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Existence of a Markov perfect equilibrium in a third market model

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Abstract

We modify Maskin and Tirole's [Maskin, E., Tirole, J., 1987. A theory of dynamic oligopoly, III. European Economic Review 31, 947–968] model to prove the existence of a Markov perfect equilibrium in a dynamic version of Brander and Spencer's [Brander, J.A., Spencer, B.J., 1985. Export subsidies and market share rivalry. Journal of International Economics 18, 83–100] 'third-market model', introducing the government as a third player. We prove the existence of a Markov perfect equilibrium for any value of the discount factor. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Maskin and Tirole (1987) have studied a dynamic duopoly game of two firms in which Firm 1 plays at even times and Firm 2 at odd times. They proved the existence of a unique Markov perfect equilibrium (MPE) under certain assumptions and computed its value under the same assumptions.

Here we prove the existence of, at least, one such equilibrium in the setting of a usual model in the strategic trade policy literature: the third-market model. As Brander (1995) remarks: in a third-market model one or more domestic firms compete with one or more foreign firms only in a third country. The firms export all their production. In this model the domestic government cannot influence the interaction among the firms with import tariffs or quotas. Therefore, export subsidies appear as a natural policy to help a domestic firm vis-à-vis its foreign rivals. We consider two firms, one domestic

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and one foreign, competing in quantities in a third market. Besides, we assume that the domestic government gives a subsidy to the domestic firm. Instead of the usual two stage game, we look for the existence of a MPE in a dynamic game following the general line of the study of Maskin and Tirole (1987). We prove the existence of at least one MPE under some restrictive assumptions, namely quadratic payoffs and linear reaction functions.

We describe the problem using an infinite sequential game where each player moves in a three-cycle with perfect information about the others actions where the government is treated as an ordinary player. As in Maskin and Tirole (1987) each player is ‘committed to a particular action’ meaning that it cannot change that action for a certain finite (although possibly short) amount of time during which the other players make their moves. The players maximize the discounted value of the payoffs and a ‘strategy’ is simply a dynamic reaction function. A MPE is the set of reaction functions that form a perfect equilibrium.

This paper is organized as follows: in Section 2 we present the model and in Section 3 we prove the existence of a MPE.

2. The model

Our model is constructed in the following way: Firm 1 is the domestic firm and Firm 2 the foreign firm. Each player (Firm 1, Firm 2 and government) is part of a three-cycle of play. This means that, for instance, Firm 1 plays at time $3k$, Firm 2 at time $3k + 1$ and the government at time $3k + 2$, for integer values of k . Since each player has an infinite number of moves and we assume the strategies are Markov, it does not matter who plays first. In fact, consider an external observer who looks at the game over a three-cycle. This cycle can be $(3k, 3k + 1, 3k + 2)$, $(3k + 1, 3k + 2, 3(k + 1))$ or $(3k + 2, 3(k + 1), 3(k + 1) + 1)$ hence, the observer has no information about who was the first to play.

The payoffs for each player are

- $\pi_1(x, y, s)$ for Firm 1, representing its profit ($\pi_1(x, y, s) = \pi(x, y) + sx$ where π is the profit in the absence of a subsidy);
- $\pi_2(x, y)$ for Firm 2, representing its profit; and
- $W(x, y)$ for the government, representing the welfare given by $W(x, y) = \pi_1(x, y, s) - sx$,

where x is the output of Firm 1, y the output of Firm 2 and s the subsidy given to Firm 1.

The dynamic reaction function for each player is represented by

- $x = R_1(y, s)$ for Firm 1;
- $y = R_2(x)$ for Firm 2; and
- $s = R(x)$ for the government.

Note that even though the reaction function of Firm 2 is independent of the subsidy, it can reflect its influence through the output of Firm 1, which obviously depends on the subsidy. The same is valid for the dependence of the reaction function of the government with the output y of Firm 2.

A set of reaction functions (R_1, R_2, R) constitutes a MPE if any player's reaction function maximizes its present discounted profit given the other players' reaction functions. From dynamic programming we know that for (R_1, R_2, R) to be a MPE it suffices that there exist valuation functions

$$\{v_1(y, s); w_1(x)\}, \quad \{v_2(x); w_2(y)\} \quad \text{and} \quad \{v(x, y); w(s)\},$$

such that, if $\delta = e^{-rt}$ is the discount factor for interest rate r , we have

$$\begin{cases} v_1(y, s) = \max_x \{\pi_1(x, y, s) + \delta^2 w_1(x)\} \\ R_1(y, s) \in \arg \max_x \{\pi_1(x, y, s) + \delta^2 w_1(x)\} \\ w_1(x) = \pi_1(x, R_2(x), R(x)) + \delta v_1(R_2(x), R(x)) \end{cases} \quad (1)$$

$$\begin{cases} v_2(x) = \max_y \{\pi_2(x, y) + \delta^2 w_2(y)\} \\ R_2(x) \in \arg \max_y \{\pi_2(x, y) + \delta^2 w_2(y)\} \\ w_2(y) = \pi_2(R_1(y, R(x)), y) + \delta v_2(R_1(y, R(x))) \end{cases} \quad (2)$$

$$\begin{cases} v(x, y) = \max_s \{W(x, y) + \delta^2 w(s)\} \\ R(x) \in \arg \max_s \{W(x, y) + \delta^2 w(s)\} \\ w(s) = W(R_1(y, s), R_2(R_1(y, s))) + \delta v(R_1(y, s), R_2(R_1(y, s))) \end{cases} \quad (3)$$

Starting with the first-order conditions to optimize the first of each set of equations we obtain

$$\frac{\partial \pi_1}{\partial x}(R_1(y, s), y, s) + \delta^2 \frac{dw_1}{dx}(R_1(y, s)) = 0 \quad (4)$$

for Firm 1 and analogous equations for the other two players.

