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Mário Cunha; Hélder Valente; Paulo B. Vasconcelos

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PONZI SCHEMES: COMPUTER SIMULATION

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>> RESUMO

Os nomes de Ponzi e Madoff, assim como no caso português D. Branca, são conhecidos como associados a esquemas de investimentos. Trata-se de um tipo de planos de investimentos profusamente espalhados e que continuam a existir, com maiores ou menores adaptações, produzindo graves danos a muitas pessoas e à sociedade em geral. Consistindo num fenómeno de fácil explicação após a sua implosão, a sua perceção não é fácil num momento prévio e oportuno. Existem alguns trabalhos interessantes nesta área mas num número reduzido.

Neste trabalho é apresentada uma abordagem computacional ao modelo matemático desenvolvido por Artzrouni (2009) para a análise dos esquemas de Ponzi. O modelo descreve a dinâmica de um fundo de investimento que promete taxas de rendibilidade superiores aquelas que pode na realidade oferecer. Na génese do esquema existem uma taxa de rendibilidade prometida pelo fundo, a taxa nominal presente, uma taxa irrealista de captura de mercado para novos investimentos e a taxa de resgate para os depósitos acumulados.

São apresentadas simulações resultantes dos choques nos parâmetros do modelo, com o objetivo de ilustrar o seu impacto no sucesso ou no colapso do fundo de investimento. Para a calibração do modelo são utilizados dados disponíveis referentes ao mais famoso esquema fraudulento deste tipo: Charles Ponzi, 1920. É igualmente apresentada para discussão uma versão filantrópica do modelo, tendo em mente os modelos de segurança social.

A aptidão das técnicas de simulação na deteção de padrões insustentáveis pode ser do interesse de reguladores financeiros e investidores quando confrontados com situações em que os fundos apresentam desempenhos irrealistas face às restrições económicas e financeiras.

Palavras-chave: Esquemas de Ponzi; investimento; taxa de rendibilidade; equações diferenciais ordinárias; métodos numéricos; simulação.

>> ABSTRACT

Ponzi and Madoff names, as well as the Portuguese D. Branca, are so-called investment schemes that have become well-known. These scams are widespread and continue to exist, with more or less modifications, depicting serious damage to many people and society in general. Being a phenomenon of easy explanation after the implosion, its perception is not easy in a timely manner. There are some interesting studies on this subject, although in reduced numbers.

In this paper we present a computational approach to the mathematical model developed by Artzrouni (2009), to study Ponzi schemes. The model describes the dynamics of an investment fund that promises higher incomes than those it can effectively offer. In the genesis there are a promised return rate, the actual nominal rate, unrealistic market capture rate of new investment and the rate of removal of accumulated deposits.

Simulations resulting from shocks on the parameters of the model will be presented, in order to illustrate the impact on the success or the collapse of the investment fund. For the model calibration, data available for one of the most famous fraudulent financial schemes was used: Charles Ponzi, 1920. A philanthropic version of the model is also presented for discussion, bearing in mind social security models.

The aptitude of simulation in the detection of unsustainable patterns may be of interest to financial regulators and investors when confronted with situations where funds show unrealistic performances vis-à-vis the economic and financial constraints.

Keywords: Ponzi schemes*; investment; rate of return; ordinary differential equations; numerical methods for odes; simulation*

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>> 1. INTRODUCTION

The aim of this work is the study of Ponzi schemes using mathematical modeling and numerical simulation. It is intended to describe the mathematics of this type of scams, going beyond the simplistic explanation that its sustainability depends only on rapid (exponential) growth on the number of customers and investors. Furthermore, it aims at exploring the numerical simulation of Ponzi schemes, allowing for the fast evaluation of funds showing unrealistic performances.

Ponzi schemes, chain letters, pyramid schemes and bubbles are used terms, often indistinctively and improperly, to refer to unsustainable financial behavior where the current and promised prices of the assets are not consistent with their value in the future. These schemes generally involve promises to pay very high interest rates. The developer of these schemes generally claims to have discovered a new secret formula that allows to earn a lot of money and to share high rates of return.

