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Limiting behaviour of a geometric-type estimator for tail indices

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Abstract

We propose a consistent estimator for the exponential tail coefficient of a d.f., that is directly related to least squares estimators of Schultze and Steinebach [Statist. Decis. 14 (1996) 353]. We investigate here the weak asymptotic properties of this geometric-type estimator, showing in particular that, under general conditions, its distribution is asymptotically normal. The results are then applied to the related problem of estimating the adjustment coefficient in risk theory [Insur.: Math. Econ. 10 (1991) 37]. A simulation study is performed in order to illustrate the finite sample behaviour of the proposed estimator.

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JEL classification: C13; C14

Keywords: Adjustment coefficient; Least squares estimators; Parameter estimation; Tail indices; Universal asymptotic normality

1. Introduction

Let $Z_1, Z_2, \ldots$ be independent, non-negative random variables with common distribution function (d.f.) F satisfying

$$1 - F(z) = P(Z_1 > z) = r(z)\,\mathrm{e}^{-Rz}, \quad z > 0, \tag{1}$$

where r is a regularly varying function at infinity and R a positive constant. Denoting by F^{-1} the left continuous inverse of F, i.e., $F^{-1}(s) := \inf\{x : F(x) \geq s\}$, (1) is equivalent to

$$F^{-1}(1-s) = -\frac{1}{R}\log s + \log \tilde{L}(s), \quad 0 < s < 1, \tag{2}$$

where $\tilde{L}$ is a slowly varying function at zero (see, e.g. Schultze and Steinebach, 1996 and references therein).

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We shall be concerned here with the estimation of the tail coefficient R in (1) or, equivalently, in (2). The problem of estimating R or other related tail indices has received considerable attention and common applications may be found in a big variety of domains. We consider here an important application in risk theory, namely the estimation of the adjustment coefficient (see Csörgő and Steinebach, 1991). For a comprehensive overview of this subject we refer to Csörgő and Viharos (1998).

Based on least squares considerations, Schultze and Steinebach (1996) proposed three estimators for the exponential tail coefficient R, given as follows. Let $Z_{1,n} \le Z_{2,n} \le \cdots \le Z_{n,n}$ denote the order statistics of the sample $Z_1, Z_2, \ldots, Z_n$ and assume that (k_n) is a sequence of positive integers satisfying

$$1 \le k_n < n, \quad \lim_{n\to\infty} k_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{k_n}{n} = 0. \tag{3}$$

The Schultze and Steinebach estimators are defined by

$$\hat{R}_1(k_n) = \frac{\sum_{i=1}^{k_n} \log^2(n/i) - (1/k_n)\left(\sum_{i=1}^{k_n} \log(n/i)\right)^2}{\sum_{i=1}^{k_n} \log(n/i) Z_{n-i+1,n} - (1/k_n)\left(\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)\left(\sum_{i=1}^{k_n} \log(n/i)\right)},$$

$$\hat{R}_2(k_n) = \frac{\sum_{i=1}^{k_n} \log^2(n/i)}{\sum_{i=1}^{k_n} \log(n/i) Z_{n-i+1,n}} \tag{4}$$

and

$$\hat{R}_3(k_n) = \frac{\sum_{i=1}^{k_n} \log(n/i) Z_{n-i+1,n} - (1/k_n)\left(\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)\left(\sum_{i=1}^{k_n} \log(n/i)\right)}{\sum_{i=1}^{k_n} Z^2_{n-i+1,n} - (1/k_n)\left(\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)^2}. \tag{5}$$

Recently, Brito and Moreira (2001) have introduced a new estimator of R, $\hat{R}(k_n)$, related to $\hat{R}_1(k_n)$ and $\hat{R}_3(k_n)$. This estimator arises in a natural way from a geometrical adaptation of the procedure used by Schultze and Steinebach in the construction of $\hat{R}_i(k_n)$, $i = 1, 3$. These estimators are motivated by the fact that, for large z, $-\log(1-F(z))$ is approximately linear with slope R, since $z^{-1}\log r(z) \to 0$ as $z \to \infty$. If the regularly varying function r was constant, say $r(z) = \mathrm{e}^d$, $d \in R$, then

$$-\log(1-F(z)) = Rz - d.$$

We thus expect that the above linear relation approximately holds for the largest observations realized in the sample $(Z_1, Z_2, \ldots, Z_n)$, which we simply denote by $z_{(i)} = z_{n-i+1,n}$, $i = 1, \ldots, k_n$. Approximating $F(z_{(i)})$ by $F_n(z^-_{(i)})$, where F_n is the empirical d.f., this gives that $-\log(1-F_n(z^-_{(i)})) = \log(n/i)$ is "close" to $Rz_{(i)} - d$, or $z_{(i)}$ is "close" to $R^{-1}\log(n/i) + R^{-1}d$, $i = 1, \ldots, k_n$. Setting $a = R^{-1}$ and $b = R^{-1}d$, a least squares estimator may then be obtained by minimizing $f_1(a,b) = \sum_{i=1}^{k_n}(z_{(i)} - a\log(n/i) - b)^2$, leading to the estimator $\hat{R}_1(k_n) \equiv \hat{a}^{-1}$. In the particular case where $d = 0$, the minimization of $f_2(a) = f_1(a, 0)$ yields the estimator $\hat{R}_2(k_n)$. On the other hand, the direct minimization of $f_3(R,d) = \sum_{i=1}^{k_n}(\log(n/i) - Rz_{(i)} + d)^2$, leads to the least squares estimator $\hat{R}_3(k_n)$.

Considering the two points of view simultaneously, by minimizing the global sum of the areas of the rectangles indicated in Fig. 1, we obtain the estimator $\hat{R}(k_n)$.

In this way, $\hat{R}(k_n)$ results from minimizing $f(R,d) = \sum_{i=1}^{k_n}(\log(n/i) - Rz_{(i)} + d)(R^{-1}\log(n/i) + R^{-1}d - z_{(i)})$, and is given by the geometric mean of $\hat{R}_1(k_n)$ and $\hat{R}_3(k_n)$, that is

$$\hat{R}(k_n) = \sqrt{\frac{\sum_{i=1}^{k_n} \log^2(n/i) - (1/k_n)\left(\sum_{i=1}^{k_n} \log(n/i)\right)^2}{\sum_{i=1}^{k_n} Z^2_{n-i+1,n} - (1/k_n)\left(\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)^2}}. \tag{6}$$

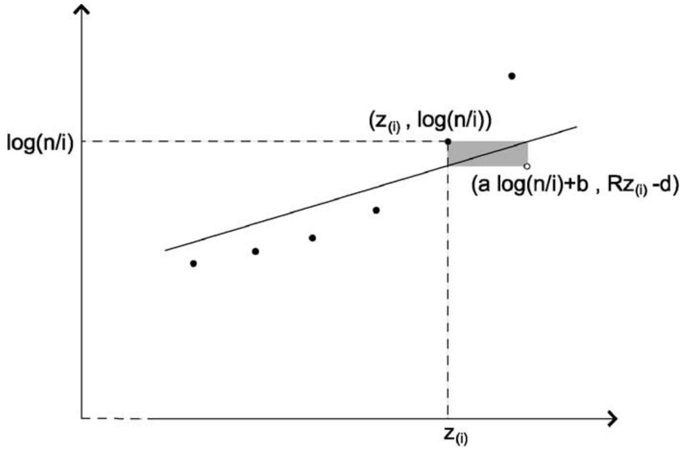

Fig. 1.

