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2012 Nonlinearity 25 3155

(http://iopscience.iop.org/0951-7715/25/11/3155)

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Nonlinearity **25** (2012) 3155–3187 doi:10.1088/0951-7715/25/11/3155

Evolution of synchrony under combination of coupled cell networks

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Received 27 April 2012, in final form 18 September 2012
Published 15 October 2012
Online at stacks.iop.org/Non/25/3155

Recommended by J A Glazier

Abstract
A natural way of modelling large coupled cell networks is to combine smaller networks through binary network operations. In this paper, we consider several non-product binary operations on networks such as join and coalescence, and examine the evolution of the lattice of synchrony subspaces under these operations. Classification results are obtained for synchrony subspaces of the combined network, which clarify the relation between the lattice of synchrony subspaces of the combined network and its components. Yet, in the case when the initial networks have the same edge type, this classification only applies to those synchrony subspaces that are compatible with respect to the considered operation. Based on the classification results, we give examples to show how the lattice of synchrony subspaces of the combined network can be reconstructed using the initial ones. Also, we show how the classification results can be applied to analyse the evolutionary fitness of synchrony patterns.

Mathematics Subject Classification: 34C15, 34D06, 05C76

(Some figures may appear in colour only in the online journal)

1. Introduction

A *network* is a set of units that are organized by mutual interactions. Networks are utilized in modeling many real world problems in a wide range of scientific fields and have become a subject of vast research interest in recent years.

Among others, networks can be used to graphically represent coupled dynamical systems, where the time evolutions of individual systems influence one another. One of the widely observed and most studied collective dynamics in coupled dynamical systems is *synchronization*, where phase trajectories of two or more coupled units coincide over time. For the importance of synchronization and its ubiquitous presence in nature, we refer to [6, 19] and references therein. In [18], Pikovsky *et al* propose to study various synchronization phenomena using a common framework based on modern nonlinear dynamics, where a variety of approaches using coupled periodic and coupled chaotic systems is discussed. Restrepo *et al* in [20] point out the crucial effect of network structure on the emergence of collective synchronization in heterogeneous systems, in terms of eigenvalues of network adjacency matrices.

Besides synchronization, in the theory of networks of coupled oscillators, there is a rich variety of other collective dynamics such as resonance and quasi-periodicity that can occur in a persistent way. These are usually related to the ratio of frequencies of the oscillators and are subject to a continuous change of parameters. We refer to Broer [9] for a survey, where resonance phenomena are modeled in terms of dynamical systems (depending on parameters) and their bifurcations, and the overall fractal geometry of the complement of all the resonances in the parameter space is underlined.

A theory of coupled cell networks and their associated systems was proposed by Golubitsky and Stewart [12], and by Field [11], where nodes of a network represent individual dynamical systems or *cells*, and edges indicate the mutual interactions. The collective dynamics of the time evolution at nodes then gives the *dynamics* on *the network*. A key advantage of these formalisms is that they allow theoretical deduction of collective dynamics based only on the network structure, without referring to specific dynamics at every cell. Fully synchronized states where all cells are synchronized are rare instances. A more common phenomenon is *partial synchronizations*, where a number of subsets of cells are synchronized within each subset. Conditions for the occurrence of robust patterns of partial synchronization in terms of network structure have been established in [13, 14].

Another kind of network dynamics, the *dynamics* of *the network*, occurs when the topology of the network changes over time. Most real world networks are *evolving networks*, that is, their topology evolves with time, either due to a rewiring of a link, the appearance or disappearance of a link or node, or by a merging of small networks into a larger one. The dynamics of network topology reflects frequent changes in the interactions among network components and translates into a rich variety of evolutionary patterns. Evolution of network topology can be described by a sequence of static networks and the topology of the networks can be regarded as a discrete dynamical system. Evolving networks are ubiquitous in nature and science (see Albert *et al* [4] and Dorogovstev *et al* [7], and references therein for examples in many diverse fields).

For many networks, both dynamics of the network and dynamics on the network can be defined simultaneously. Evolution of the two dynamics does not necessarily have the same time scale. The rate at which the network structure changes is usually much slower than the rate at which the state variables of the cells change. Naturally, in most cases there is an important interaction between the changes in the network topology and in the cells' dynamics. Under this interplay between the two dynamics, the networks are said to be *coevolutionary or adaptive* (see Gross *et al* [15] for more details). Both the formalisms of the coupled cell networks of Golubitsky and Stewart [12], and of Field [11], have so far considered networks with static structure. However, as shown in Gorochowski *et al* [16], they can be extended to a new framework, the *evolving dynamical networks formalism*, which incorporates topology, dynamics and evolution in an integrated way.