The term "Ponzi" is due to Charles Ponzi, an Italian immigrant in the USA, who operated a small deposit taking company in a suburb of Boston in the 20's. Since most of the economies in Europe were depressed after the First World War, he thought of buying International Reply Coupons (IRC) in Europe and selling them in the U.S. Certainly it was not possible to buy a required large amount of coupons neither to cope with the logistic to take them back to the U.S. He soon realized that the only way to become rich was by convincing others that they too could be rich if they follow his scheme and promised a 50% profit within 45 days, and soon 100% within 90 days (DeWitt (2009)). As already known, the payments to the old investors were being paid by the new ones.

Bernard Madoff was responsible for the greater and long-lasting Ponzi scheme. By the time of the collapse in December of 2008, reported liabilities amounted to values close to the 65 billion. Madoff kept his scheme active for approximately 20 years and affected private and institutional investors around the world. An exhaustive description of the event is reported in Kindleberg and Aliber (2011).

Directly affecting Portuguese investors, two cases are in the recent memory of all. The case “Dona Branca” - the people's banker: 10% monthly interest rate compared with the 30% rate paid by the (state) bank annually. Portugal was living a depression period in the 80's. Also Afinsa/Philatelic Forum case: a pyramid scam detected in Spain in May 2006, where the remuneration to clients will have varied, as the years, between 10 and 16 percent per year, J.M.R (2008).

These events are becoming more and more frequent and their social and economic consequences subject of many news stories, papers and research. Fraudulent schemes makes their place in the news and articles, for example, in Albania are referred by Jarvis (2000); the global and well-known Madoff case is pictured in great detail by Sander (2009) and an extensive review with recent stories of swindlers (individual and institutional) are presented in Fisher (2010).

However, there is a lack of scientific literature providing mathematical background on these schemes helping to predict their initial success and final collapse of fraudulent financial instruments. Two of the most prominent exceptions describing macroeconomic models of rational Ponzi games are the works of Blanchard and Weil (2001) and Bhattacharya (2003). Micro Ponzi schemes have proliferated on the Internet or at a smaller scale beginning with a group of friends. These schemes are usually described via a simplistic explanation based on an exponential boom on the number of new investors. A mathematical model that captures the main characteristics of a Ponzi scheme was recently introduced by Artzrouni (2009).

We will briefly describe the modeling and introduce a simulation procedure able to be easily adapted to new variations on the model parameters and possibly extended to cope with other features not yet considered.

>> 2. THE ARTZROUNI'S MODEL

The fund motion equation, defining the fund value accumulation along time, is given by

$$\frac{dS(t)}{dt} = r_n S(t) + s(t) - W(t) \tag{1}$$

where $S(t)$ is the amount in the fund at t, r_n is the nominal interest rate, $s(t)$ is a continuous cash inflow (new money) and $W(t)$ the withdrawals. The inflow assumes an exponential behavior

$$s(t) = s_0 e^{r_i t} \tag{2}$$

where r_i is the investment rate and s_0 the initial density of the deposits. For the withdrawals, the following formulation is considered

$$W(t) = r_w K e^{t(r_p - r_w)} + r_w \int_0^t s(u) e^{(r_p - r_w)(t-u)} du \tag{3}$$

where r_p is the promised rate and r_w a constant withdrawal rate along t. The first term refers to withdrawals from those investing an initial deposit of $K \geq 0$ (at $t = 0$) and the second from those who invested $s(u)$ at time u and want to withdraw a quantity $r_w s(u) e^{(r_p - r_w)(t-u)}$ at time $t > u$. If $r_w < r_p$ ($r_w > r_p$) then the withdrawals increase (decrease) exponentially.

Putting (2) in (3) gives

$$W(t) = r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{e^{t(r_w + r_i - r_p)} - 1}{r_w + r_i - r_p} \right) \tag{4}$$

For further details see Artzrouni (2009). The differential equation is thus given by (1) with $W(t)$ defined by (4).

The idea behind the simulation is to tackle the solution dynamics of the model along with the dynamics resulting from a sudden parameter change at a certain moment in time. We will consider the solution path of the actual amount in the fund (its real value), $S_a(t)$, and compare it with the solution path of the promised amount (theoretical value of the fund), $S_p(t)$. The latter equation results from the former by taking the nominal interest rate equal to the promised one, $r_n = r_p$ and the initial condition $S(0) = K$, initial deposit, or consider additionally an "in-house" deposit invested at rate r_n, $S(0) = K0 + K$.

>> 3. MODEL CALIBRATION

For the calibration we follow closely DeWitt (2009) and Artzrouni (2009).