Schultze and Steinebach (1996) established the consistency of the estimators $\hat{R}_i(k_n)$, $i = 1, 2, 3$ and their corresponding asymptotic behaviour was subsequently investigated by Csörgő and Viharos (1997). Independent of these authors, Kratz and Resnick (1996) introduced an equivalent form of $1/\hat{R}_1(k_n)$, designated by qq-estimator, in reference to the quantile–quantile plots (for this interpretation and application of qq-plots in this estimation problem, see also Beirlant et al., 1996). Kratz and Resnick proved the consistency and the asymptotic normality of the qq-estimator centred at $1/R$. Not forcing the centring at $1/R$, Csörgő and Viharos (1997) have shown that, for suitable sequences (k_n), $1/\hat{R}_i(k_n)$, $i = 1, 2, 3$, are universally asymptotically normal over the family (1), in the usual sense, that is, with deterministic centring sequences converging to $1/R$. Moreover, for $1/\hat{R}_i(k_n)$, $i = 1, 3$, the norming sequence is $k_n^{1/2}$, and as Csörgő and Viharos (1997) pointed out, these were the first estimators asymptotically normal over the whole family (1), with the ideal factor $k_n^{1/2}$.

The above estimation problem is equivalent to the estimation of the tail index of a Pareto type distribution. In fact, setting $X_i = e^{Z_i}$ with Z_i, $i = 1, 2, \ldots$ as above, we have

$$1 - G(x) = P(X_1 > x) = x^{-1/\alpha}L(x), \quad x > 0, \tag{7}$$

where $\alpha = 1/R$ and $L(x) = r(\log x)$ is slowly varying at infinity. The qq-estimator was actually introduced under (7). In this context, several estimators have been proposed. One of the most commonly used estimators for α, is the Hill estimator (1975), defined by

$$H_n(k_n) = \frac{1}{k_n}\sum_{i=1}^{k_n} \log X_{n-i+1,n} - \log X_{n-k_n,n}, \tag{8}$$

where $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ denote the order statistics of the sample $X_1, X_2, \ldots, X_n$ (for related estimators, see, e.g. De Haan and Resnick, 1980; Csörgő et al., 1985; Bacro and Brito, 1993). The asymptotic properties of the Hill estimator have been much studied and it is well known that, under certain conditions, $H_n(k_n)$ is a strongly consistent estimator (cf. Deheuvels et al., 1988) with asymptotic normal distribution (cf. Haeusler and Teugels, 1985).

In this paper, we investigate the asymptotic properties of the geometric-type estimator $\hat{R}(k_n)$. In particular, we shall give conditions which ensure the asymptotic normality of $\hat{R}(k_n)$ when centred at R. We shall also see that $1/\hat{R}(k_n)$ is universally asymptotically normal over the family (1). We recall that this property is not shared by the Hill estimator (see, e.g. Csörgő and Viharos, 1998). Moreover, the norming sequence is again the ideal factor $k_n^{1/2}$.

This specific property, jointly with the fact that $\hat{R}(k_n)$ takes values between those of $\hat{R}_1(k_n)$ and $\hat{R}_3(k_n)$, makes the use of the estimator $\hat{R}(k_n)$ specially attractive for the case where R is expected to be small. The application in risk theory considered here is of this kind. Our results are given in Section 2 and the proofs are collected in Section 3. The application in the estimation of the adjustment coefficient is discussed in Section 4. One complex practical problem is the choice of the number of observations included in the estimation of R. We consider here an heuristic method suggested by Schultze and Steinebach (1996) and adapt it to our estimator $\hat{R}(k_n)$. This procedure is applied in a small-scale simulation study and the corresponding results are contained in Section 5.

2. Results

We begin by considering the consistency of the estimator $\hat{R}(k_n)$ defined by (6). In the sequel, $\stackrel{D}{\to}$ and $\stackrel{D}{=}$ stand, respectively, for convergence and equality in distribution. In the same way, $\stackrel{P}{\to}$ denotes convergence in probability.

Theorem 1. *Assume that F satisfies condition* (1) *and* k_n *is a sequence of positive integers satisfying* (3) *and such that* $\lim_{n\to\infty} \log^2 n/k_n = 0$. *If* F^{-1} *is continuous on* $(s_0, 1)$ *for some* $s_0 \in (0, 1)$, *then,*

$$\hat{R}(k_n)\stackrel{P}{\to}R.$$

As noted in Section 1, $\hat{R}(k_n)$ is the geometric mean of the estimators $\hat{R}_1(k_n)$ and $\hat{R}_3(k_n)$, and so takes values between them. In the following theorem the order relation between the three estimators is made explicit.

Theorem 2. *Let* $\hat{R}_1(k_n)$, $\hat{R}_3(k_n)$ *and* $\hat{R}(k_n)$ *be the estimators defined by* (4)–(6), *respectively. Then,*

$$\hat{R}_3(k_n) \le \hat{R}(k_n) \le \hat{R}_1(k_n).$$

Before giving our main results, we summarize below the results on the universal asymptotic normality of $1/\hat{R}_i(k_n)$, $i = 1, 3$, obtained by Csörgő and Viharos (1997).

Theorem 3 (Csörgő and Viharos, 1997, Theorem 1.1). *If* k_n *is a sequence of positive integers such that* (3) *holds and* $\lim_{n\to\infty} k_n/\log^4 n = \infty$, *then, whenever F satisfies* (1),

$$k_n^{1/2}\left\{\frac{1}{\hat{R}_1(k_n)} - \mu_n^{(1)}(k_n)\right\}\stackrel{D}{\to}N\left(0, \frac{2}{R^2}\right),$$

where $\mu_n^{(1)}(k_n) := -(n/k_n)\int_0^{k_n/n} F^{-1}(1-t)\{1+\log(nt/k_n)t\}\,\mathrm{d}t \to 1/R$ *as* $n \to \infty$.

Theorem 4 (Csörgő and Viharos, 1997, Theorem 1.3). *If* k_n *is a sequence of positive integers such that* (3) *holds and* $\lim_{n\to\infty} k_n/\log^4 n = \infty$, *then, whenever F satisfies* (1),

$$k_n^{1/2}\left\{\frac{1}{\hat{R}_3(k_n)} - \mu_n^{(3)}(k_n)\right\}\stackrel{D}{\to}N\left(0, \frac{2}{R^2}\right),$$

where $\mu_n^{(3)}(k_n) := \left((n/k_n)\int_0^{k_n/n}(F^{-1})^2(1-t)\,\mathrm{d}t - \left((n/k_n)\int_0^{k_n/n} F^{-1}(1-t)\,\mathrm{d}t\right)^2\right) \Big/ \left(-(n/k_n)\int_0^{k_n/n} F^{-1}(1-t)\{1+\log(nt/k_n)\}\,\mathrm{d}t\right) \to 1/R$ *as* $n \to \infty$.

