collusion.

The most closely related paper to ours is Engelbrecht-Wiggans (1994), who considers an auction game in which each bidder receives an equal share α of the revenue. He characterizes equilibrium bid functions for both FPSB and SPSB, and gives a revenue comparison between these two auction types.[6] The comparative statics in our model and Engelbrecht-Wiggans' model (the effect of φ respectively α on the equilibrium bids and the seller's revenue) turn out to be different. Engelbrecht-Wiggans shows that the equilibrium bid functions of FPSB and SPSB are increasing in α. In our model, the effect of φ on the equilibrium bids can be both increasing and decreasing. In addition to Engelbrecht-Wiggans' study, we analyze the effect of the reserve price on the equilibrium bids.

[4] See, e.g., McAfee and McMillan (1992).

[5] United States versus Chas. Pfizer Co., Inc. (1963).

[6] Several other papers make use of Engelbrecht-Wiggans' model. Ettinger (2003) extends the model by allowing the revenue shares to differ among the bidders and by introducing reserve prices. Engers and McManus (2004) study *charity auctions*, in which bidders receive a *warm glow* from the auction revenue, so that their utility depends on the auction revenue. Goeree et al. (2005) compare standard auctions with k-th price all-pay auctions in Engelbrecht-Wiggans' environment.

There are several other papers related to ours. Goeree et al. (2005) study winner-pay and all-pay auctions in Engelbrecht-Wiggans' setting in the context of fund-raising mechanisms. They show that winner-pay auctions are inept fund-raising mechanisms because of the positive externality bidders forgo if they top another's high bid. Revenues are suppressed as a result and remain finite even when bidders value a dollar donated to the public good the same as a dollar kept. This problem does not occur in all-pay auctions where bidders have to pay irrespective of whether they win or lose. They show that the optimal fund-raising mechanism is among the all-pay formats they consider.

Jehiel and Moldovanu (1996, 2000), and Jehiel et al. (1996, 1999) study auctions in which losing bidders receive positive or negative allocative externalities from the winner. Since the utility of the bidders is affected by the identity of the winner and not by how much she pays, these externalities are clearly different from financial externalities. Jehiel and Moldovanu (2000) derive equilibrium bid strategies that involve some pooling at the reserve price for SPSB with a reserve price and positive externalities. This equilibrium structure is similar to the one we found in the case of financial externalities.

Benoît and Krishna (2001) study a two-bidder model with complete information in which two objects are sold sequentially. As bidders are budget constrained, a particular bidder's payoff is affected by the price paid by a rival bidder, so that their model can be interpreted as a model with endogenously determined financial externalities.

2 The model

We consider a situation with $n \geq 2$ risk neutral bidders, numbered $1, 2, ..., n$, who bid for one indivisible object. The auction being used is either FPSB or SPSB. Each of these auction types may or may not have a reserve price.

As a starting point, we take Bulow et al.'s (1999) model of independent signals, which is a special case of Milgrom and Weber's (1982) affiliated signals model. We assume that each bidder i receives a one-dimensional private signal t_i (we also say that bidder i is of type t_i.). We will let $v_i(\mathbf{t})$ denote the value of the object for bidder i given the vector $\mathbf{t} \equiv (t_1, ..., t_n)$ of all signals. Special cases are independent private value models ($v_i(\mathbf{t})$ only depends on t_i), and common value models ($v_i(\mathbf{t}) = v_j(\mathbf{t})$ for all $i, j, \mathbf{t}$). We assume that the signals t_i are drawn independently from the same distribution function. Without further loss of generality, we assume that the distribution function is the uniform distribution on the interval $[0, 1]$.

We make the following assumptions on the functions v_i.

Value Differentiability: v_i is differentiable in all its arguments, for all $i, \mathbf{t}$.
Value Monotonicity: $v_i(\mathbf{t}) \geq 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_j} \geq 0$, and $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > \frac{\partial v_i(\mathbf{t})}{\partial t_j}$ for all $i, j, \mathbf{t}$.
Symmetry: $v_i(..., t_i, ..., t_j, ...) = v_j(..., t_j, ..., t_i, ...)$ for all t_i, t_j, i, j.

Value Differentiability is imposed to make the calculations on the equilibria tractable. *Value Monotonicity* indicates that all bidders are serious, and that bidders' values are strictly increasing in their own signal, and weakly in the signals of the others. Moreover, it includes a single crossing property. *Symmetry* may be crucial for the existence of efficient equilibria in standard auctions.[7] *Value Differentiability, Value Monotonicity*, and *Symmetry* together ensure that the bidder with the highest signal is also the bidder with the highest value, so that these assumptions imply that the seller assigns the object efficiently if and only if the bidder with the highest signal gets it.

Also, let us define $v(x, y)$ as the expected value that bidder i assigns to the object, given that her signal is x, and that the highest signal of all the other bidders is equal to y:

$$v(x, y) \equiv E\{v_i(\mathbf{t}) | t_i = x, \max_{j \neq i} t_j = y\}.$$

By *Symmetry*, v does not depend on i.

The bidders are expected utility maximizers. Each bidder is risk neutral, and cares about what other bidders pay in the auction. More specifically, the utility function of bidder i is defined as follows:

$$u_i(j, b) = \begin{cases} v_i - b & \text{if } j = i \\ \varphi b & \text{if } j \neq i, \end{cases}$$

[7] Bulow et al. (1999) show that a slight asymmetry in value functions may have dramatic effects on bidding behavior in the English auction in a common value setting, as the bidder with the lower value function faces a strong winner's curse, and therefore bids zero in equilibrium.

where v_i is the value that i attaches to the object, j is the winner of the object and b is the payment by j. It is a natural assumption to let a bidder's interest in her own payments be larger than her interest in the payments by the other bidders, so that we assume $\varphi \leq 1/(n-1)$.

A specific interpretation of the model is a situation of an auction in which all losing bidders receive an equal share of the auction revenue. In particular, when $\varphi = 1/(n-1)$, the entire auction revenue is divided among all losing bidders, which may be the case in situations of dissolving partnerships, or heirs bidding for a family estate. If $n = 2$ and $\varphi = 1$, then FPSB and SPSB are special cases of the k-double auction with $k = 0$ and $k = 1$ respectively.[8,9]

3 Zero reserve price

Consider FPSB and SPSB with a zero reserve price.

3.1 First-price sealed-bid auction

The following proposition characterizes the equilibrium bid function for FPSB.[10] To derive equilibrium bidding, we suppose that in equilibrium, all bidders use the same bid function. By a standard argument, this bid function must be strictly increasing and continuous. Let $U(t, s)$ be the utility for a bidder with signal t who behaves as if having signal s, whereas the other bidders play according to the equilibrium bid function. A necessary equilibrium condition is that

$$\frac{\partial U(t,s)}{\partial s} = 0$$

[8]The k-double auction has the following rules. Both bidders submit a bid. The highest bidder wins the object, and pays the loser an amount equal to $kb_L + (1-k)b_W$, where b_L is the loser's bid, b_W the winner's bid, and $k \in [0, 1]$.