In this paper, we focus on evolving networks where new networks are formed by combining existing ones using binary network operations. The product operations are omitted as they are being considered in Aguiar *et al* [2]. Our goal is to clarify the relation between the lattice of synchrony subspaces of the combined network and that of the initial ones. One can expect, in general, that some synchrony subspaces of the initial networks disappear, some remain and some new synchrony subspaces appear. With our results, we hope to explain the choice of the way two networks are combined, given the desired evolutionary patterns of synchrony.

Although our results are presented in the framework of the theory of coupled cell networks of Golubitsky and Stewart, they have a broader application, for example, in the theory of coupled cell networks of Field [3]. For that, one only needs to take into account that the cell equivalence relation in Field's theory corresponds to the input equivalence relation in Golubitsky and Stewart's theory. Note that in Field's formalism the number and types of inputs of every cell is defined *a priori* and all the inputs of every cell in a network have to be filled.

As shown in [14], synchrony subspaces and balanced equivalence relations of a coupled cell network are in a one-to-one correspondence. An equivalence relation on the set of cells of the network is called *balanced*, if by coloring cells from the same equivalence class with the same color, any two cells with the same color a receive the same number of edges from cells of color b, for every two colors a and b. Given the isomorphism between the lattice of synchrony subspaces and the lattice of balanced equivalence relations of a network (see Stewart [21]), we will analyze the evolution of synchrony subspaces using balanced equivalence relations.

For the initial networks, we only consider networks having one type of cell and one type of edge. The assumption of one cell type is based on the fact that a balanced equivalence relation is a refinement of cell types. Thus, the lattice of balanced equivalence relations on a network with different cell types is a join of all lattices of balanced equivalence relations on the network considered with a single cell type. Similarly, since a balanced equivalence relation is exactly balanced if it is balanced with respect to every edge type of the network, the lattice of balanced equivalence relations on a network with different edge types (but with one cell type) is the intersection of all lattices of balanced equivalence relations on the network considered with a single edge type. Therefore, the case of networks with identical cell type and identical edge type is generic for our purpose of studying lattices of balanced equivalence relations.

The paper is organized as follows: section 2 introduces preliminary definitions and results on lattices of balanced equivalence relations in coupled cell networks. Section 3 gives the definition of the two binary network operations of our consideration: the f-join and the coalescence. In section 4, we introduce compatibility conditions and derive classification results of the lattice of balanced equivalence relations for both f-join and coalescence (see subsection 4.1–4.2). In subsection 4.3, we apply our results to analyze the evolutionary fitness of synchrony types. In subsection 4.4, we present a reconstruction procedure of the lattice of synchrony subspaces of the combined network using those of the initial ones. We end with some discussion in section 5.

2. Preliminary definitions and results

In this section, we give preliminary definitions and results on lattices of balanced equivalence relations in coupled cell networks.

2.1. Equivalence relations and lattices

An *equivalence relation* $\sim$ on a set X is a binary relation among the elements of X such that the axioms of *reflexivity*, *symmetry* and *transitivity* are satisfied. The *equivalence class* of $x \in X$, usually denoted by $[x]_\sim$, is the set of elements $y \in X$ such that $x \sim y$. Denote by $\#A$ or $\#(A)$ the cardinality of a finite set A. An equivalence class $[x]_\sim$ is called *trivial*, if $\#\,[x]_{\bowtie} = 1$. The equivalence relation $\sim$ is called *trivial*, if every equivalence class is trivial.

The set of all equivalence relations on X is partially ordered by the *refinement relation*. For two equivalence relations $\bowtie_i$ and $\bowtie_j$, we say that $\bowtie_i$ *refines* $\bowtie_j$, denoted by $\bowtie_i \prec \bowtie_j$, if $[x]_{\bowtie_i} \subseteq [x]_{\bowtie_j}$, for all $x \in X$. Moreover, define the *meet* and *join* of two equivalence relations as follows.

Definition 2.1. Let $\bowtie_i$ and $\bowtie_j$ be two equivalence relations on a set X.