We shall prove a similar result for the estimator $1/\hat{R}(k_n)$.

Theorem 5. *If k_n is a sequence of positive integers such that (3) holds and $\lim_{n\to\infty} k_n/\log^4 n = \infty$, then, whenever F satisfies (1),*

$$k_n^{1/2}\left\{\frac{1}{\hat{R}(k_n)} - \mu_n(k_n)\right\} \xrightarrow{D} N\left(0, \frac{2}{R^2}\right),$$

where $\mu_n(k_n) := (\mu_n^{(1)}(k_n)\mu_n^{(3)}(k_n))^{1/2} \to 1/R$ as $n \to \infty$, and $\mu_n^{(1)}(k_n)$ and $\mu_n^{(3)}(k_n)$ are defined as in the above two theorems.

Thus, the estimator $1/\hat{R}(k_n)$ is, like the estimators $1/\hat{R}_1(k_n)$ and $1/\hat{R}_3(k_n)$, universally asymptotically normal over the whole family (1).

We turn now to the investigation of conditions under which $k_n^{1/2}\{\hat{R}(k_n) - R\}$ is asymptotically normal.

Theorem 6. *Assume that F satisfies (1) and k_n is a sequence of positive integers such that (3) holds. If we suppose that, as $n \to \infty$,*

$$k_n^{1/2} \sup_{1/k_n \le y \le 1}\left|\log\left(\frac{\tilde{L}(yt(k_n/n))}{\tilde{L}(k_n/n)}\right)\right| \to 0 \tag{9}$$

uniformly in t on compact sets of $(0,\infty)$, then

$$\frac{1}{\sqrt{2}R} k_n^{1/2}(\hat{R}(k_n) - R) \xrightarrow{D} N(0,1).$$

In order to get more explicit conditions, it is necessary to specify the asymptotic behaviour of the slowly varying function $L(x) = r(\log x)$ introduced in (7). For that we use the notion of slow variation with remainder (see Bingham et al., 1987, Chapter 3), and, in particular, we consider the following asymptotic relation:

$$(\mathbf{SR1})\quad \frac{L(tx)}{L(x)} - 1 = \mathrm{O}(g(x)) \quad \text{as } x \to \infty \text{ for each } t > 0,$$

where g is a positive function satisfying $g(x) \to 0$ as $x \to \infty$. For the sake of simplicity, we shall assume in the sequel that g is regularly varying with index $\gamma < 0$. As shown in Bacro and Brito (1998), under the above assumptions, condition (9) may be much simplified, leading to the following result.

Corollary 1. *Assume that the slowly varying function L in (7) satisfies* (**SR1**) *with g regularly varying at infinity with index $\gamma < 0$. Then, if*

$$k_n^{1/2} g\left(\exp\left(F^{-1}\left(1 - \frac{k_n}{n}\right)\right)\right) \to 0 \quad \text{as } n \to \infty,$$

we have

$$\frac{1}{\sqrt{2}R} k_n^{1/2}(\hat{R}(k_n) - R) \xrightarrow{D} N(0,1).$$

3. Proofs

Throughout this section we shall assume that (1) holds. We assume also that $U_1, U_2, \ldots$ is a sequence of independent uniform $U(0,1)$ random variables. The order statistics of the sample $(U_1, U_2, \ldots, U_n)$ are denoted by $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$.

Proof of Theorem 1. Schultze and Steinebach (1996) proved that if k_n satisfies (3) and $\log^2 n/k_n \to 0$ as $n \to \infty$, then $\hat{R}_1(k_n)$ is a consistent estimator of R. Moreover, if F^{-1} is continuous on $(s_0, 1)$ for some $s_0 \in (0, 1)$, then $\hat{R}_3(k_n)$ is also a consistent estimator of R.

Thus, since $\hat{R}(k_n) = \sqrt{\hat{R}_1(k_n)\hat{R}_3(k_n)}$, the result follows by applying Slutsky's theorem. □

Proof of Theorem 2. We recall that $\hat{R}(k_n)$ is the geometric mean of the estimators $\hat{R}_1(k_n)$ and $\hat{R}_3(k_n)$. Hence, for proving this result it suffices to show that $\hat{R}_3(k_n) \le \hat{R}_1(k_n)$, which follows easily applying the Cauchy–Schwarz inequality:

$$\left(\sum_{i=1}^{k_n}\left(\log\left(\frac{n}{i}\right) - \frac{1}{k_n}\sum_{i=1}^{k_n}\log\left(\frac{n}{i}\right)\right)\left(Z_{n-i+1,n} - \frac{1}{k_n}\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)\right)^2$$
$$\le \sum_{i=1}^{k_n}\left(\left(\log\left(\frac{n}{i}\right) - \frac{1}{k_n}\sum_{i=1}^{k_n}\log\left(\frac{n}{i}\right)\right)^2\right)\sum_{i=1}^{k_n}\left(\left(Z_{n-i+1,n} - \frac{1}{k_n}\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)^2\right).$$

□

For the proof of Theorem 5 we need the following two lemmas.

Lemma 1 (Csörgő and Viharos, 1997, Lemma 5.6). *Assume that* (1) *holds and let k_n be a sequence of positive integers satisfying* (3) *and such that* $\lim_{n\to\infty} k_n/\log^4 n = \infty$. *Define* $W_n(k_n) := (1/k_n)\sum_{i=1}^{k_n} Z_{n-i+1,n}^2 - \left((1/k_n)\sum_{i=1}^{k_n} Z_{n-i+1,n}\right)^2$ *and let* $\mu_n(k_n) = (\mu_n^{(1)}(k_n)\mu_n^{(3)}(k_n))^{1/2}$ *be as in* Theorem 5. *Then,* $k_n^{1/2}\{W_n(k_n) - \mu_n^2(k_n)\} = N_n^\star + o_P(1)$, *for a certain sequence* $N_n^\star = N_n^\star(l_n, k_n)$ (*where l_n is a sequence conveniently chosen*) *such that*

$$N_n^\star \xrightarrow{D} N\left(0, \frac{8}{R^4}\right).$$

Lemma 2. *Let k_n be a sequence of positive integers such that* $1 \le k_n \le n$, *and consider the sequence* $i_n(k_n) := (1/k_n)\sum_{i=1}^{k_n}\log^2(n/i) - \left((1/k_n)\sum_{i=1}^{k_n}\log(n/i)\right)^2$. *Then*

$$i_n(k_n) = 1 + \mathrm{O}\left(\frac{\log^2 k_n}{k_n}\right).$$

Proof. Note that

$$i(k_n) \equiv i_n(k_n) = \frac{1}{k_n}\sum_{i=1}^{k_n}\log^2(i) - \left(\frac{1}{k_n}\sum_{i=1}^{k_n}\log(i)\right)^2.$$