[9]Cramton et al. (1987) study k-double auctions in a private values environment with symmetric value distributions. It is shown that partners with equal shares may dissolve a partnership efficiently using these auctions. McAfee and McMillan (1992) show that the 0-double auction is a mechanism that allows a bidding ring to allocate the obtained object efficiently among the ring members. Van Damme (1992) shows that k-double auctions may lead to unfair equilibrium outcomes. Angeles de Frutos (2000) and Kittsteiner (2003) generalize the model of Cramton et al. (1987) allowing for asymmetric value distributions and interdependent valuations respectively.

[10]The proof of this and all other propositions are relegated to the Appendix.

at $s = t$. From this condition, a differential equation can be derived, from which the equilibrium bid function is uniquely determined. The auction outcome is efficient. Observe that in the case of independent private values ($v(x, y)$ only depends on x), the bid function is strictly increasing in n.

Proposition 1 *The unique Bayesian Nash equilibrium of FPSB is characterized by*

$$B_1(\varphi, t) = v(t, t) - \frac{\varphi}{1+\varphi} v(t, t) - \frac{1}{1+\varphi} \int_0^t \frac{dv(y, y)}{dy} \left(\frac{y}{t}\right)^{(n-1)(1+\varphi)} dy, \tag{1}$$

where $B_1(\varphi, t)$ is the bid of a bidder with signal t. The outcome of this auction is efficient.

Each of the terms of the RHS of (1) has an attractive interpretation. The first term is the equilibrium bid for a bidder with type t in SPSB without financial externalities, as in the absence of financial externalities, in SPSB, a bidder will submit a bid equal to her maximal willingness to pay given that her strongest opponent has the same signal as she (Milgrom and Weber, 1982). The second term can be interpreted as the bid shading because of financial externalities. The reason for bid shading is that in the case of financial externalities, the willingness to pay of a bidder with type t bidding against an opponent who has the same signal is given by $\frac{1}{1+\varphi} v(t, t)$. This can be seen as follows. When a bidder wins at a bid of b, her utility is $v(t, t) - b$. When her opponent wins at the same bid, her utility is φb. Equating these utilities results in a bid of $\frac{1}{1+\varphi} v(t, t)$. The third term can be interpreted as the strategic bid shading because in FPSB, a bidder has to pay her own bid rather than the second highest bid which she has to pay in SPSB.

This interpretation of the equilibrium bid function suggests that this function is decreasing in φ, which in fact follows directly from (1):

Corollary 1 *Increasing φ decreases $B_1(\varphi, t)$.*

From Corollary 1, it immediately follows that the expected revenue is decreasing in φ.

Corollary 2 *Increasing φ decreases the seller's expected revenue.*

Observe that all of the above results also apply to direct financial externalities. Equilibrium bidding is not affected by whether the financial externalities are direct or indirect. Moreover, the seller's revenue in the case of direct financial externalities is a fraction $(1 - n\varphi)$ of the revenues under indirect financial externalities. Corollary 2 indicates that revenue under indirect financial externalities is a decreasing function of φ, which also holds true for direct financial externalities as $(1 - n\varphi)$ is decreasing in φ as well. Therefore, it is not attractive for the seller to have the losers share a fraction of the auction's revenue.

3.2 Second-price sealed-bid auction

Equilibrium bids for SPSB are obtained using the same logic as for FPSB. The analysis reveals uniqueness and efficiency of the equilibrium bid function. Observe that in the case of private values, the bid function does not depend on n.[11]

Proposition 2 *The unique Bayesian Nash equilibrium of SPSB is characterized by*

$$
\begin{aligned}
B_2(\varphi, t) &= v(t,t) - \frac{\varphi}{1+\varphi} v(t,t) + \\
&\quad + \frac{\varphi}{(1+\varphi)(1+2\varphi)} \int_t^1 \frac{dv(y,y)}{dy} \left(\frac{1-y}{1-t} \right)^{\frac{1+\varphi}{\varphi}} dy
\end{aligned} \tag{2}
$$

where $B_2(\varphi, t)$ is the bid of a bidder with signal t. The outcome of this auction is efficient.

Each term of the RHS of (2) has its attractive interpretation. From the discussion of FPSB, it follows that the first term is the bid in SPSB in the absence of financial externalities. The second term is the bid shading due to positive externalities from the payment of the winning bidder. The third term increases the bid due to the fact that

[11] This is actually a quite subtle observation, as n does not appear in the expression for the equilibrium bid. However, in general, $v(t,t)$ depends on n.

each bidder is willing to drive up the final price, as it is the second highest bid that is paid by the winner.

In contrast to FPSB, the effect of an increase in φ on the equilibrium bids in SPSB is dependent on a bidder's type. From (2), it is clear that the equilibrium bid of the highest type is decreasing in φ. The reason is that as this bidder does not have a type above her, she does not have an incentive to drive up the price. However, the effect of φ on the equilibrium bids of the other types is not clear. The effect of the second term of the RHS of (2) may be larger as well as smaller than the third term. The following example illustrates how equilibrium bidding is affected when φ is varied.

Example 1 ***(Effect of*** φ ***on equilibrium bidding)*** *Let* $v(t,t) = t$ *(independent private values) for all* $t \in [0,1]$. *The equilibrium bid function is given by*

$$B_2(\varphi, t) = \frac{\varphi}{(1+\varphi)(1+2\varphi)} + \frac{1}{1+2\varphi} t, \; t \in [0,1].$$

As B_2 *is a continuous function in both* φ *and* t, *the following can be derived. First, there is a strictly positive mass of types close to zero for which the effect of* φ *is ambiguous in the sense that for* φ *close to 0, an increase in* φ *leads to higher bids and for* φ *close to 1, an increase in* φ *leads to lower bids. This follows from the following observations.*

$$\frac{\partial B_2(0,0)}{\partial \varphi} = 1 > 0,$$

and

$$\frac{\partial B_2(1,0)}{\partial \varphi} = -\frac{1}{36} < 0.$$

Intuitively, if φ *is large enough,* $B_2(\varphi, t)$ *decreases as for each bidder, losing becomes more interesting due to higher financial externalities. Second, the equilibrium bids of types close to 1 are decreasing in* φ. *This follows from the fact that* $B_2(\varphi, 1) = \frac{1}{1+\varphi}$.

Also, the effect of φ on the expected revenue may be ambiguous. This follows from Example 2, in which the expected revenue is increasing if φ is small, and decreasing if φ is large.