- *Meet:* a relation $\bowtie$ is the *meet* of $\bowtie_i$ and $\bowtie_j$, denoted by $\bowtie = \bowtie_i \wedge \bowtie_j$, if for all $x, y \in X$, we have $x \bowtie y$ if and only if $x \bowtie_i y$ and $x \bowtie_j y$.
- *Join:* a relation $\bowtie$ is the *join* of $\bowtie_i$ and $\bowtie_j$, denoted by $\bowtie = \bowtie_i \vee \bowtie_j$, if for all $x, y \in X$, we have $x \bowtie y$ if and only if there exists a finite chain $x = x_q, \ldots, x_s = y$ such that for all t with $q \leqslant t \leqslant s-1$ either $x_t \bowtie_i x_{t+1}$ or $x_t \bowtie_j x_{t+1}$. $\diamond$

For a partially ordered set $(\mathcal{X}, \leqslant)$ and a subset $\mathcal{Y} \subseteq \mathcal{X}$, an element $a \in \mathcal{X}$ is called an *upper bound* of $\mathcal{Y}$, if $b \leqslant a$ for all $b \in \mathcal{Y}$; an upper bound a of $\mathcal{Y}$ is called the *least upper bound* of $\mathcal{Y}$ if $a \leqslant a'$, for every upper bound a' of $\mathcal{Y}$. Dually, one defines a *lower bound* and the *greatest lower bound*. In the case $(\mathcal{X}, \prec)$ is the set of equivalence relations on X, the least upper bound of $\mathcal{Y}$ is the join of all $\bowtie \in \mathcal{Y}$ and the greatest lower bound of $\mathcal{Y}$ is the meet of all $\bowtie \in \mathcal{Y}$. In fact, $(\mathcal{X}, \prec)$ is a complete lattice.

A *lattice* is a partially ordered set $\mathcal{X}$ such that every pair of elements $a, b \in \mathcal{X}$ has a unique least upper bound or *join*, denoted by $a \vee b$, and a unique greatest lower bound or *meet*, denoted by $a \wedge b$. A *complete lattice* is a lattice such that every subset $\mathcal{Y} \subseteq \mathcal{X}$ has a unique least upper bound or *join*, and a unique greatest lower bound or *meet*. Note that every finite lattice is complete. A subset of a lattice $\mathcal{X}$ is called a *sublattice*, if it is a lattice on its own right. More details about lattices and complete lattices can be found in Davey and Priestley [10].

2.2. Coupled cell networks

Definition 2.2. A *coupled cell network* consists of a finite nonempty set $\mathcal{C}$ of *nodes* or *cells* and a finite nonempty set $\mathcal{E} = \{(c, d) : c, d \in \mathcal{C}\}$ of *edges* or *arrows* and two equivalence relations: $\sim_C$ on $\mathcal{C}$ and $\sim_E$ on $\mathcal{E}$ such that the *consistency condition* is satisfied: if $(c_1, d_1) \sim_E (c_2, d_2)$, then $c_1 \sim_C c_2$ and $d_1 \sim_C d_2$. We write $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. $\diamond$

A coupled cell network can be represented by a directed graph, where the cells are placed at vertices, edges are depicted by directed arrows and the equivalence relations are indicated by different types of vertices or edges in the graph. Note that a coupled cell network may have multiple edges and loops.

A *multiset* is a generalized notion of set, in which elements are allowed to appear more than once. For a multiset A and $x \in A$, define the *multiplicity* of x as the number of copies of x contained in A, denoted by $\mathfrak{m}(x, A)$; for a subset $B \subset A$, define the *multiplicity* of B as $\mathfrak{m}(B, A) := \sum_{x \in B} \mathfrak{m}(x, A)$.

Definition 2.3. For an edge $e := (c, d) \in \mathcal{E}$, the cell c is called the *tail cell* and d is called the *head cell* of e. The edge e is called an *input edge* of d. The set of all tail cells of input edges of d, which is a multiset, is called the *input set* of d, usually denoted by $I(d)$. For an edge type $\mathfrak{e}$ of $\mathcal{G}$, denote by $I^{\mathfrak{e}}(d) \subset I(d)$ the tail cells of input edges of d that are of type $\mathfrak{e}$. Two cells $d_1, d_2 \in \mathcal{C}$ are called *input equivalent*, denoted by $d_1 \sim_I d_2$, if $\#I^{\mathfrak{e}}(d_1) = \#I^{\mathfrak{e}}(d_2)$, for all edge type $\mathfrak{e}$, where $\#I^{\mathfrak{e}}(d_i)$ denotes de cardinality of the multiset $I^{\mathfrak{e}}(d_i)$, $i = 1, 2$. $\diamond$

It follows from the consistence condition that the input-equivalence relation $\sim_I$ refines the cell-equivalence relation $\sim_C$.

Definition 2.4. A coupled cell network is called *homogeneous*, if it has only one input-equivalence class. A *regular* network is a homogeneous network with only one edge-equivalence class. A coupled cell network is called *uniform*, if it contains no multiple edges nor loops. ◇

In a homogeneous network, all cells are of identical type and receive the same number of input edges per edge type. The number, which is the cardinality of the input set, is called the *valency* of the network.