Since $\sum_{i=1}^{k_n}\log^2(i) \le \int_1^{k_n+1}\log^2(t)\,\mathrm{d}t$ and $\sum_{i=1}^{k_n}\log(i) = \sum_{i=2}^{k_n}\log(i) \ge \int_1^{k_n}\log(t)\,\mathrm{d}t$, we have

$$i(k_n) \le \frac{k_n+1}{k_n}[\log^2(k_n+1) - 2\log(k_n+1)] + 2 - \frac{1}{k_n^2}[k_n\log k_n - (k_n-1)]^2,$$

which implies the result by a routine calculation. □

Proof of Theorem 3. We may write

$$k_n^{1/2}\left\{\frac{1}{\hat{R}(k_n)}-\mu_n(k_n)\right\}=\frac{1}{i_n^{1/2}(k_n)}k_n^{1/2}\{W_n^{1/2}(k_n)-\mu_n(k_n)\}-\frac{\mu_n(k_n)}{i_n^{1/2}(k_n)}k_n^{1/2}\{i_n^{1/2}(k_n)-1\}. \tag{10}$$

First note that we have

$$W_n^{1/2}(k_n)-\mu_n(k_n)=(W_n(k_n)-\mu_n^2(k_n))\frac{1}{2\xi_n^{1/2}}$$

with $\min(\mu_n^2(k_n), W_n(k_n)) < \xi_n < \max(\mu_n^2(k_n), W_n(k_n))$.

Now, by Lemma 1, $k_n^{1/2}\{W_n(k_n)-\mu_n^2(k_n)\} \xrightarrow{D} N(0, 8/R^4)$ and so $W_n(k_n)-\mu_n^2(k_n) \xrightarrow{P} 0$. Moreover, we know from Theorems 3 and 4 that $\mu_n^2(k_n) \to 1/R^2$ as $n \to \infty$. Thus $W_n(k_n)$ is a consistent estimator of $1/R^2$, and consequently $\xi_n \xrightarrow{P} 1/R^2$.

Hence,

$$k_n^{1/2}\{W_n^{1/2}(k_n)-\mu_n(k_n)\} \xrightarrow{D} N\left(0, \frac{2}{R^2}\right).$$

The result then follows from Eq. (10), using Lemma 2. □

Instead of proving directly Theorem 6, we will establish the following proposition, from which we may immediately obtain the desired result, by considering the transformation $h(x) = 1/\sqrt{x}$.

Proposition 1. *Assume that F satisfies (1) and k_n is a sequence of positive integers such that (3) holds. If we suppose that, as $n \to \infty$,*

$$k_n^{1/2} \sup_{1/k_n \le y \le 1}\left|\log\left(\frac{\tilde{L}(yt(k_n/n))}{\tilde{L}(k_n/n)}\right)\right| \to 0 \tag{11}$$

uniformly in t on compact sets of $(0, \infty)$, then

$$\frac{R^2}{\sqrt{8}}k_n^{1/2}\left(\frac{1}{\hat{R}^2(k_n)}-\frac{1}{R^2}\right) \xrightarrow{D} N(0, 1).$$

Proof. Consider the sequence $(W_i)_{1\le i\le k_n}$ defined by

$$W_i = Z_{n-k_n+i,n} - Z_{n-k_n,n}, \quad 1 \le i \le k_n.$$

With the above notation,

$$\frac{1}{\hat{R}^2(k_n)} = \frac{1}{i_n(k_n)}\left(\frac{1}{k_n}\sum_{i=1}^{k_n}W_i^2 - \left(\frac{1}{k_n}\sum_{i=1}^{k_n}W_i\right)^2\right), \tag{12}$$

where $i_n(k_n)$ is the sequence introduced in Lemma 2:

$$i_n(k_n) := \frac{1}{k_n}\sum_{i=1}^{k_n}\log^2\left(\frac{n}{i}\right) - \left(\frac{1}{k_n}\sum_{i=1}^{k_n}\log\left(\frac{n}{i}\right)\right)^2.$$

Since $Z_i \stackrel{D}{=} F^{-1}(U_i)$, $i \ge 1$, we write, without loss of generality,

$$W_i = F^{-1}(U_{n-k_n+i,n}) - F^{-1}(U_{n-k_n,n}).$$

As in Bacro and Brito (1998) (cf. Theorem 1) we shall make use of the following equivalent representation for W_i, $i = 1, \dots, k_n$:

$$W_i = -\frac{1}{R}\log Y_i + \log\frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}, \tag{13}$$

where

$$Y_i = \frac{1 - U_{n-k_n+i,n}}{1 - U_{n-k_n,n}} \quad \text{for } i = 1, \dots, k_n.$$

We recall also that $(Y_i)_{1\le i\le k_n}$ is distributed as the vector of the order statistics of an i.i.d. k_n-sample from an uniform $(0, 1)$ distribution.

Using Eqs. (12) and (13) we may write

$$\begin{aligned}
\frac{i_n(k_n)}{\hat{R}^2(k_n)} - \frac{1}{R^2} \stackrel{D}{=} & \frac{1}{R^2}\left(\frac{1}{k_n}\sum_{i=1}^{k_n}\left(\log Y_i - \frac{1}{k_n}\sum_{i=1}^{k_n}\log Y_i\right)^2 - 1\right)\\
&+ \frac{1}{k_n}\sum_{i=1}^{k_n}\left(\log\frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})} - \frac{1}{k_n}\sum_{i=1}^{k_n}\log\frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}\right)^2\\
&+ \frac{2}{R}\frac{1}{k_n}\sum_{i=1}^{k_n}\log Y_i \log\frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})} - \frac{2}{R}\frac{1}{k_n^2}\sum_{i=1}^{k_n}\log Y_i \sum_{i=1}^{k_n}\log\frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}\\
=:& \, I(n) + II(n) + III(n).
\end{aligned}$$

Notice that $(1/k_n)\sum_{i=1}^{k_n}\left(\log Y_i - (1/k_n)\sum_{i=1}^{k_n}\log Y_i\right)^2$ is the sample variance of an unit exponential i.i.d. k_n-sample, and consequently we have

$$\frac{R^2}{\sqrt{8}}k_n^{1/2}I(n) \xrightarrow{D} N(0, 1).$$

Next we will show that $k_n^{1/2}(II(n) + III(n))$ converges, in probability, to zero.