Example 2 ***(Effect of φ on the expected revenue)*** *Let $v(t,t) = t$ (independent private values) and $n = 2$ (two bidders). The expected revenue is equal to the expectation of $B_2(\varphi, t^{(2)})$ with respect to the second highest signal $t^{(2)}$, which is given by*

$$E_{t^{(2)}}\{B_2(\varphi, t^{(2)})\} = \frac{1+4\varphi}{3(1+\varphi)(1+2\varphi)}.$$

This continuous function is increasing for φ close to 0 and decreasing for φ close to 1, as

$$\frac{\partial E_{t^{(2)}}\{B_2(0, t^{(2)})\}}{\partial \varphi} = \frac{1}{3} > 0,$$

and

$$\frac{\partial E_{t^{(2)}}\{B_2(1, t^{(2)})\}}{\partial \varphi} = -\frac{11}{108} < 0.$$

Example 2 might suggest that the seller can gain more revenue by guaranteeing the losers a fraction of the auction revenue. This, however, turns out not to be the case. We prove this proposition by using the famous revenue equivalence theorem (Myerson, 1981), which states that the expected utility of the lowest type is a sufficient statistic for the ranking of efficient auctions is: the higher the utility of the lowest type, the lower the expected revenue. First, consider the standard case in which the bidders obtain no indirect financial externalities. If the seller pays the losers a fraction of the auction revenue, the lowest type's expected utility goes up from zero to a strictly positive number. The reason is that the lowest type gets a fraction of the second highest bid, which is strictly positive (see Proposition 2). As SPSB is efficient, the seller's expected revenue decreases. This result turns out to remain valid if the bidders do experience financial externalities, indirect or direct (e.g., the bidders own a share in the seller), as the lowest type's utility is strictly increasing in φ.[12]

Proposition 3 *The seller cannot increase its revenue by guaranteeing the losing bidders a fraction of the auction's revenue.*

[12] Maasland and Onderstal (2002) show that the revenue equivalence theorem remains valid in the case that financial externalities are introduced.

3.3 Revenue comparison

Let us compare the expected revenue from FPSB and SPSB. As said, the auction for which the lowest type obtains the highest expected utility generates the highest expected revenue. It turns out that if $0 < \varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB.[13] For $\varphi = \frac{1}{n-1}$, both auctions are revenue equivalent, which follows as the utility of the lowest type is the same for both auctions.

Proposition 4 *For $\varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB. For $\varphi = \frac{1}{n-1}$, FPSB and SPSB are revenue equivalent.*

4 Positive reserve price

Consider FPSB and SPSB with a reserve price $R > 0$. In order to keep the model tractable, we assume that the independent private values model holds. With some abuse of notation, we write $v_i(\mathbf{t}) = v(t_i)$ for all i, $\mathbf{t}$, where v is a strictly increasing function.

This section focuses, among other things, on the existence of *weakly separating Bayesian Nash equilibria*, for which the following definition applies.

Definition 1 *A weakly separating Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function.*

4.1 First-price sealed-bid auction

FPSB has a symmetric equilibrium that involves pooling at the reserve price. We assume that $R < v(1)$, as otherwise, none of the bidders has an incentive to submit a bid. Proposition 5 describes a Bayesian Nash equilibrium in which bidders with a type below a threshold type L do not bid, bidders with a type t above a threshold type H bid $g^R(t)$,

[13] Engelbrecht-Wiggans (1994) claims the same result, but his proof is not correct.

where g^R is a strictly increasing function, and types in the interval $[L, H]$ submit a bid equal to R. More specifically, let

$$H = \min\{1, v^{-1}((1+\varphi)R)\}, \tag{3}$$

L be the unique solution of

$$1 - \frac{n\left(HL^{n-1} - L^n\right)}{H^n - L^n} = \frac{v(L) - R}{\varphi R}, \tag{4}$$

and

$$g^R(t) = (n-1)t^{-(n-1)(1+\varphi)}\left[\int_H^t y^{(n-1)\varphi+n-2}v(y)dy + v(H)H^{(n-1)(1+\varphi)}\right].$$

This is an equilibrium, as (1) L is indifferent between abstaining from bidding, and submitting a bid equal to the reserve price, (2) H is indifferent between bidding R (and therefore pool with all types in the interval $[L, H]$), and bidding marginally higher than R, and (3) the incentive compatibility constraint for types above H results in the same differential equation as the bid function for FPSB without reserve price, of which g^R is the solution satisfying the boundary condition $g^R(H) = R$.

Proposition 5 *Assume independent private values and $R < v(1)$. Let $B_1^R(\varphi, t)$, the bid of a bidder with value t, be given by*

$$B_1^R(\varphi, t) = \begin{cases} g^R(t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{“no bid”} & \text{if } t < L \end{cases}$$

where H and L follow from (3) and (4) respectively. Then $B_1^R(\varphi, t)$ constitutes a symmetric Bayesian Nash equilibrium of FPSB if $R > 0$.[14]

To get an intuition why pooling at R occurs in equilibrium, consider a situation in which $n = 2$, $v(t) = t$, and $R \geq \frac{1}{1+\varphi}$. The threshold level H, above which bidders bid according to a strictly increasing bid function, lies above 1, so that bidders either abstain

[14] Note that $B_1^R(\varphi, t)$ is continuous at H. This must be the case in equilibrium. Suppose, on the contrary, that the bid function has a jump at H. Then a bidder with a type slightly higher than H has an incentive to deviate from the bid strategy to a bid of just above R.

from bidding, or bid R. Why is this an equilibrium? Suppose that one of the two bidders submits a bid $b \geq R$. Then the other bidder prefers losing to winning. This can be seen as follows. If she loses, then her utility is

$$\varphi b \geq \varphi R \geq \frac{\varphi}{1+\varphi} \geq \frac{\varphi t}{1+\varphi},$$

whereas winning gives her a utility of at most

$$t - R \leq t - \frac{1}{1+\varphi} \leq t - \frac{t}{1+\varphi} = \frac{\varphi t}{1+\varphi}.$$

Low types are then willing to lose the opportunity of getting the object by abstaining from bidding. High types bid R, assuring themselves the object if the other bidder does not bid, but also making sure that if the other bids, to lose as often as possible.

In contrast to a situation without financial externalities, there exists no weakly separating Bayesian Nash equilibrium for FPSB:

Proposition 6 *Assume independent private values. There exists no weakly separating Bayesian Nash equilibrium of FPSB if $R > 0$ and $\varphi > 0$.*

The proof of Proposition 6 is by contradiction. If a weakly separating Bayesian Nash equilibrium existed, $v^{-1}(R)$ would be the threshold type. The equilibrium bid function can be constructed analogous to the equilibrium bid function for FPSB without reserve price. But then a contradiction is established, as a bidder with type $v^{-1}(R)$ turns out to submit a bid below the reserve price.