The coupling structure of an identical-cell network having s edge types $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ is given by s adjacency matrices $A_1, A_2, \ldots, A_s$, for $A_l := (a_{ij}^{(l)})$ and $a_{ij}^{(l)} = \mathfrak{m}(c_j, I^{\mathfrak{e}_l}(c_i))$, $l \in \{1, \ldots, s\}$, where c_i denotes the i-th cell of the network.

Definition 2.5. Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be an identical-cell network. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset. An *interior symmetry* of $\mathcal{G}$ on $\mathcal{S}$ is a permutation σ on $\mathcal{C}$ such that σ fixes every element in $\mathcal{C} \setminus \mathcal{S}$, and there is a bijection between edges $(\sigma(a), \sigma(b))$ and (a, b), which preserves edge-equivalence relation $\sim_E$, for $a \in \mathcal{S}$, $b \in \mathcal{C}$. ◇

Let $\mathcal{G}$ be an identical-cell network with adjacency matrices $A_1, A_2, \ldots, A_s$. Then, a permutation σ is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$, if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)}, \quad \forall i \in \mathcal{S},\ j \in \mathcal{C},\ l = 1, \ldots, s. \tag{2.1}$$

For more on coupled cell networks see Golubitsky and Stewart [12] and Field [11].

2.3. Balanced equivalence relations

It is well known that the set $\boldsymbol{E}_{\mathcal{G}}$ of all equivalence relations on a network $\mathcal{G}$ is a complete lattice with the partial order given by the refinement relation and where the meet and the join are as in definition 2.1. For our purpose, we only consider the equivalence relations that are balanced in the sense of definition 2.6.

Definition 2.6. Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. An equivalence relation $\bowtie$ on $\mathcal{G}$ is called *balanced*, if for $c, d \in \mathcal{C}$ with $c \bowtie d$, $\mathfrak{m}\big([\alpha]_{\bowtie}, I^{\mathfrak{e}}(c)\big) = \mathfrak{m}\big([\alpha]_{\bowtie}, I^{\mathfrak{e}}(d)\big)$ holds for every $\bowtie$-equivalence class $[\alpha]_{\bowtie}$ and every edge type $\mathfrak{e}$ in $\mathcal{G}$. ◇

Note that a balanced equivalence relation refines the input-equivalence relation $\sim_I$. For a coupled cell network $\mathcal{G}$, denote by

$$\Lambda_{\mathcal{G}} := \{\bowtie : \bowtie \text{ is a balanced equivalence relation on } \mathcal{G}\}.$$

Theorem 2.7 (see theorem 5.7 in Stewart [21] and chapter 4 in Aldis [5]). The set $\Lambda_{\mathcal{G}}$ of all balanced equivalence relations on a coupled cell network $\mathcal{G}$ is a complete lattice with the partial order given by the refinement relation, and the join is as defined in definition 2.1.

As shown in Stewart [21], $\Lambda_{\mathcal{G}}$ is not a sublattice of $\boldsymbol{E}_{\mathcal{G}}$. The join operation is the same for both lattices, but the meet of two balanced equivalence relations as defined in definition 2.1, even though is an equivalence relation, may be not balanced. Apparently, there is no general form for the meet operation in $\Lambda_{\mathcal{G}}$, although it can be defined in terms of the join. In [1], Aguiar and Dias describe the lattice of balanced equivalence relations of a network in terms of the eigenvalue structure of the network adjacency matrices and present an algorithm to compute the lattice.

To every balanced equivalence relation, there is an associated quotient network obtained by the identification of equivalent cells.

Definition 2.8 (see [14]). Let $\bowtie$ be a balanced equivalence relation on a coupled cell network $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. Define the *quotient network* $\mathcal{G}_{\bowtie} = (\mathcal{C}_{\bowtie}, \mathcal{E}_{\bowtie}, \sim_{C_{\bowtie}}, \sim_{E_{\bowtie}})$ as follows: the cells of $\mathcal{G}_{\bowtie}$ are the $\bowtie$-equivalence classes $[c]_{\bowtie}$ of cells $c \in \mathcal{C}$ and, for every edge type $\mathfrak{e}$ in $\mathcal{G}$ and cells $[c]_{\bowtie}$, $[d]_{\bowtie}$ in $\mathcal{C}_{\bowtie}$, there are m edges $([c]_{\bowtie}, [d]_{\bowtie}) \in \mathcal{E}_{\bowtie}$ of type $\mathfrak{e}$, with $m = \mathfrak{m}\big([d]_{\bowtie}, I^{\mathfrak{e}}(c)\big)$. The cell-equivalence relation $\sim_{C_{\bowtie}}$ and the edge-equivalence relation $\sim_{E_{\bowtie}}$ are induced by $\sim_C$ and $\sim_E$, respectively. $\diamond$