First observe that

$$k_n^{1/2}|III(n)| \le k_n^{1/2}\frac{2}{R}\sqrt{\frac{1}{k_n}\sum_{i=1}^{k_n}\left(\log Y_i - \frac{1}{k_n}\sum_{i=1}^{k_n}\log Y_i\right)^2}\sqrt{II(n)},$$

and hence

$$k_n^{1/2}|III(n)| = O_P(1)k_n^{1/2}\sqrt{II(n)}. \tag{14}$$

Considering now the term $II(n)$, it is easily verified that

$$k_n^{1/2}|II(n)| \le 2k_n^{1/2}\left(\sup_{Y_{k_n}\le y\le 1}\left|\log\frac{\tilde{L}(y(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}\right|\right)^2,$$

which implies

$$k_n^{1/2}|II(n)| \le 8k_n^{1/2}\left(\sup_{Y_{k_n}\le y\le 1}\left|\log\frac{\tilde{L}(y(1 - U_{n-k_n,n}))}{\tilde{L}(k_n/n)}\right|\right)^2. \tag{15}$$

Next, choose any $(\lambda_1, \lambda_2) \in]1, +\infty[\times]1, +\infty[$ and consider the event:

$$A_n(\lambda_1, \lambda_2) = \left\{ \lambda_1^{-1} \le \frac{n}{k_n}(1 - U_{n-k_n,n}) \le \lambda_1, \lambda_2^{-1} \le k_n Y_{k_n} \le \lambda_2 \right\}.$$

We have on $A_n(\lambda_1, \lambda_2)$:

$$k_n^{1/2} \sup_{Y_{k_n} \le y \le 1} \left| \log \frac{\tilde{L}(y(1 - U_{n-k_n,n}))}{\tilde{L}(k_n/n)} \right| \le k_n^{1/2} \sup_{1/k_n \le y \le 1} \sup_{1/\lambda_1\lambda_2 \le t \le \lambda_1} \left| \log \frac{\tilde{L}(ytk_n/n)}{\tilde{L}(k_n/n)} \right|,$$

which converges to zero as $n \to \infty$, by condition (11).

Since $P\{A_n(\lambda_1, \lambda_2)\} \to 1$ as $n \to \infty$, then it follows from (14) and (15) that $k_n^{1/2}(II(n) + III(n))$ converges, in probability, to zero.

Now, to complete the proof it is enough to note that

$$\frac{R^2}{\sqrt{8}} k_n^{1/2} \left(\frac{1}{\hat{R}^2(k_n)} - \frac{1}{R^2} \right) = \frac{1}{i_n(k_n)} \frac{R^2}{\sqrt{8}} k_n^{1/2} (I(n) + II(n) + III(n)) + \frac{k_n^{1/2}}{\sqrt{8}} \left(\frac{1}{i_n(k_n)} - 1 \right)$$

and to recall that $i_n(k_n) = 1 + O(\log^2 k_n / k_n)$, by Lemma 2. □

Proof of Corollary 1. See the proof of Corollary 1 in Bacro and Brito (1998). □

4. Estimating the adjustment coefficient in risk theory

The problem of estimating the coefficient R in Eq. (1) is motivated by an important problem in risk theory. Consider the Sparre Andersen model for claims arriving at an insurance company, and assume that the sequence $C_1, C_2, \dots$ of claims occur at times $T_1, T_1 + T_2, \dots$, where $\{C_i\}$ and $\{T_i\}$ are independent sequences of i.i.d. r.v.'s. Starting with initial capital x and with incoming premiums in the time interval $[0, t]$ equal to γt, the risk reserve is

$$S(t) = x + \gamma t - \sum_{i=1}^{N(t)} C_i,$$

where $N(t) = \max\{n \ge 0 : \sum_{i=1}^n T_i \le t\}$ is the number of claims observed up to time t. The probability of ruin is then given by

$$U(x) = P\left\{ \inf_{t>0} S(t) < 0 \right\} = P\left\{ \max_{n\ge1} \sum_{i=1}^n (C_i - \gamma T_i) > x \right\}.$$

Let $D_i = C_i - \gamma T_i$, $i = 1, 2, \dots$ be i.i.d. r.v.'s with $E(D_1) < 0$ and assume that the moment generating function

$$M(r) = E(e^{rD_1})$$

of these r.v.'s is finite in an interval $[0, r_0)$. The unique positive solution R (if it exists) of the equation

$$M(r) = 1$$

is called the *adjustment coefficient*. The importance of this coefficient in risk theory follows from the well-known Cramér–Lundberg inequality:

$$U(x) \le e^{-Rx} \quad \text{for all} \;\; x > 0$$

and the asymptotic relationship:

$$U(x) \sim \lambda\, \mathrm{e}^{-Rx} \quad \text{as } x \to \infty,$$

where λ is a positive constant (see, e.g. Grandell, 1991, Chapter 3).

Csörgő and Steinebach (1991) suggested to estimate R by means of a sequence of auxiliary r.v.'s $\{Z_k\}$, recursively defined as follows:

$$M_0 = 0, \qquad M_n = \max\{M_{n-1} + D_n, 0\} \quad \text{for } n = 1, 2, \dots, \qquad \nu_0 = 0,$$
$$\nu_k = \min\{n \geq \nu_{k-1} + 1 : M_n = 0\} \quad \text{for } k = 1, 2, \dots, \qquad Z_k = \max_{\nu_{k-1} < j \leq \nu_k} M_j \quad \text{for } k = 1, 2, \dots .$$

Then $Z_1, Z_2, \dots$ is an i.i.d. sequence satisfying

$$P(Z_1 > z) = \exp\{-(1 + \mathrm{o}(1))Rz\} \tag{16}$$

as $z \to \infty$. Moreover, if $C(t) := \sum_{i=1}^{N(t)} C_i$ is a compound Poisson process, or if the claims C_i are exponentially distributed, then (16) can even be refined to

$$P(Z_1 > z) = c\, \mathrm{e}^{-Rz}\{1 + \mathrm{O}(\mathrm{e}^{-Az})\} \tag{17}$$

as $z \to \infty$, with positive constants c and A (cf. Csörgő and Steinebach, 1991). So the adjustment coefficient may be estimated by any of the estimators for the tail index previously presented (for alternative estimators, see Deheuvels and Steinebach, 1990; Pitts et al., 1996 and references therein).

Next we will see that Corollary 1 may be applied to the family given by (17).

Note that for this family, $L(x) = c\{1 + \mathrm{O}(x^{-A})\}$. The relation (**SR1**) is then satisfied with the regularly varying function $g(x) = x^{-A}$, and consequently Corollary 1 can be applied. In this case,

$$F^{-1}(s) = \frac{1}{R}\log c - \frac{1}{R}\log(1 - s) + \mathrm{O}((1 - s)^{A/R}) \quad \text{as } s \to 1,$$

and so $F^{-1}(1 - k_n/n) \sim (1/R)\log c - (1/R)\log(k_n/n)$ as $n \to \infty$. It follows that $k_n^{1/2}(\exp(F^{-1}(1 - k_n/n)))^{-A} \sim c^{-A/R}k_n^{1/2}(k_n/n)^{A/R}$ as $n \to \infty$, and this converges to zero if $k_n = \mathrm{o}(n^{2A/(2A+R)})$. Then by Corollary 1, we have

$$\frac{1}{\sqrt{2}R}k_n^{1/2}(\hat{R}(k_n) - R) \xrightarrow{D} N(0, 1)$$

for $k_n \to \infty$ such that $k_n = \mathrm{o}(n^{2A/(2A+R)})$.