4.2 Second-price sealed-bid auction

SPSB has no less than five different types of equilibria when the seller imposes a reserve price. The starting point is that all bidders who submit a bid above the reserve price, do so according to the same bid function as in the absence of a reserve price. This implies that, in contrast to FPSB, SPSB has a weakly separating Bayesian Nash equilibrium if the reserve price is not too high. This observation follows trivially when the reserve price is smaller than the lowest submitted equilibrium bid, which is strictly positive according to Proposition 2. However, also in nontrivial cases weakly separating Bayesian Nash equilibria exist. According to Proposition 7, for low R, types up to a threshold type $\widehat{t}$ abstain from bidding, and types above $\widehat{t}$ submit the same bid as in the case of no reserve price.

Proposition 7 *Assume independent private values. SPSB with reserve price R has a weakly separating Bayesian Nash equilibrium if and only if $B_2(\varphi, v^{-1}(R)) \geq R$. If such an equilibrium exists, then it is given by:*

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t \geq \widehat{t} \\ \text{“no bid”} & \text{if } t < \widehat{t} \end{cases}$$

where $B_2^R(\varphi, t)$ is the bid of a bidder with value t. If $R < B_2(\varphi, 0)$, $\widehat{t} = 0$, otherwise $\widehat{t}$ is the unique solution of

$$\varphi B_2(\varphi, \widehat{t})(1 - \widehat{t}^n) + \widehat{t}^n \left[v(\widehat{t}) - R\right] = \varphi(1 - \widehat{t})R. \tag{5}$$

The threshold type $\widehat{t}$ is indifferent between bidding $B_2(\varphi, \widehat{t})$ and abstaining from bidding, and hence follows from equation (5). Observe that $\widehat{t}$ jumps to a bid above the reserve price. An explanation for this is the following. In a weakly separating Bayesian Nash equilibrium, each bidder who submits a bid, submits a bid as if there were no reserve price. Then, it may be the case that $B_2(\varphi, \widehat{t}) > R$. Observe that this type of equilibrium may exist if $B_2(\varphi, v^{-1}(R)) > R$. In that case, there is a type $\widetilde{t} < R$ for which $B_2(\varphi, \widetilde{t}) = R$. As a reserve price does not affect equilibrium bidding of types that submit a bid, it follows that if type $\widetilde{t}$ would submit a bid in equilibrium, she would submit a bid equal to R. However, type $v^{-1}(R)$ is indifferent between bidding R and not submitting a bid (as its value is exactly R), so that $\widetilde{t} < v^{-1}(R)$ strictly prefers to abstain from bidding. Therefore, as $\widehat{t}$ exceeds $\widetilde{t}$, $B_2(\varphi, \widehat{t}) > B_2(\varphi, \widetilde{t}) = R$.

An intuition for the condition $B_2(\varphi, v^{-1}(R)) \geq R$ being necessary is the following. In a weakly separating Bayesian Nash equilibrium, a bidder with type R is always prepared to submit a bid of at least R. To see this, observe that for this bidder, in a weakly separating

Bayesian Nash equilibrium, a bid equal to R yields the same revenue as abstaining from bidding. However, in equilibrium, each type that submits a bid, does so according to the equilibrium bid function for the situation with no reserve price. This implies that if $B_2(\varphi, v^{-1}(R)) < R$, a bidder with type R would submit a bid below the reserve price, which is not possible, so that a contradiction is established.

The condition $B_2(\varphi, v^{-1}(R)) \geq R$ is sufficient for the following reason. As said, in a weakly separating Bayesian Nash equilibrium, each bidder who submits a bid, submits a bid as if there were no reserve price. Then, for the existence of a weakly separating equilibrium, it remains to be checked that $B_2(\varphi, \widehat{t}) \geq R$. If $B_2(\varphi, v^{-1}(R)) \geq R$, then there is a type $\widetilde{t} \leq R$ for which $B_2(\varphi, \widetilde{t}) = R$. As a reserve price does not affect equilibrium bidding of types that submit a bid, it follows that if type $\widetilde{t}$ would submit a bid in equilibrium, she would submit a bid equal to R. However, type R is indifferent between bidding R and not submitting a bid, so that $\widetilde{t}$ prefers not to submit a bid. Therefore, $\widehat{t}$ must exceed $\widetilde{t}$, so that indeed $B_2(\varphi, \widehat{t}) \geq B_2(\varphi, \widetilde{t}) = R$.

The necessary and sufficient condition $B_2(\varphi, v^{-1}(R)) \geq R$ implies that only for small R, a weakly separating Bayesian Nash equilibrium exists. As said, the existence of such an equilibrium is trivial in the case of small R. However, for R close to 1, $B_2(\varphi, v^{-1}(R)) < R$, as, by Proposition 2, $B_2(\varphi, 1) < v(1)$.

If the condition $B_2(\varphi, v^{-1}(R)) \geq R$ is violated, there may be an equilibrium with pooling at R. If $B_2(\varphi, 1) > R$ and $v(1) \leq R$ all types above a threshold L submit a bid equal to the reserve price. Type L is indifferent between bidding and not bidding, and follows uniquely from a similar condition as in FPSB:

$$1 - \frac{n\left(L^{n-1} - L^{n}\right)}{1 - L^{n}} = \frac{v(L) - R}{\varphi R}. \tag{6}$$

Moreover, if $v(1) > R$, none of the bidders submit a bid in equilibrium, as the value of winning for none of the bidders exceeds the reserve price.

Proposition 8 *Assume independent private values. Consider SPSB with reserve price R. If $B_2(\varphi, v^{-1}(R)) < R$ and $B_2(\varphi, 1) < R$, the following bidding strategies constitute a Bayesian Nash equilibrium:*

$$B_2^R(\varphi, t) = \begin{cases} R & \text{if } L < t \leq 1 \\ \text{“no bid”} & \text{if } t \leq L \end{cases}$$

where $L = 1$ if $v(1) > R$, and L follows from (6) otherwise.

The most involved case is $B_2(\varphi, 1) \geq R$. If this condition holds, types below a certain type L abstain from bidding, bidders between types L and H bid the reserve price, and types $t > H$ bid $B_2(\varphi, t)$. The threshold types L and H respectively follow from the following indifference relations:

$$\left[p(L,H) - L^n\right] \varphi R = p(L,H) \left(v(L) - R\right) \tag{7}$$

$$\left[q(H) + r(L,H)\right] \varphi R = q(H)\varphi B_2(\varphi, H) + r(L,H) \left(v(H) - R\right). \tag{8}$$

with

$$\begin{aligned} q(H) &= (n-1)(1-H) H^{n-2} \\ p(L,H) &= \frac{1}{n(H-L)} \left(H^n - L^n\right) \\ r(L,H) &= H^{n-1} - p(L,H). \end{aligned}$$

where $q(H)$ is the probability that exactly one bidder has a type above H, $p(L,H)$ is the probability that a bidder wins given that she bids the reserve price, and $r(L,H)$ is the probability that a bidder does *not* win given than she bids R and that the highest type of the other bidders does not exceed H.