Let $\bowtie\!\!\!\!\bowtie \in \Lambda_{\mathcal{G}}$ and $\mathcal{G}_{\bowtie\!\bowtie}$ be the quotient network. Then, $\Lambda_{\mathcal{G}_{\bowtie\!\bowtie}}$ is isomorphic to a sublattice of $\Lambda_{\mathcal{G}}$ defined by (see proposition 6.3 in Stewart [21])

$$\Lambda_{\mathcal{G}}^{\bowtie\!\bowtie} := \{\bowtie \in \Lambda_{\mathcal{G}} \; : \; \bowtie\!\bowtie \prec \bowtie\}. \tag{2.2}$$

Since the result was originally stated without proof, we give a brief proof here.

For $\bowtie \in \Lambda_{\mathcal{G}}^{\bowtie\!\bowtie}$, define an equivalence relation $\bowtie_r$ on $\mathcal{G}_{\bowtie\!\bowtie}$ by

$$[c]_{\bowtie\!\bowtie} \;\bowtie_r\; [d]_{\bowtie\!\bowtie} \;\Leftrightarrow\; \exists\, c' \in [c]_{\bowtie\!\bowtie}, \qquad d' \in [d]_{\bowtie\!\bowtie} \;\; s.t. \;\; c' \;\bowtie\; d', \tag{2.3}$$

which is called the *restriction* of $\bowtie$ to $\mathcal{G}_{\bowtie\!\bowtie}$. Since $\bowtie\!\bowtie \prec \bowtie$, (2.3) is equivalent to

$$[c]_{\bowtie\!\bowtie} \;\bowtie_r\; [d]_{\bowtie\!\bowtie} \;\Leftrightarrow\; c \;\bowtie\; d. \tag{2.4}$$

For $\bowtie \in \Lambda_{\mathcal{G}_{\bowtie\!\bowtie}}$, define an equivalence relation $\bowtie_l$ on $\mathcal{G}$ by

$$c \;\bowtie_l\; d \;\Leftrightarrow\; [c]_{\bowtie\!\bowtie} \;\bowtie\; [d]_{\bowtie\!\bowtie}, \tag{2.5}$$

which is called the *lifting* of $\bowtie$ to $\mathcal{G}$.

Proposition 2.9. *Let* $\bowtie\!\bowtie \in \Lambda_{\mathcal{G}}$ *and* $\mathcal{G}_{\bowtie\!\bowtie}$ *be the quotient network. Let* $\Lambda_{\mathcal{G}_{\bowtie\!\bowtie}}$ *be the lattice of balanced equivalence relations of* $\mathcal{G}_{\bowtie\!\bowtie}$ *and* $\Lambda_{\mathcal{G}}^{\bowtie\!\bowtie}$ *be given by (2.2). Then,*

$$\mathrm{res} : \; \Lambda_{\mathcal{G}}^{\bowtie\!\bowtie} \to \Lambda_{\mathcal{G}_{\bowtie\!\bowtie}}, \qquad \bowtie \;\mapsto\; \bowtie_r$$

is an isomorphism whose inverse is given by the lifting operation given by (2.5).

Proof. By definition, the restriction and the lifting are inverse operations to each other (see (2.4) and (2.5)). We only need to show that $\bowtie_r$ is balanced for every balanced $\bowtie$. For convenience, write $\bar{c} = [c]_{\bowtie\!\bowtie}$ for $\bowtie\!\bowtie$-equivalence classes on $\mathcal{G}$. Then, the input sets $I^{\mathfrak{e}}(c)$ and $I^{\mathfrak{e}}(\bar{c})$ are isomorphic as multiset, since $I^{\mathfrak{e}}(\bar{c}) = \{\bar{x} \;:\; x \in I^{\mathfrak{e}}(c)\}$, for every edge type $\mathfrak{e}$. Also, by definition of $\bowtie_r$,

$$\mathfrak{m}([x]_{\bowtie}, I^{\mathfrak{e}}(c)) = \mathfrak{m}([\bar{x}]_{\bowtie_r}, I^{\mathfrak{e}}(\bar{c})), \qquad \forall\, c \in \mathcal{C}. \tag{2.6}$$