We will finish this section with the following example.

Example 1. Consider the case of a compound Poisson claim process with exponentially distributed claims, and put $\alpha := E(\gamma T_1) > E(C_1) =: \beta$. Then

$$M(t) = E(\mathrm{e}^{t(C_1 - \gamma T_1)}) = \frac{1}{(1 - \beta t)(1 + \alpha t)}, \quad t \in \left(-\frac{1}{\alpha}, \frac{1}{\beta}\right).$$

In this case the adjustment coefficient is the solution of the equation $(1 + \alpha r)(1 - \beta r) = 1$, i.e.

$$R = \frac{\alpha - \beta}{\alpha\beta},$$

and the exact d.f. of the r.v.'s $Z_1, Z_2, \dots$ is

$$F(z) = \frac{1 - a\exp(-(1 - a)z/\beta)}{1 - a^2\exp(-(1 - a)z/\beta)}, \quad z > 0,$$

where $a := \beta/\alpha$ (see, e.g. Csörgő and Steinebach, 1991 and references therein). From a routine calculation we see that this distribution is a particular case of the family (17) with $A = R$. So, Corollary 1 still applies with $k_n = \mathrm{o}(n^{2/3})$.

5. Simulation results

Below we extend the simulation study of Schultze and Steinebach (1996) to the estimator $\hat{R}(k_n)$, where samples $Z_1, Z_2, \dots, Z_n$ have been simulated making use of the exact distribution F of the above example, or more precisely, its quantile function

$$F^{-1}(1-s) = \begin{cases} \dfrac{\beta}{1-a} \log\left(\dfrac{a(1-a)}{s} + a^2\right), & 0 < s < \dfrac{a}{1+a}, \\ 0 & \text{elsewhere,} \end{cases} \tag{18}$$

where $a = \beta/\alpha < 1$. For sake of comparison with related studies (see Schultze and Steinebach, 1996 and references therein) we take $(\alpha, \beta) = (24\,000, 10\,000)$, resulting in $R = 5.8(3) \times 10^{-5}$.

In this section we illustrate the finite sample behaviour of the new estimator $\hat{R}(k_n)$ and compare it to the estimators $\hat{R}_1(k_n)$, $\hat{R}_3(k_n)$ and $H_n^{-1}(k_n)$.

We present three figures (Figs. 2–4) that correspond to Z-sample sizes $n = 50, 100, 500$. Simulations have been repeated 100 times. In each figure, we plot the mean estimates of the 100 runs, as a function of $k = k_n$. The exact value of $R = 5.8(3) \times 10^{-5}$ is also indicated. The corresponding standard deviations are given in Tables 1–3.

As it was expected, the finite sample behaviour of all the estimators considered, heavily depends on the number of largest observations taken into account for estimation. The irregular behaviour is accentuated for the Hill estimator, specially for small samples. In view of the results of Csörgő and Viharos (1997, 1998) and the small value of R, this is not surprising. In fact, as pointed out by the above mentioned authors, the least squares-type estimators seem to be more "robust" against deviations of the slowly varying function L from a constant. We may also note that this advantage gradually disappears as the sample size increases. Finally, we may note that the estimates given by $\hat{R}(k_n)$ are quite reliable, even for very small sample sizes. For the basic practical problem of choosing

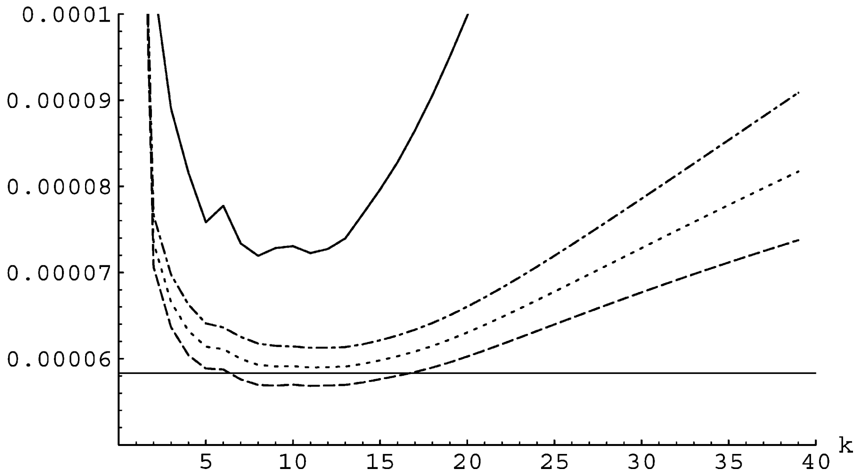

Fig. 2. Mean of the $\hat{R}$ estimates (dotted line), $\hat{R}_3$ estimates (dashed line), $\hat{R}_1$ estimates (dash-dotted line) and Hill estimates (full line) as a function of $k = 2, \dots, 40$, obtained from 100 samples of size $n = 50$ from the distribution defined by (18) with $(\alpha, \beta) = (24\,000, 10\,000)$ $(R = 5.8(3) \times 10^{-5})$.

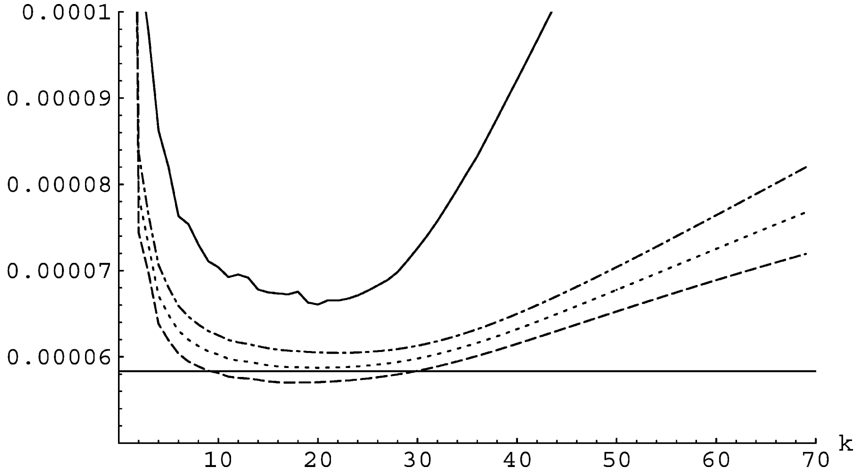

Fig. 3. Mean of the $\hat{R}$ estimates (dotted line), $\hat{R}_3$ estimates (dashed line), $\hat{R}_1$ estimates (dash-dotted line) and Hill estimates (full line) as a function of $k = 2, \ldots, 70$, obtained from 100 samples of size $n = 100$ from the distribution defined by (18) with $(\alpha, \beta) = (24\,000, 10\,000)$ $(R = 5.8(3) \times 10^{-5})$.