Ponzi's scheme:

The model is calibrated to replicate the fraudulent scheme perpetrated by Charles Ponzi in 1920, and evaluate the effects resulting from considering the investment constant from a certain instant of time, $r_i = 0$. The data is supported on DeWitt (2009) and the time is expressed in "years" and the monetary values in million USD.

Since Ponzi had no funds to invest, it is assumed $S(0) = K = 0$. Considering that he paid 100% (the money doubles) of the invested money after 90 days (0.25 years), on an interest rate with continuous capitalization this would result in $e^{0.25r_p} = 2$, that is $r_p = 2.773$, The scheme worked from the 26th December to the 26th June, that is, for 213 days (0.58 years). At day 213, Ponzi collected 0.2 million USD from a total of 10 million USD invested in the fund:

$$\begin{cases} s_0 e^{0.58 r_i} = 0.2 * 365 \\ s_0 \left(\dfrac{e^{0.58 r_i} - 1}{r_i} \right) = 10 \end{cases}$$

which yields s0 = 1.130 and ri = 7.187. This represents an instantaneous rate of 7% and an initial flow of money of $\frac{1.130}{365} \cong 3.095$ USD a day. Regarding the nominal interest rate, it is assumed $r_n = 0.01$; as referred by Artzrouni (2009), the profit from the IRC is negligible given the high rate of entry of new investments (one could consider r_n ranging from 0 to 5%). To compute r_w it is taking into account the data available at the time of the first trial of Ponzi: he was able to pay 5 million but still was pending payment to investors other 7 million USD, so $S_p(0,58) = 12$. Numerically we can evaluate the value of $r_w = 1.470$.

From the 26th June, probably the growing flux of new investments had stopped and so, from $t = 0.58$ onwards, new parameters occurred. The daily flow remains constant and equal to 0.2 million USD, as in the last day, $r_i = 0$ and $S(0) = S_a(0.58) \cong 7.758$. The initial values for investment are and (numerically computed). The remaining parameters are kept the same (see Table 1. for a summary).

Table 1: Model parameters for the Ponzi simulation

	K	S(0)	s_0	r_i	r_w	r_p	r_n
$t \le 0.58$	0	0	1.130	7.187	1.470	2.773	0.01
$t > 0.58$	12	7.758	73	0	1.470	2.773	0.01

Philanthropic scheme:

The model can be easily used to mimic a philanthropic (Ponzi) scheme. Following Artzrouni (2009), let us track a manager (meritorious profile) paying return upon one million dollars contribution per investor a year. It is assumed that the investors withdraw 12% of their accumulated capital (continuously), the manager can make investments at the nominal rate of 4%, and it is not request any initial investment to investors. Under these conditions, the minimum value for in-house investment needed to keep the fund solvent is (computed numerically) $S(0) \cong 275$. In the following, the value $S(0) = 280$ will be used for simplicity.

The philanthropic character of this fund is based on the fact that, although the actual value of the fund $S_a(t)$ and of the fund's profit, $S_a(t) - S_p(t)$, grow asymptotically at a rate equal to r_n, the lucrativeness that the manager of the fund would had if he had invested $S(0)$ at a rate r_n would be higher, since $S(0) > 0$.