Let $\bar{c}_1, \bar{c}_2$ be such that $\bar{c}_1 \bowtie_r \bar{c}_2$. Then, $c_1 \bowtie c_2$ and thus $\mathfrak{m}([x]_{\bowtie}, I^{\mathfrak{e}}(c_1)) = \mathfrak{m}([x]_{\bowtie}, I^{\mathfrak{e}}(c_2))$, since $\bowtie$ is balanced. It then follows from (2.6) that

$$\mathfrak{m}([\bar{x}]_{\bowtie_r}, I^{\mathfrak{e}}(\bar{c}_1)) = \mathfrak{m}([\bar{x}]_{\bowtie_r}, I^{\mathfrak{e}}(\bar{c}_2)),$$

for every edge type $\mathfrak{e}$ and equivalence class $[\bar{x}]_{\bowtie_r}$. Consequently, $\bowtie_r$ is balanced. $\square$

For more on equivalence relations and the lattice of equivalence relations of a coupled cell network, see Stewart *et al* [22], Golubitsky *et al* [14], Stewart [21] and Aguiar and Dias [1].

3. Binary network operations

In this section, we define two binary network operations on coupled cell networks, which can be used to describe evolution of networks. We omit the product of networks since it is being considered in Aguiar and Dias [2].

Given two coupled cell networks $\mathcal{G}_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ and $\mathcal{G}_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$, we define a binary operation on $\mathcal{G}_1, \mathcal{G}_2$ to obtain a new network $\mathcal{G}$. For simplicity, we assume that $\mathcal{G}_i$ has one cell type $\mathfrak{c}_i$ and one edge type $\mathfrak{e}_i$ for $i = 1, 2$ such that $\mathfrak{c}_1 = \mathfrak{c}_2$.

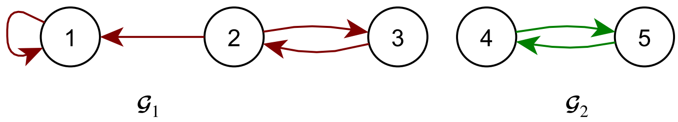


Figure 1. Two networks $\mathcal{G}_1$ and $\mathcal{G}_2$ for $\mathfrak{e}_1 \neq \mathfrak{e}_2$.

3.1. Join

The usual definition of *join of graphs* is given by the disjoint union of all graphs together with additional arrows added between every two cells from distinct graphs. We introduce a generalized version of join on coupled cell networks.

Recall that a *multimap* is a generalized notion of map, where an element from the domain is assigned to a set of values from the range. Let $\tilde{\mathcal{C}}_1 \subset \mathcal{C}_1$ and $\tilde{\mathcal{C}}_2 \subset \mathcal{C}_2$ be nonempty subsets of cells. Denote by $P(\tilde{\mathcal{C}}_2)$ the set of all subsets of $\tilde{\mathcal{C}}_2$. Consider a multimap f from $\tilde{\mathcal{C}}_1$ to $\tilde{\mathcal{C}}_2$ given by

$$\begin{aligned} f : \tilde{\mathcal{C}}_1 &\to P(\tilde{\mathcal{C}}_2) \\ c &\mapsto f(c) \subset \tilde{\mathcal{C}}_2. \end{aligned} \tag{3.7}$$

We define the *f-join* of $\mathcal{G}_1$ and $\mathcal{G}_2$ as follows.

Definition 3.1. Let $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. A network $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ is called the *f-join* of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$, if

- $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$;
- $c_1 \sim_C c_2$, for all $c_1, c_2 \in \mathcal{C}$;
- $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{F}$, where $\mathcal{F} = \{(c, d), (d, c) : c \in \tilde{\mathcal{C}}_1 \wedge d \in f(c)\}$ and f is defined by (3.7);
- $e_1 \sim_E e_2$, for all $e_1, e_2 \in \mathcal{E}$ if $\mathfrak{e}_1 = \mathfrak{e}_2$; otherwise $e_1 \sim_E e_2$ if and only if $e_1, e_2 \in \mathcal{E}_1$ or $e_1, e_2 \in \mathcal{E}_2$ or $e_1, e_2 \in \mathcal{F}$.

If $\tilde{\mathcal{C}}_1 = \mathcal{C}_1, \tilde{\mathcal{C}}_2 = \mathcal{C}_2$ and $f(c) \equiv \mathcal{C}_2$ for all $c \in \mathcal{C}_1$, then $\mathcal{G}_1 *_f \mathcal{G}_2 := \mathcal{G}_1 * \mathcal{G}_2$ is called the *join* of $\mathcal{G}_1$ and $\mathcal{G}_2$; if $f(c) \equiv \tilde{\mathcal{C}}_2$ for all $c \in \tilde{\mathcal{C}}_1$, then $\mathcal{G}_1 *_f \mathcal{G}_2 := \mathcal{G}_1 *_p \mathcal{G}_2$ is called a *partial join* of $\mathcal{G}_1$ and $\mathcal{G}_2$; if $f : \tilde{\mathcal{C}}_1 \to \tilde{\mathcal{C}}_2$ is a bijection, then $\mathcal{G}_1 *_f \mathcal{G}_2 := \mathcal{G}_1 *_{pp} \mathcal{G}_2$ is called a *point-wise partial join* of $\mathcal{G}_1$ and $\mathcal{G}_2$. $\diamond$