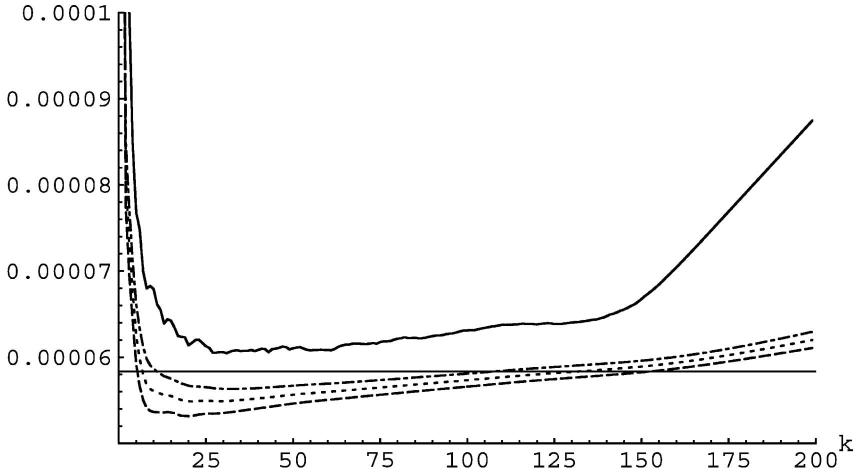

Fig. 4. Mean of the $\hat{R}$ estimates (dotted line), $\hat{R}_3$ estimates (dashed line), $\hat{R}_1$ estimates (dash-dotted line) and Hill estimates (full line) as a function of $k = 2, \ldots, 200$, obtained from 100 samples of size $n = 500$ from the distribution defined by (18) with $(\alpha, \beta) = (24\,000, 10\,000)$ $(R = 5.8(3) \times 10^{-5})$.

the value of k_n, say $\tilde{k}$, we consider one of the heuristic methods proposed by Schultze and Steinebach (1996): choose $\tilde{k} \in \{l, l+1, \ldots, n\}$ $(l > 2)$ such that the mean squared residuals in the linear regression based on the points $(z_{n,n}, \log(n/1))$, $(z_{n-1,n}, \log(n/2))$, $\ldots$, $(z_{n-\tilde{k}+1,n}, \log(n/\tilde{k}))$ is minimal. Motivated by the encouraging simulation results obtained by Schultze and Steinebach (1996), we adapt here this technique for the estimator $\hat{R}(k_n)$. The means of $\tilde{k}$ (plus standard deviations) obtained by this procedure are presented in Tables 4–6. In Tables 4 and 5 we take $l = 5$ and in Table 6 we take $l = 10$.

We have also performed a small simulation study, in order to examine the finite sample behaviour of the normalized $\hat{R}(k_n)$, centred at R.

Table 1
Standard deviations of $\hat{R}(k)$, $\hat{R}_3(k)$, $\hat{R}_1(k)$ and $H_{50}^{-1}(k)$ for some values of k, over the 100 runs, with $n = 50$

k	S.D.($\hat{R}(k)$)	S.D.($\hat{R}_3(k)$)	S.D.($\hat{R}_1(k)$)	S.D.($H_{50}^{-1}(k)$)
5	3.3742×10^{-5}	3.2406×10^{-5}	3.5438×10^{-5}	4.6344×10^{-5}
10	2.6029×10^{-5}	2.5319×10^{-5}	2.6856×10^{-5}	3.0092×10^{-5}
15	2.3875×10^{-5}	2.3444×10^{-5}	2.4377×10^{-5}	3.0570×10^{-5}
20	2.4304×10^{-5}	2.3801×10^{-5}	2.4953×10^{-5}	4.0844×10^{-5}
25	2.6014×10^{-5}	2.5164×10^{-5}	2.7168×10^{-5}	5.1857×10^{-5}
30	2.7979×10^{-5}	2.6624×10^{-5}	2.9848×10^{-5}	6.2661×10^{-5}
35	2.9953×10^{-5}	2.8008×10^{-5}	3.2657×10^{-5}	7.3465×10^{-5}
40	3.1874×10^{-5}	2.9290×10^{-5}	3.5498×10^{-5}	8.4268×10^{-5}

Table 2
Standard deviations of $\hat{R}(k)$, $\hat{R}_3(k)$, $\hat{R}_1(k)$ and $H_{100}^{-1}(k)$ for some values of k, over the 100 runs, with $n = 100$

k	S.D.($\hat{R}(k)$)	S.D.($\hat{R}_3(k)$)	S.D.($\hat{R}_1(k)$)	S.D.($H_{100}^{-1}(k)$)
10	2.1887×10^{-5}	2.1397×10^{-5}	2.2475×10^{-5}	2.5266×10^{-5}
20	1.6860×10^{-5}	1.6846×10^{-5}	1.6905×10^{-5}	1.5737×10^{-5}
30	1.4812×10^{-5}	1.4924×10^{-5}	1.4733×10^{-5}	1.6340×10^{-5}
40	1.4569×10^{-5}	1.4632×10^{-5}	1.4574×10^{-5}	2.3680×10^{-5}
50	1.5350×10^{-5}	1.5237×10^{-5}	1.5611×10^{-5}	2.9858×10^{-5}
60	1.6359×10^{-5}	1.5984×10^{-5}	1.6991×10^{-5}	3.5951×10^{-5}
70	1.7400×10^{-5}	1.6716×10^{-5}	1.8473×10^{-5}	4.2045×10^{-5}

Table 3
Standard deviations of $\hat{R}(k)$, $\hat{R}_3(k)$, $\hat{R}_1(k)$ and $H_{500}^{-1}(k)$ for some values of k, over the 100 runs, with $n = 500$

k	S.D.($\hat{R}(k)$)	S.D.($\hat{R}_3(k)$)	S.D.($\hat{R}_1(k)$)	S.D.($H_{500}^{-1}(k)$)
25	1.4154×10^{-5}	1.3985×10^{-5}	1.4391×10^{-5}	1.2546×10^{-5}
50	1.0367×10^{-5}	1.0456×10^{-5}	1.0306×10^{-5}	8.5985×10^{-6}
75	8.8380×10^{-6}	8.9895×10^{-6}	8.7083×10^{-6}	7.4255×10^{-6}
100	7.9829×10^{-6}	8.1635×10^{-6}	7.8246×10^{-6}	6.8193×10^{-6}
125	7.4078×10^{-6}	7.6004×10^{-6}	7.2368×10^{-6}	6.1600×10^{-6}
150	7.0046×10^{-6}	7.2027×10^{-6}	6.8280×10^{-6}	7.0714×10^{-6}
175	6.8494×10^{-6}	7.0602×10^{-6}	6.6649×10^{-6}	8.7296×10^{-6}
200	6.8930×10^{-6}	7.1060×10^{-6}	6.7163×10^{-6}	9.9838×10^{-6}

Table 4
Means and standard deviations of the proposed $\tilde{k}$ and of the corresponding estimators, for $n = 50$ and $l = 5$