These are all we need to prove the existence of a MPE in this game setting.

The above three sets of equations are derived in the following way, using the arguments in Maskin and Tirole (1987):

1. For the first set of equations, suppose Firm 1 plays at time $3k$, which is the present time. Then v_1 represents the present profit of Firm 1 plus all the discounted profits in the future — these are given by $\delta^2 w_1(x)$, since Firm 1's next play will be at time $3k + 2$. The function w_1 is then calculated using the reaction functions of the other two players for times between the present and $3k + 2$. To include in w_1 all future profits we discount for time $3k + 3$ using v_1 calculated at that instant.
2. The remaining two sets are obtained in a similar way, assuming that Firm 2 plays at time $3k + 1$, which is the present when deriving the second set of equations, and the government plays at $3k + 2$, which is again taken to be the present for the third set of equations.

3. Existence of a Markov perfect equilibrium

Using the same arguments as Fudenberg and Tirole (1995) (p. 523), Maskin and Tirole (1987) and Brander (1995) we use quadratic payoffs and linear reaction functions ensuring thus that the second order conditions for maximization are satisfied. In what follows the payoff functions are, considering that the inverse demand function can be represented by $p = d - x - y$,

- $\pi_1(x, y, s) = x(d - x - y) - cx + sx - F$,
- $\pi_2(x, y) = y(d - x - y) - cy - G$,
- $W(x, y) = x(d - x - y) - cx - F$

and the reaction functions

- $R_1(y, s) = a - b_1y + es$,
- $R_2(x) = a - b_2x$,
- $R(x) = \beta x$.

Note that there is no constant term in the reaction function for the government since, because the subsidy is given to the production, the government does not give a subsidy unless the output x of Firm 1 is non-zero.

All the constants are positive, except possibly β . This means that we allow for the subsidy to be, in fact, a tariff. This also means that the reaction function for Firm 2 is downward sloping and that the reaction function for Firm 1 is downward sloping only for y . We assume that the firms are identical hence, the constant term is the same for their reaction functions (they produce the same in the absence of all other players).

Next we state and prove our main result.

Theorem 3.1. *For any discount factor δ there exists at least one Markov perfect equilibrium.*

Proof. Because the reaction functions are linear and the payoffs quadratic, the valuation functions $\{v_1, w_1\}$, $\{v_2, w_2\}$ and $\{v, w\}$ are quadratic. By Theorem 13.2 in Fudenberg and Tirole (1995) (p. 515), we know that if the objective functions defined by the first equations of (1), (2) and (3) are continuous at infinity then there exists a MPE. Furthermore, we know that concave functions are continuous at infinity and this is what we prove.

We treat the objective functions as functions of one variable (x , y or s , respectively) only and use the first-order conditions written in Eq. (4) for Firm 1 and their analogues for the two other players. The function in the left-hand side of (4) is a function of the two variables y and s . Calculating the partial derivative with respect to, for instance, y (the result is the same for the other variable) we obtain

$$(\pi_{1,xx}(R_1, y, s) + \delta^2 w_{1,xx}(R_1))R_{1,y} + \pi_{1,xy}(R_1, y, s) = 0.$$

The notation used is $F_{\alpha\beta}$ to represent the second-order derivative of F with respect to α and β , in this order, without distinguishing partial from total derivatives.

With the payoff and reaction functions we have chosen, we have

$$R_{1,y} < 0 \quad \text{and} \quad \pi_{1,xy}(R_1, y, s) < 0$$

hence, we must have

$$\pi_{1,xx}(R_1, y, s) + \delta^2 w_{1,xx}(R_1) < 0,$$

proving that, as a function of x the objective function is concave.

Using the first-order conditions for Firm 2 we conclude that the objective function is concave by an entirely analogous process.

Let us now consider the first-order conditions for the government, describing a function of x and y which we differentiate with respect to x to obtain

$$(W_{ss}(x, y) + \delta^2 w_{ss}(R))R_x(x) + W_{xs}(R) = 0.$$

Since W does not depend explicitly on s , we have $W_s = 0$ which implies

$$\delta^2 w_{ss}(R)R_x(x) = 0$$

and, because both δ and $R_x(x)$ are non-zero, $w_{ss}(R) = 0$. This means we have

$$W_{ss}(x, y) + \delta^2 w_{ss}(R) = 0.$$

If we calculate higher-order derivatives we conclude that all derivatives are zero and hence, the function is constant. It is trivial to conclude that then it is continuous at infinity. \square

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References

- Brander, J.A., 1995. Strategic trade policy. In: Grossman, G., Rogoff, K. (Eds.), *Handbook of International Economics*, Vol. 3, Elsevier, pp. 1395–1455.
- Fudenberg, D., Tirole, J., 1995. In: *Game Theory*, MIT Press.
- Maskin, E., Tirole, J., 1987. A theory of dynamic oligopoly, III. *European Economic Review* 31, 947–968.