Remark 3.2. Let $\mathfrak{e}_f$ denote the edge type of edges from $\mathcal{F}$ in $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$. Besides the two possibilities of edge types given by definition 3.1:

(E1) $\mathfrak{e}_1 = \mathfrak{e}_2 = \mathfrak{e}_f$,
(E2) $\mathfrak{e}_1 \neq \mathfrak{e}_2 \neq \mathfrak{e}_f$,

one can also consider other possible combinations of edge types in $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$:

(E3) $\mathfrak{e}_1 \neq \mathfrak{e}_2 = \mathfrak{e}_f$, or alternatively, $\mathfrak{e}_1 = \mathfrak{e}_f \neq \mathfrak{e}_2$,
(E4) $\mathfrak{e}_1 = \mathfrak{e}_2 \neq \mathfrak{e}_f$.

As we will see later, in terms of balanced equivalence relations on $\mathcal{G}$, the case (E3) is similar to (E1) and the case (E4) is similar to (E2) (see remark 4.24). $\diamond$

Note that the f-join of two uniform networks is again uniform. We give an example of join, partial join and point-wise partial join of two networks.

Example 3.3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two coupled cell networks given in figure 1 with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Then, the join of $\mathcal{G}_1$ and $\mathcal{G}_2$ is given by figure 2(a). For $\tilde{\mathcal{C}}_1 = \{2, 3\}$ and $\tilde{\mathcal{C}}_2 = \{5\}$, the partial join of $\mathcal{G}_1$ and $\mathcal{G}_2$ is given by figure 2(b). For the bijection $f : \tilde{\mathcal{C}}_1 = \{1, 3\} \to \tilde{\mathcal{C}}_2 = \{4, 5\}$ with $f(1) = 5$ and $f(3) = 4$, the point-wise partial join is given by figure 2(c).

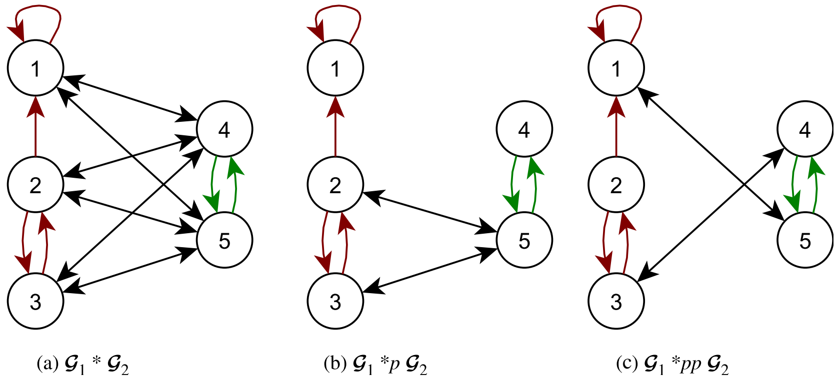


Figure 2. *(a)* $\mathcal{G}_1 * \mathcal{G}_2$; *(b)* $\mathcal{G}_1 *_p \mathcal{G}_2$; *(c)* $\mathcal{G}_1 *_{pp} \mathcal{G}_2$, for $\mathcal{G}_1$ and $\mathcal{G}_2$ given in figure 1.

For the purpose of our proof later, we show that $\mathcal{G}_1 *_f \mathcal{G}_2$ can be rewritten as $\mathcal{G}_2 *_g \mathcal{G}_1$ for another multimap g. Let f be a multimap given by (3.7). Define the *inverse* of f by

$$\begin{aligned} f^{-1} &: \tilde{\mathcal{C}}_2 \to P(\tilde{\mathcal{C}}_1) \\ &d \mapsto f^{-1}(d) := \{c \in \tilde{\mathcal{C}}_1 \;:\; d \in f(c)\} \subset \tilde{\mathcal{C}}_1. \end{aligned} \tag{3.8}$$

Lemma 3.4. *For two networks $\mathcal{G}_1$ and $\mathcal{G}_2$, we have $\mathcal{G}_1 *_f \mathcal{G}_2 = \mathcal{G}_2 *_{f^{-1}} \mathcal{G}_1$, where f^{-1} is the inverse of f given by (3.8).*