The indifference relation (7) for type L is constructed as follows. L is indifferent between bidding R and abstaining from bidding. The outcome is different in two situations: first, if the bidder wins against another bidder who bids R (which occurs with probability $p(L,H) - L^n$), and if no other bidder bids (which happens with probability L^n).

For type H, the indifference relation (**??**) follows as she is indifferent between bidding R and submitting a bid $B_2(\varphi, H)$. These two bids result in a different outcome if (1) exactly one bid exceeds R, so that H determines the price of the winner (this event has probability $q(H)$), and (2) H, when bidding R, loses against another bidder who also bids R (probability $r(L,H)$).

Proposition 9 *Assume independent private values. Consider SPSB with reserve price R. If $B_2(\varphi, v^{-1}(R)) < R$ and $B_2(\varphi, 1) \geq R$, the following bidding strategies constitute a*

Bayesian Nash equilibrium:

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{"no bid"} & \text{if } t < L \end{cases}$$

where (L, H) is a solution of the system of equations (7) and (8).

The intuition for pooling at the reserve price is analogous to FPSB. Let $\widetilde{t}$ be the unique solution to $B_2(\varphi, \widetilde{t}) = R$. A bidder with type $\widetilde{t}$ considers the following when deciding to submit a bid. Bidding R and abstaining from bidding only makes a difference if at most one bidder bids more than R. If none of the other bidders bids, bidding R yields her

$$v(\widetilde{t}) - R$$

which is positive (as otherwise $\widetilde{t}$ would not be willing to bid R in the absence of a reserve price), so that bidding could be interesting. However, when exactly one other bidder bids, it is not profitable to submit is bid. To see this, note that type $\widetilde{t}$'s expected revenue from losing against this bidder are

$$\varphi R$$

while her expected gains from winning are

$$\begin{aligned} v(\widetilde{t}) - R &< (1+\varphi)\, B_2(\varphi, \widetilde{t}) - R \\ &= \varphi R. \end{aligned}$$

The inequality follows from the property that $(1+\varphi)\, B_2(\varphi, t) > v(t)$ for all t (see the proof of Proposition 2 in the Appendix). The equality holds as by definition, $B_2(\varphi, \widetilde{t}) = R$. In other words, type $\widetilde{t}$, when bidding, prefers to bid as low as possible (the reserve price) in order to minimize the probability that she wins if another bidder is bidding as well. The same reasoning holds true for types in the neighborhood of $\widetilde{t}$, so that these types pool at the reserve price.

To summarize: SPSB has no less than five types of equilibria if the seller requires a minimum bid R. First, if $R < B_2(\varphi, 0)$, all bidders submit a bid according to $B_2(\varphi, t)$. Second, if $R > v(1)$, no bidder bids. Third, if $B_2(\varphi, 1) < R < v(1)$, all bidders above a threshold bid exactly the reserve price. For $R \in (B_2(\varphi, 0), B_2(\varphi, 1))$, the condition $B_2(\varphi, v^{-1}(R)) \geq R$ becomes crucial. If this condition holds true, types up to a threshold do not bid, and types above this threshold submit bids according to $B_2(\varphi, t)$. Otherwise, low types abstain from bidding, intermediate types pool at R, and high types t bid $B_2(\varphi, t)$.

5 Concluding remarks

We have studied auctions in which losing bidders obtain financial externalities from the winning bidder. We have derived bidding equilibria for FPSB and SPSB, and we have shown that SPSB dominates FPSB in terms of expected auction revenue if $\varphi < \frac{1}{n-1}$ and that both auctions are revenue equivalent if $\varphi = \frac{1}{n-1}$. Moreover, we have studied equilibrium bidding for FPSB and SPSB when a reserve price is imposed. We have observed pooling at the reserve price for FPSB, and for SPSB if the reserve price is sufficiently high.

Motivated by the observation that in SPSB, low signal bidders increase their bids when φ is increased (for φ not too large), a model with asymmetries in the valuation function may be fruitful to study. One may imagine that with one bidder with a low value, and one bidder with a high value, the price in SPSB may be higher with financial externalities than without financial externalities, as the bidder with the low value has an incentive to push up the price when φ is strictly positive. This indicates that the seller may have an incentive to promise low bidders a share of the auction revenue. Indeed, Goeree and Offerman (2004) show that in asymmetric environments, the seller may obtain more revenue by rewarding one of the losing bidder with a fraction of the auction revenue.

6 Appendix

Proof of Proposition 1. A higher type of a bidder cannot submit a lower bid than a lower type of the same bidder. (If the low type gets the same expected surplus from strategies with two different probabilities of being the winner of the object, the high type strictly prefers the strategy with the highest probability of winning, so the high type will not submit a lower bid than the low type.) Also, $B_1(\varphi, t)$ cannot be constant on an interval $[t', t'']$. (By bidding slightly higher, a type t'' can largely improve its probability of winning, while only marginally influencing the payments by her and the other bidders.) Moreover, $B_1(\varphi, t)$ cannot be discontinuous at any t. (Suppose that $B_1(\varphi, t)$ makes a jump from $\underline{b}$ to $\overline{b}$ at t^*. A type just above t^* has an incentive to deviate from $\overline{b} + \delta$ to $\underline{b}$. Doing so, she is able to decrease the auction price, while just slightly affecting its probability of winning the object. As φ is small enough, this type is able to improve its utility.) Hence, a symmetric equilibrium bid function must be strictly increasing and continuous. Define the utility $U(t, s)$ for a bidder with signal t who misrepresents herself as having signal s, whereas the other bidders report truthfully, if the bid function is indeed strictly increasing. Moreover, let $F^{[1]}$ $[f^{[1]}]$ denote the cumulative distribution function [density function] of the highest order statistic from $n - 1$ draws from the uniform distribution. Then,

$$U(t, s) = \int_0^s v(t, y) dF^{[1]}(y) - F^{[1]}(s) B_1(\varphi, s) + \varphi \int_s^1 B_1(\varphi, y) dF^{[1]}(y).$$

The first two terms of the RHS of this expression refer to the case that this bidder wins the object. The third term refers to the case that she does not win. Assume that $B_1(\varphi, s)$ is differentiable in s. Maximizing $U(t, s)$ with respect to s and equating s to t gives the FOC of the equilibrium

$$f^{[1]}(t) v(t, t) - f^{[1]}(t) B_1(\varphi, t) - F^{[1]}(t) B_1'(\varphi, t) - \varphi B_1(\varphi, t) f^{[1]}(t) = 0.$$