Table *2: Model parameters for the philanthropic (Ponzi) simulation*

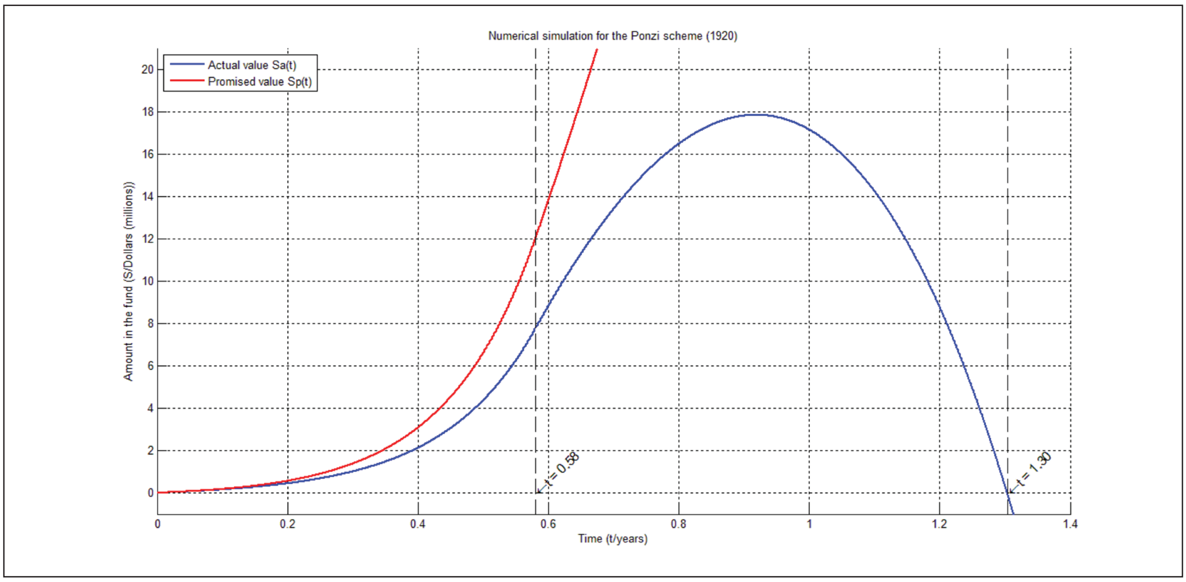
K	$S(0)$	s_0	r_i	r_w	r_p	r_n
0	280	1	0	0.12	0.15	0.04

>> 4. MODEL SIMULATION AND RESULTS

Ponzi's scheme (1920):

Based on the parameters of the model presented in Table 1., the transition dynamics, computed numerically, are plotted in Fig. 1., for the actual $S_a(t)$ and promised $S_p(t)$ values. The numerical algorithm used to solve the ordinary differential equation was the Runge-Kutta pair (4,5), embodied in the MATLAB ODE45 function (convergence criterium defined for a tolerance of 10^{-9}).

Fig. 1: numerical simulation for the Ponzi scheme (1920)

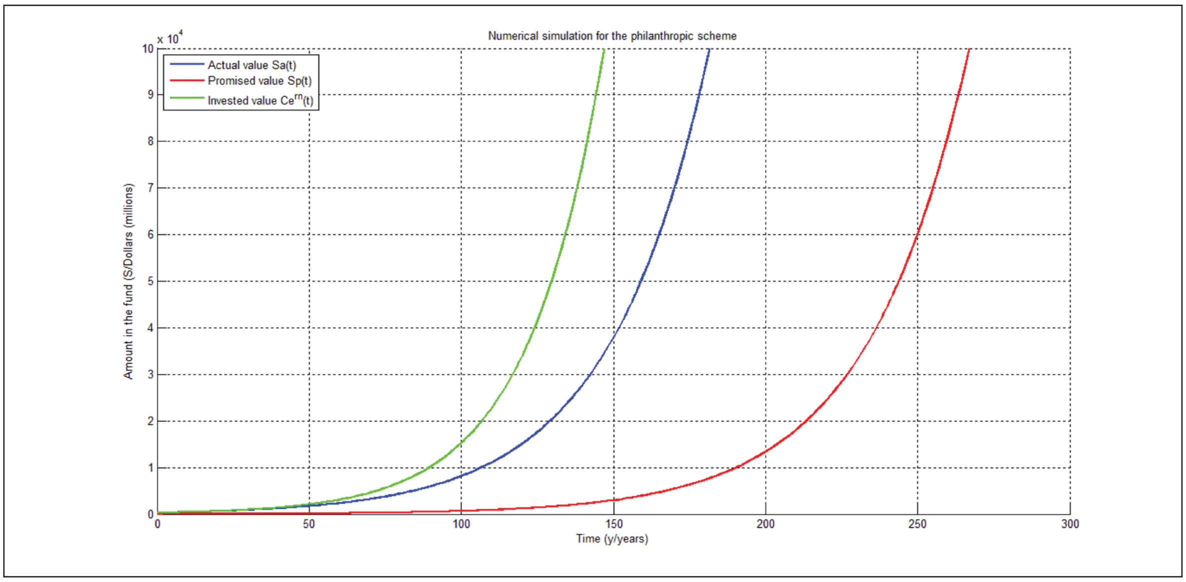


Graphically it turns out that, at the date of the arrest of Charles Ponzi, t = 0.58, the scheme is yet sustainable, although, as expected, the real value is growing at a lower rate than the promised. From t = 0.58 onwards, with the change of the rate of investment to $r_i = 0$, the scheme enters in collapse: increases until 0.9 years but then decays to collapse, $S_a(t) = 0$, 1.3 years later, that is, after only 8.5 months after Ponzi be arrested.