Mean($\hat{R}(\tilde{k})$)	6.0181×10^{-5}
S.D.($\hat{R}(\tilde{k})$)	3.2539×10^{-5}
Mean($\tilde{k}$)	11.68
S.D.($\tilde{k}$)	6.62
Mean($\hat{R}_3(\tilde{k})$)	5.5778×10^{-5}
S.D.($\hat{R}_3(\tilde{k})$)	3.1756×10^{-5}
Mean($\tilde{k}$)	10.58
S.D.($\tilde{k}$)	6.34
Mean($\hat{R}_1(\tilde{k})$)	6.7150×10^{-5}
S.D.($\hat{R}_1(\tilde{k})$)	3.3879×10^{-5}
Mean($\tilde{k}$)	14.63
S.D.($\tilde{k}$)	8.66

Table 5
Means and standard deviations of the proposed $\tilde{k}$ and of the corresponding estimators, for $n = 100$ and $l = 5$

Mean($\hat{R}(\tilde{k})$)	5.9135×10^{-5}
S.D.($\hat{R}(\tilde{k})$)	2.7928×10^{-5}
Mean($\tilde{k}$)	19.94
S.D.($\tilde{k}$)	14.42
Mean($\hat{R}_3(\tilde{k})$)	5.3088×10^{-5}
S.D.($\hat{R}_3(\tilde{k})$)	2.7185×10^{-5}
Mean($\tilde{k}$)	18.10
S.D.($\tilde{k}$)	13.73
Mean($\hat{R}_1(\tilde{k})$)	6.3247×10^{-5}
S.D.($\hat{R}_1(\tilde{k})$)	2.7936×10^{-5}
Mean($\tilde{k}$)	21.62
S.D.($\tilde{k}$)	16.56

Table 6
Means and standard deviations of the proposed $\tilde{k}$ and of the corresponding estimators, for $n = 500$ and $l = 10$

Mean($\hat{R}(\tilde{k})$)	5.7807×10^{-5}
S.D.($\hat{R}(\tilde{k})$)	1.4755×10^{-5}
Mean($\tilde{k}$)	126.29
S.D.($\tilde{k}$)	38.57
Mean($\hat{R}_3(\tilde{k})$)	5.4136×10^{-5}
S.D.($\hat{R}_3(\tilde{k})$)	1.6161×10^{-5}
Mean($\tilde{k}$)	109.91
S.D.($\tilde{k}$)	43.36
Mean($\hat{R}_1(\tilde{k})$)	6.0596×10^{-5}
S.D.($\hat{R}_1(\tilde{k})$)	1.5509×10^{-5}
Mean($\tilde{k}$)	130.72
S.D.($\tilde{k}$)	30.08

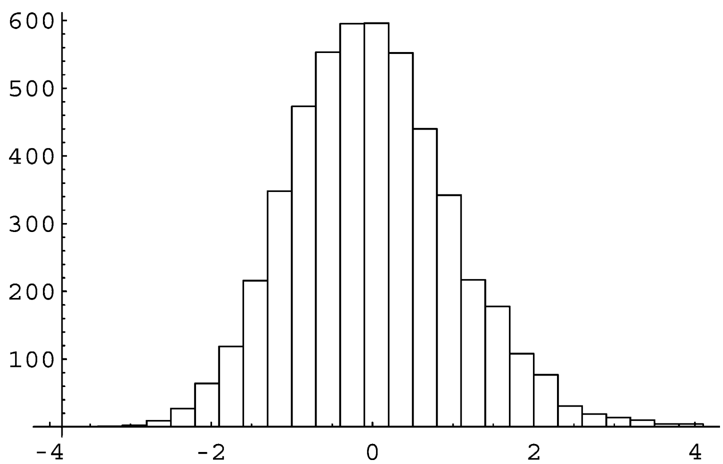

Fig. 5. Histogram of 5000 runs for the estimator $\hat{R}(k_n)$ with $n = 500$ and $k_n = 120$. Mean $= -0.0431$; S.D. $= 1.0372$.

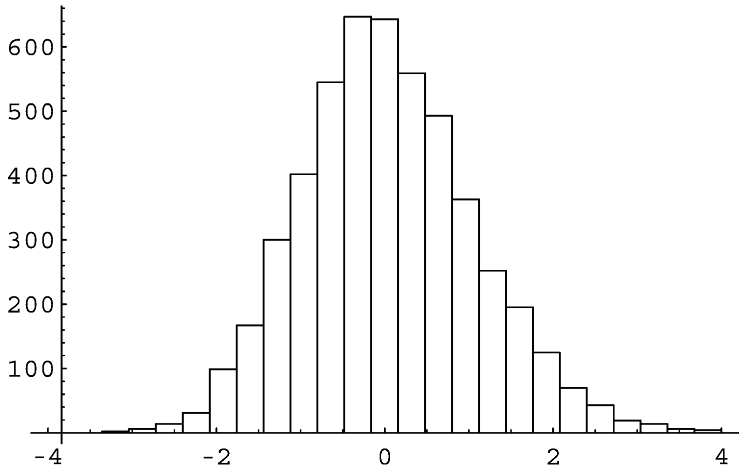

Fig. 6. Histogram of 5000 runs for the estimator $\hat{R}(k_n)$ with $n = 1000$ and $k_n = 200$. Mean $= 0.0253$; S.D. $= 1.0278$.

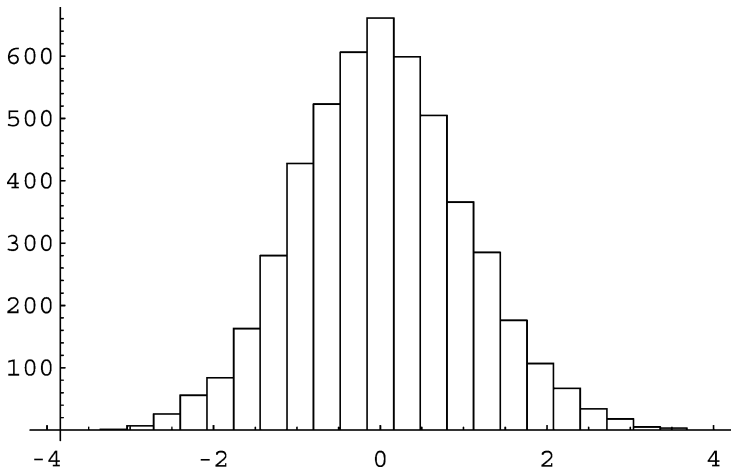

Fig. 7. Histogram of 5000 runs for the estimator $\hat{R}(k_n)$ with $n = 2000$ and $k_n = 300$. Mean $= 0.0184$; S.D. $= 1.0054$.

We consider again the family defined by (18) with $(\alpha, \beta) = (24\,000, 10\,000)$.

We successively take $n = 500$ and $k_n = 120$, $n = 1000$ and $k_n = 200$, $n = 2000$ and $k_n = 300$, with a number of runs equal to 5000.

The resulting histograms are given in Figs. 5–7. They look quite normal as predicted by the theory.

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