With some manipulation we get

$$F^{[1]}(t)^{\varphi} f^{[1]}(t) v(t, t) = (1 + \varphi) B_1(\varphi, t) f^{[1]}(t) F^{[1]}(t)^{\varphi} + B_1'(\varphi, t) F^{[1]}(t)^{1+\varphi}, \tag{9}$$

or, equivalently

$$C_1 + \int_0^t F^{[1]}(y)^{\varphi} f^{[1]}(y) v(y, y) dy = F^{[1]}(t)^{1+\varphi} B_1(\varphi, t),$$

where C_1 is a constant. Substituting $t = 0$ gives $C_1 = 0$, so that the bid function is given by

$$\begin{aligned} B_1(\varphi, t) &= \frac{1}{F^{[1]}(t)} \int_0^t \left(\frac{F^{[1]}(y)}{F^{[1]}(t)} \right)^{\varphi} f^{[1]}(y) v(y, y) dy \\ &= (n-1)\, t^{-n+1-\varphi} \int_0^t y^{\varphi(n-1)+n-2} v(y, y) dy. \end{aligned} \tag{10}$$

It is readily checked that the second order condition $sign\left(\frac{\partial U(t,s)}{\partial s}\right) = sign(t-s)$ is fulfilled. Using integration by parts, (10) can be rewritten as (1). From (9), we infer that $\frac{\partial B_1(\varphi,t)}{\partial t} > 0$ if and only if $B_1(\varphi, t) < \frac{v(t,t)}{1+\varphi}$, so that indeed $B_1(\varphi, t)$ is strictly increasing in t, as *Value Monotonicity* implies that $\frac{dv(y,y)}{dy} > 0$ for all y. Finally, by *Value Differentiability, Value Monotonicity*, and *Symmetry*, the efficiency of the auction outcome is established. Uniqueness follows from a standard argument (see Bulow et al., 1999, who show uniqueness in a similar setting). ■

PROOF OF PROPOSITION 2. Let $F^{[1]}$ $[F^{[2]}]$ denote the cumulative distribution function of the first [second] order statistic from $n-1$ draws from the uniform distribution, and $f^{[1]}$ $[f^{[2]}]$ their corresponding density functions. Following the lines of the proof of Proposition 1 it can be established that a symmetric equilibrium function must be strictly increasing and continuous. The utility for a bidder with signal t acting as if she had signal s is given by

$$U(t, s) = \int_0^s [v(t, y) - B_2(\varphi, y)] dF^{[1]}(y) + \varphi\pi(s) B_2(\varphi, s) + \varphi \int_{y=s}^1 B_2(\varphi, y) dF^{[2]}(y),$$

where $\pi(s) \equiv F^{[2]}(s) - F^{[1]}(s)$ denotes the probability that there is exactly one opponent with a signal larger than s. The first term of the RHS refers to the case that this bidder wins, the second term to the case that she submits the second highest bid, and the third term to her bid being the third or higher. Assume that $B_2(\varphi, s)$ is differentiable in s. The FOC of the equilibrium is

$$[v(t, t) - B_2(\varphi, t)] f^{[1]}(t) + \varphi \frac{\partial \pi(t) B_2(\varphi, t)}{\partial t} - \varphi B_2(\varphi, t) f^{[2]}(t) = 0$$

or, equivalently

$$v(t,t)f^{[1]}(t) = -B_2'(\varphi,t)\varphi\pi(t) + B_2(\varphi,t)[(1+\varphi)f^{[1]}(t)]. \tag{11}$$

The general solution to the above differential equation is equal to

$$B_2(\varphi,t)(1-t)^{\frac{1+\varphi}{\varphi}} = C_2 - \int_0^t (1-y)^{\frac{1}{\varphi}} v(y,y)dy,$$

where C_2 is a constant. Substituting $t = 1$ yields a unique solution for C_2:

$$C_2 = \int_0^1 (1-y)^{\frac{1}{\varphi}} v(y,y)dy.$$

The only possible differentiable bid function that may constitute a symmetric equilibrium is given by

$$B_2(\varphi,t) = \frac{1}{\varphi}(1-t)^{-1-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y,y)dy. \tag{12}$$

It is readily checked that the second order condition $sign\left(\frac{\partial U(t,s)}{\partial s}\right) = sign(t-s)$ holds. Using integration by parts on $B_2(\varphi,t)$, we see that (12) can also be written as (2). To complete the proof, we must show that $B_2(\varphi,t)$ is indeed increasing in t. From (12), it follows that

$$B_2(\varphi,t) > \frac{v(t,t)\int_t^1 (1-y)^{\frac{1}{\varphi}}dy}{\varphi(1-t)^{\frac{1+\varphi}{\varphi}}} = \frac{v(t,t)}{1+\varphi}.$$

As (11) implies that $B_2'(\varphi,t) > 0$ if and only if $B_2(\varphi,t) > \frac{v(t,t)}{1+\varphi}$, $B_2(\varphi,t)$ is indeed strictly increasing in t. Then, by *Value Differentiability, Value Monotonicity*, and *Symmetry*, it follows that the outcome of the auction is efficient. Uniqueness follows from a standard argument (see Bulow et al., 1999, who show uniqueness in a similar setting). ■

Proof of Proposition 3. As stated in the text, it is sufficient to show that the expected utility $U_2(0)$ of the lowest type is decreasing in φ. If $n = 2$, the price paid equals the bid of the lowest type. Therefore,

$$\begin{aligned} U_2(0) &= \varphi B_2(\varphi,0) \\ &= \int_0^1 (1-t)^{\frac{1}{\varphi}} v(t,t)dt. \end{aligned}$$

which is strictly increasing in φ. If $n > 2$, the bidder receives financial externalities equal to φ times the second highest bid. Let $F^{[2]}$ be the cumulative distribution function of the second order statistic from $n-1$ draws from the uniform distribution. Then, using the expression for the bid function in (12),

$$\begin{aligned} U_2(0) &= \varphi \int_0^1 B_2(\varphi, t) dF^{[2]}(t) \\ &= (n-1)(n-2) \int_0^1 t^{n-3}(1-t)^{-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y,y) dy dt \qquad (13) \\ &= (n-1)(n-2) \int_0^1 t^{n-3} \int_t^1 \left(\frac{1-y}{1-t} \right)^{\frac{1}{\varphi}} v(y,y) dy dt \end{aligned}$$

which is strictly increasing in φ as in the inner integral, $y > t$. ■

Proof of Proposition 4. (The proof follows the same logic as Bulow et al., 1999). Let $U_1(0)$ and $U_2(0)$ be the equilibrium utility of the lowest type in FPSB and SPSB respectively. According to Maasland and Onderstal (2002), for SPSB to generate higher [the same] expected revenue than [as] FPSB, it is sufficient to show that $U_1(0) > U_2(0)$ $[U_1(0) = U_2(0)]$. We split the proof in two cases: $n = 2$ and $n > 2$.