Philanthropic scheme:

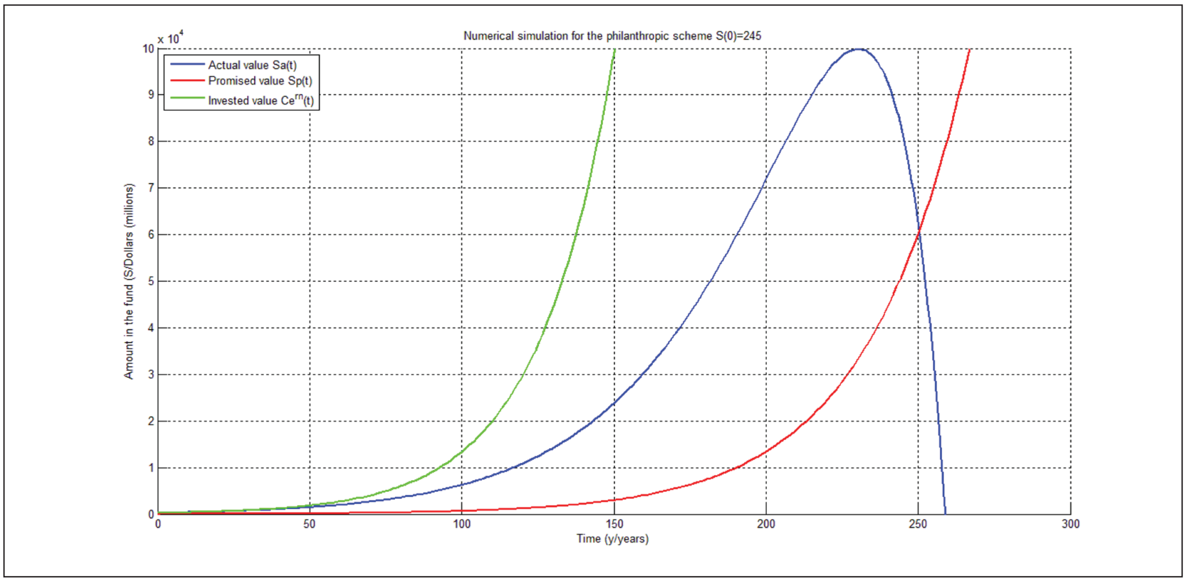
Based on the parameters on Table 2., Fig. 2 depicts the time evolution of the actual and promised values for the philanthropic (Ponzi) scheme.

Fig. 2: numerical simulation for the philanthropic scheme



It can be seen that, even in the long run, a fund within these conditions is always solvent. The initial in-house entry of capital is sufficient to guarantee the sustainability. The actual value grows at a higher rate than the promised one; but it is lower than the value that the manager would earn if he had instead invested $S(0)$ at rate r_n (green line).

Fig. 3: numerical simulation for the philanthropic scheme S(0) = 245:



To illustrate the importance of $S(0)$, corresponding to the initial in-house investment, we access what happens if it decays from 280 to 245 million USD, below the threshold of 275 (the threshold for the sustainability of the fund). Fig. 3, indicates that the schema will collapse after 250 years of existence.

Table 3. shows the time for the collapse for several values of $S(0)$, emphasizing that solvency depends on a philanthropic fund manager who is willing to invest a significant initial amount, giving away a share of her profits.

Table 3: Several parameters for the philanthropic simulation: K=0; s_0=1, r_i=0; r_n=0,04

$S(0)$	r_w	r_p	r_n	time to collapse
280	0.12	0.15	0.04	N/A (Fig. 2)
275	0.12	0.15	0.04	N/A
265	0.12	0.15	0.04	369
245	0.12	0.15	0.04	259 (Fig. 3)
175	0.12	0.15	0.04	138
100	0.12	0.15	0.04	80
245	*0.12*	*0.155*	*0.04*	*146*

Further enlightening examples:

We extend further our sensitivity analysis by considering two additional situations, and analyze the impact on the sustainability, due to changes in the initial conditions (of launching and operating).

A. What is the effect due to changes in the rate of investment in a fund, considering the parameters in Table 4. and keeping the remaining conditions unchanged?