We start with the case $n = 2$. As the outcome of both auctions is efficient, a bidder with type 0 loses the auction with probability 1, and gets financial externalities as the other bidder has to pay. For FPSB, the expected price paid by the other bidder is the expectation of his bid (10). Hence,

$$\begin{aligned} U_1(0) &= \varphi \int_0^1 B_1(\varphi, t) dt \\ &= \varphi \int_0^1 t^{-1-\varphi} \int_0^t y^{\varphi} v(y,y) dy dt \\ &= \varphi \int_0^1 y^{\varphi} v(y,y) \int_y^1 t^{-1-\varphi} dt dy \\ &= \int_0^1 v(y,y) \left[1 - y^{\varphi}\right] dy. \end{aligned}$$

$U_2(0)$ is already determined in the proof of Proposition 3:

$$U_2(0) = \int_0^1 (1-t)^{\frac{1}{\varphi}} v(t,t)dt.$$

The difference between $U_1(0)$ and $U_2(0)$ can be expressed as

$$U_1(0) - U_2(0) = \int_0^1 \left\{1 - t^{\varphi} - (1-t)^{\frac{1}{\varphi}}\right\} v(t,t)dt$$

For $\varphi < 1$ $[\varphi = 1]$, $U_1(0) - U_2(0) > 0$ $[U_1(0) - U_2(0) = 0]$, as the expression in curly brackets has zero expected utility, and is strictly negative for all $t \in (0, \hat{t})$ and positive for all $t \in (\hat{t}, 1)$ for some $\hat{t}$ [is zero for all $t \in [0,1]$].

Let us now consider the case of n bidders. We define $F^{[1]}$ $[F^{[2]}]$ as the cumulative distribution function of the first [second] order statistic from $n-1$ draws from the uniform distribution. Using the expression for the bid functions in (10) and $U_2(0)$ in (13), we derive that

$$\begin{aligned}
U_1(0) &= \varphi \int_0^1 B_1(\varphi, t) dF^{[1]}(t) \\
&= (n-1)^2 \varphi \int_0^1 t^{-1-\varphi(n-1)} \int_0^t y^{n-2+\varphi(n-1)} v(y,y) dy dt \\
&= (n-1)^2 \varphi \int_0^1 y^{\varphi(n-1)+n-2} v(y,y) \int_y^1 t^{-(n-1)\varphi - 1} dt dy \\
&= (n-1) \int_0^1 v(y,y) y^{n-2} \left[1 - y^{\varphi(n-1)}\right] dy
\end{aligned}$$

and

$$\begin{aligned}
U_2(0) &= (n-1)(n-2) \int_0^1 t^{n-3} (1-t)^{-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y,y) dy dt \\
&= (n-1)(n-2) \int_0^1 (1-y)^{\frac{1}{\varphi}} v(y,y) \int_0^y t^{n-3} (1-t)^{-\frac{1}{\varphi}} dt dy \\
&= (n-1) \left\{ y^{n-2} - \frac{1}{\varphi} (1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} dt \right\}.
\end{aligned}$$

The difference between $U_1(0)$ and $U_2(0)$ can be expressed as

$$\frac{U_1(0) - U_2(0)}{n-1} = \int_0^1 \left\{ -y^{\varphi(n-1)+n-2} + \frac{1}{\varphi}(1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1} dt \right\} v(y,y) dy$$

For $\varphi < \frac{1}{n-1}$ $[\varphi = \frac{1}{n-1}]$, $U_1(0) - U_2(0) > 0$ $[U_1(0) - U_2(0) = 0]$, as the expression in curly brackets has zero expected utility, and is strictly negative for all $y \in (0, \hat{y})$ and positive for all $y \in (\hat{y}, 1)$ for some $\hat{y}$ [is zero for all $y \in [0,1]$].

The following observations prove the last statement. Let

$$g(\varphi, n, y) \equiv -y^{\varphi(n-1)+n-2} + \frac{1}{\varphi}(1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1} dt.$$

The expectation of $g(\varphi, n, y)$ w.r.t. y is

$$\int_0^1 g(\varphi, n, y) dy = -\frac{1}{(\varphi+1)(n-1)} + \frac{1}{\varphi} \int_0^1 t^{n-2}(1-t)^{-\frac{1}{\varphi}-1} \int_t^1 (1-y)^{\frac{1}{\varphi}} dy dt = 0.$$

Define

$$\begin{aligned} f(y) &\equiv -\varphi(1-y)^{-\frac{1}{\varphi}} g(\varphi, n, y) \\ &= -\varphi(1-y)^{-\frac{1}{\varphi}} y^{\varphi(n-1)+n-2} + \int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1} dt. \end{aligned}$$

Note that $f(0) = 0$, and $f(y)$ is negative for positive y close to 0, as the first [second] term on the RHS is of the order $y^{\varphi(n-1)+n-2}$ $[y^{n-1}]$ and $\varphi(n-1) + n - 2 < n - 1$. Moreover,

$$f'(y) = -[\varphi(n-1) + n - 2]\varphi(1-y)^{-\frac{1}{\varphi}} y^{\varphi(n-1)+n-3} + y^{n-2}\left(1 - y^{\varphi(n-1)}\right)(1-y)^{-\frac{1}{\varphi}-1}.$$

Note that $\lim_{y \uparrow 1} f'(y) = +\infty$, and that for $y \in (0,1)$, $f'(y) = 0$ implies

$$\{y + [\varphi(n-1) + n - 2]\varphi(1-y)\} = y^{1-\varphi(n-1)}$$

As the function on the LHS is linear, the one on the RHS concave, and the equality holds for $y = 1$, there is at most one point $y \in (0,1)$ at which $f'(y) = 0$. This implies that f is strictly negative for all $y \in (0, \hat{y})$ and positive for all $y \in (\hat{y}, 1)$ for some $\hat{y}$, so that the same holds true for g.

For $\varphi = \frac{1}{n-1}$, we prove that $g\left(\varphi = \frac{1}{n-1}, n, y\right) = 0$ for all $y \in [0,1]$ by induction to n. It straightforwardly checked that $g(\varphi = 1, 2, y) = 0$. Suppose that $g\left(\varphi = \frac{1}{n-1}, n, y\right) = 0$ for some $n \geq 2$. We now show that this implies that $g\left(\varphi = \frac{1}{n}, n+1, y\right) = 0$:

$$\begin{aligned}
g\left(\frac{1}{n}, n+1, y\right) &= -y^n + n\left(1-y\right)^n \int_0^y t^{n-1}(1-t)^{-n-1}dt \\
&= -y^n + y^{n-1} - (n-1)\left(1-y\right)^n \int_0^y t^{n-2}(1-t)^{-n}dt \\
&= -y^n + y^{n-1} - (1-y)\left[g\left(\frac{1}{n+1}, n, y\right) + y^{n-1}\right] \\
&= 0.
\end{aligned}$$

■

Proof of Proposition 5. Assume that threshold types L and H exist such that in equilibrium all types $t < L$ abstain from bidding, all types $t \in [L, H]$ bid R, and all types $t > H$ bid according to a strictly increasing bid function g^R.