Table 4: parameters for scheme A

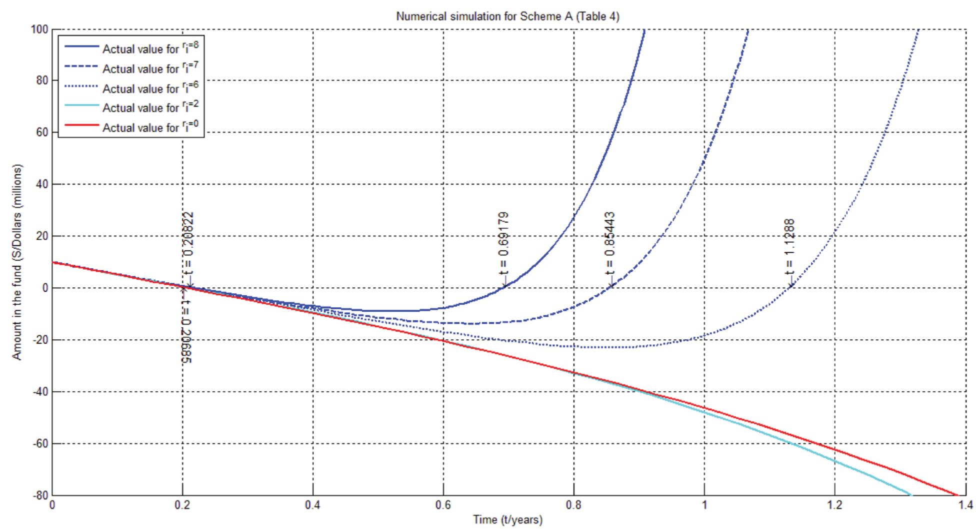
Case	K	C	s_0	r_i	r_w	r_p	r_n	t/collapse	t/recovery
1	10	10	2	8	6	5	1	0.217	0.692
2	10	10	2	7	6	5	1	0.215	0.854
3	10	10	2	6	6	5	1	0.213	1.129
4	10	10	2	2	6	5	1	0.208	N/A
5	10	10	2	0	6	5	1	0.207	N/A

This scheme is very interesting in that, initially, the fund value decreases, reaches a minimum and then, depending on the rate of investment, can retrieve at distinct speeds. Clearly the recovery is much faster as the higher the investment rate. If the investment rate is lower than the rate of withdrawal, the system does not retrieve and collapses shortly after its launch.

This schema also supports the idea that, although illegal (r_p » r_n), its sustainability would be ensured through an appropriate relation between withdraw and investment rates.

Figure 4 shows the solution paths for the cases summarized in Table 4.

Fig. 4: numerical simulation for Table 4.



B. A fund offers 10% return per month. The management entity is supposed to perform an initial investment of 50 million USD in-house. It is expected the investor deposits to increase at a 0.5 annual rate and each investor, annually, will withdraw half of his winnings. What happens if the annual density of deposits is 4 million USD? And if it is 1 million USD?

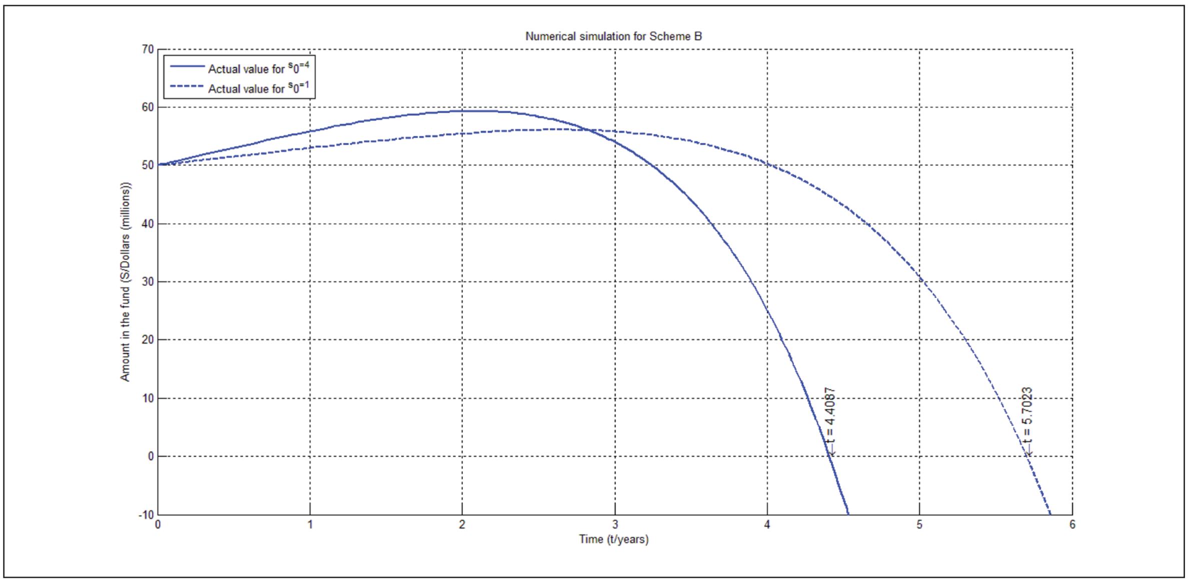
The conditions of the fund are summarized in Table 5.