Let $p(H, L)$ denote the probability that a bidder wins when bidding R:

$$\begin{aligned}
p(L, H) &= \sum_{i=0}^{n-1} \frac{1}{i+1} \binom{n-1}{i} L^{n-1-i} \left(H-L\right)^i \\
&= \frac{1}{n\left(H-L\right)} \sum_{j=1}^{n} \binom{n}{j} L^{n-j} \left(H-L\right)^j \\
&= \frac{1}{n\left(H-L\right)} \left(H^n - L^n\right)
\end{aligned}$$

For type L, the indifference relation is

$$\begin{aligned}
\varphi R\left(H^{n-1} - L^{n-1}\right) &= \varphi R\left(H^{n-1} - p(H,L)\right) + p(H,L)\left[v(L) - R\right] \Longrightarrow \\
1 - \frac{L^{n-1}}{p(H,L)} &= \frac{v(L) - R}{\varphi R} \Longrightarrow \\
1 - \frac{n\left(HL^{n-1} - L^n\right)}{H^n - L^n} &= \frac{v(L) - R}{\varphi R} \Longrightarrow \\
1 - n - \frac{n\left(HL^{n-1} - H^n\right)}{H^n - L^n} &= \frac{v(L) - R}{\varphi R} \qquad (14)
\end{aligned}$$

L is uniquely determined from (14) as the LHS of (14) is strictly decreasing in L and the RHS is strictly increasing in L for $L \geq 0$.

A type H is indifferent between bidding R and bidding an infinitesimal δ above R. These two bids only yield a different outcome if all other types are lower than H. The difference between bidding R and a bid just above R is that in the former case the bidder always wins and gains $v(H) - R$, whereas a bid equal to R may result in utility φR if another bidder also bids R. Hence, H is indifferent if and only if

$$v(H) = (1 + \varphi)R.$$

In order to complete the proof, we need to check whether types have no incentive to deviate from the proposed equilibrium. We only check if a type $t > H$ has no incentive to mimic another type $t' > H$, as by a standard argument, other deviations are not profitable. Incentive compatibility of types $t > H$ implies that g^R should follow from the same differential equation as derived in the proof of Proposition 1 with the boundary condition $g^R(H) = R$. This is indeed how g^R is constructed.

Finally, we should establish that $g^R(t)$ is strictly increasing for $t \geq H$. Analogous to the proof of Proposition 1, this is the case if and only if $g^R(t) < \frac{v(t)}{1+\varphi}$. Now, for $1 > t > H$,

$$\begin{aligned} g^R(t) &= (n-1)t^{-(n-1)(1+\varphi)} \left[\int_H^t y^{(n-1)\varphi+n-2} v(y)dy + v(H)H^{(n-1)(1+\varphi)} \right] \\ &= \frac{v(t)}{1+\varphi} - \frac{t^{-(n-1)(1+\varphi)}}{1+\varphi} \int_H^t y^{(n-1)(\varphi+1)} v'(y)dy \\ &< \frac{v(t)}{1+\varphi}. \end{aligned}$$

■

Proof of Proposition 6. The proof is by contradiction. Suppose for the moment that a weakly separating equilibrium does exist. Then it is easily derived that all bidders with a type below R abstain from bidding, and all types above R submit a bid according to a strictly increasing bid function, which we denote by h. Using similar arguments as in the proof of Proposition 1, it can be established that $h'(t) \geq 0$ if and only if $h(t) \leq \frac{t}{1+\varphi}$. Hence, for $t = R$, it holds that $h(R) \leq \frac{R}{1+\varphi}$. In other words, in a weakly separating equilibrium, a bidder with type R submits a bid strictly below R. This contradicts the fact that all submitted bids should exceed the reserve price R. ■

Proof of Propositions 7 - 9. Suppose that all types t above a threshold type $\widehat{t}$ submit a bid above R according to a strictly increasing bid function g. Analogous to the proof of Proposition 2, the utility of a type t that wishes to mimic a type $s > \widehat{t}$ is given by

$$U(t,s) = \int_{\widehat{t}}^{s} [v(t,y) - g(y)]dF^{[1]}(y) + \varphi\pi(s)g(s) + \varphi \int_{y=s}^{1} g(y)dF^{[2]}(y).$$

The equilibrium bid function follows from the condition

$$\frac{\partial U(t,s)}{\partial s} = 0$$

at $s = t$. This immediately leads to the same differential equation as derived in the proof of Proposition 2 with the same boundary condition $g(1) = \frac{v(1)}{1+\varphi}$, so that $B_2^R(\varphi, t) = B_2(\varphi, t)$ is a solution for all R and $t \geq \widehat{t}$.

Now, suppose there is an R for which a weakly separating equilibrium exists. Then there is an indifference type $\widehat{t}$ such that

$$B_2^R(\varphi,t) = \begin{cases} B_2(\varphi,t) & \text{if } t \geq \widehat{t} \\ \text{“no bid”} & \text{if } t < \widehat{t} \end{cases}$$

is an equilibrium. $\widehat{t}$ is indifferent between submitting no bid, and submitting a bid equal to $B_2(\varphi, \widehat{t})$, so that indeed $\widehat{t}$ follows from (5).

A weakly separating equilibrium exists if and only if $B_2(\varphi, \widehat{t}) \geq R$, as all bids should be above R. We will now show that $B_2(\varphi, \widehat{t}) \geq R$ is equivalent to the condition $B_2(\varphi, v^{-1}(R)) \geq R$. Define $\widetilde{t}$ such that

$$B_2(\varphi, \widetilde{t}) = R. \tag{15}$$

As $B_2(\varphi, t)$ is strictly increasing in t, $\widetilde{t}$ is uniquely determined. Consider the function h with

$$h(t) \equiv \varphi B_2(\varphi, t) + \frac{t^n [v(t) - R]}{1 - t^n}$$

for all t. Note that h is a strictly increasing function, with

$$h(\widehat{t}) = \varphi R,$$

(which follows from (5)), and

$$h(\widetilde{t}) = \varphi R + \frac{\widetilde{t}^n \left[v(\widetilde{t}) - R\right]}{1 - \widetilde{t}^n}. \tag{16}$$

Now, with (15), as B_2 is strictly increasing,

$$B_2(\varphi, v^{-1}(R)) \geq R \Longleftrightarrow B_2(\varphi, \widetilde{t}) = R \leq B_2(\varphi, v^{-1}(R)) \Longleftrightarrow \widetilde{t} \leq v^{-1}(R).$$

Moreover, with (16), as h is strictly increasing,

$$\widetilde{t} \leq v^{-1}(R) \Longleftrightarrow h(\widetilde{t}) \leq \varphi R = h(\widehat{t}) \Longleftrightarrow \widetilde{t} \leq \widehat{t}.$$

Then, as B_2 is strictly increasing, and from (15),

$$\widetilde{t} \leq \widehat{t} \Longleftrightarrow B_2(\varphi, \widehat{t}) \geq R.$$

Finally, if $B_2(\varphi, v^{-1}(R)) < R$, the pooling equilibrium is straightforwardly established.

■

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