Table 5: parameters for scheme B

Case	K	C	s_0	r_i	r_w	r_p	r_n	t/collapse
1	0	50	4	0.5	0.5	1.1437	0.04	4.409
2	0	50	1	0.5	0.5	1.1437	0.04	5.702

If the analyzes only covers the short run, 0-2 years, it seems that this scheme is sustainable. However, the combination of parameters during time does that at the medium-long term, 5-6 years, the fund collapse. Figure 5 presents the solution path for the two cases tested (see Table 5.)

Fig. 5: numerical simulation for Table 5.



Social Security systems:

One of the issues currently in focus, see for example, Mandel (2008) and Buttonwood (2011), is the easy temptation to associate Social Security systems to a Ponzi one. The usual short duration of Ponzi schemes, despite the longevity of the Madoff case, in counterpoint with the high longevity of Social Security systems, since 1889 in Germany or since 1935 in the United States, are motive enough for DeWitt (2009), to exclude a Ponzi oddity to Social Security systems.

The abovementioned philanthropic scheme clarifies some ideas on the topic, illustrating that a scheme can work sustainably "forever", provided that a high initial investment is delivered (for example, an investment in a government employment policy to create a new social security system). However, this long-term sustainability can be threatened, by investment rates, withdrawal rates and nominal rates.

We can see from Table 3. that, keeping all other parameters constant, the system is sustainable for $S(0) = 280$ and $r_p = 0.15$, whereas it collapses after 259 years for $S(0) = 245$ and $r_p = 0.15$; furthermore, for $S(0) = 245$ and $r_p = 0.115$ it collapses in almost half of the time, 146 years. The sensitivity is very high.

In periods where the beneficiaries of a social security system grow much faster than the entrance of new contributors, there may results in an operation deficit. This vulnerability to demographic fluctuations has however nothing to do with Ponzi schemes or other fraudulent financial transactions.

It is clear the relationship between the model parameters. For example, fluctuations in investment, withdraw and birth rates, the rate of individuals who retire and the presence of other individual retirement protection mechanisms (possibly more attractive).

>> 5. CONCLUSIONS

Ponzi scheme is a generic term to designate non sustainable patterns of financial operations. The creators of these schemas can only fulfill their commitments to pay high interest rates if they can collect money from new loans, namely novel clients and investors. The promised interest rates are so high that the sustainability of the scheme requires a continuous stream of new money and at a fast pace. Initially, many investors are so satisfied with their high returns that they allow recapitalization of their gains in the same scheme. The scheme can work and be sustainable provided that the fund's value output rate is lower than the rate of money entry of new investors. This work describes a model that can provide some answers to financial regulators and investors when faced with situations of funds presenting themselves as having unrealistic performances regarding the economic and the financial situation.

Since the model is built on continuous time, it does not allow carrying out the follow-up of an investor situation in accordance with its position on a pyramidal structure. This may be the (only) limitation of this model.

We provide an innovative approach by considering the numerical solution of the model. Several simulations were performed to access the behavior of the fund by considering different perturbations on the parameters of the model.

Future work may consider going beyond the concepts of investments, withdrawals and interest rates, adding new parameters to the template, such as for example, typification of customer profiles and promoters, economic indicators of conjuncture, etc. Some important references for further research can be found in, for example, Sadiraj and Schram (1999), Benson and Chumney (2011) and Kindleberg and Aliber (2011).

Another research line to extend the scope of this work is to consider stochastic modeling. In this possible framework, the work of Mayorga-Zambrano (2011) may be of great importance to explore the social impact in case of collapse of the system.

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