Proof. By definition of join, it suffices to show $\mathcal{F} = \mathcal{F}'$, where

$$\begin{aligned} \mathcal{F} &= \{(c,d),(d,c) :\; c \in \tilde{\mathcal{C}}_1 \wedge d \in f(c)\}, \\ \mathcal{F}' &= \{(d,c),(c,d) :\; d \in \tilde{\mathcal{C}}_2 \wedge c \in f^{-1}(d)\}. \end{aligned}$$

By (3.8), we have $c \in f^{-1}(d)$ if and only if $d \in f(c)$, for all $c \in \tilde{\mathcal{C}}_1$, $d \in \tilde{\mathcal{C}}_2$. Thus, $\mathcal{F} = \mathcal{F}'$ and the statement follows. □

3.2. Coalescence

A *coalescence* of two graphs is a graph obtained from the disjoint union of the two graphs by merging two vertices chosen from the two graphs respectively. Depending on the choice of the two vertices, two graphs usually have more than one coalescence. For technical reasons, we define the *coalescence* on coupled cell networks in a slightly different way, which nevertheless, leads to the same outcome of graphs.

Definition 3.5. Let $\mathcal{C}_1 \cap \mathcal{C}_2 = \{\theta\}$. A network $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ is the *coalescence* of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$, if

- $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$;
- $c_1 \sim_C c_2$, for all $c_1, c_2 \in \mathcal{C}$;
- $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$;
- $e_1 \sim_E e_2$, for all $e_1, e_2 \in \mathcal{E}$ if $\mathfrak{e}_1 = \mathfrak{e}_2$; otherwise $e_1 \sim_E e_2$ if and only if $e_1 \sim_{E_1} e_2$ or $e_1 \sim_{E_2} e_2$, for e_1, e_2 the corresponding edges in $\mathcal{E}_1$ or $\mathcal{E}_2$. ◇

Note that the coalescence of two uniform networks is again uniform. As mentioned before, our definition of coalescence leads to the same coalesced graphs. Indeed, given two disjoint networks, we can first identify one cell $c_1 \in \mathcal{G}_1$ with another cell $c_2 \in \mathcal{G}_2$ and call it 'θ', then apply the coalescence of definition 3.5. This will correspond to the coalesced graph obtained

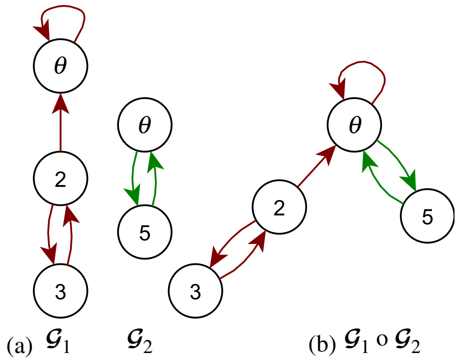


Figure 3. *(a)* $\mathcal{G}_1$ and $\mathcal{G}_2$; *(b)* $\mathcal{G}_1 \circ \mathcal{G}_2$.

by merging c_1 and c_2. Now let c_1 and c_2 run through $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, this will give rise to all possible coalescence of graphs.

Example 3.6. Consider the two coupled cell networks $\mathcal{G}_1$ and $\mathcal{G}_2$ given in figure 3, which have a common cell θ. The coalescence of $\mathcal{G}_1$ and $\mathcal{G}_2$ is then given by figure 3(*b*).

4. Synchrony under binary network operations

In this section, we discuss how the lattice of balanced equivalence relations on networks may 'evolve' when the networks evolve. We are especially interested in relating the lattice of balanced equivalence relations of the new network to those of the initial ones.

In what follows, $\Lambda_{\mathcal{G}}$ denotes the lattice of balanced equivalence relations on a coupled cell network $\mathcal{G}$.

Let $\mathcal{G}_i = (\mathcal{C}_i, \mathcal{E}_i, \sim_{C_i}, \sim_{E_i})$ be a coupled cell network with edge type $\mathfrak{e}_i$, for $i = 1, 2$. Within this subsection, $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ stands for the network obtained by applying a binary operation defined in section 3.

Notations 4.1. Let $c \in \mathcal{C}$ be a cell of $\mathcal{G}$ and $I(c)$ be the input set of c in $\mathcal{G}$. Denote by $I^{\mathfrak{e}}(c)$ the input set of c corresponding to edge type $\mathfrak{e}$. For $c \in \mathcal{C}_i$ with $i \in \{1, 2\}$, denote by $I_i(c)$ the input set of c in $\mathcal{G}_i$. Then, $I_i(c) = I(c) \cap \mathcal{C}_i$ as multiset, in case of f-join or coalescence, unless θ has a self-directed edge in the latter case. Note that all input edges of c in $\mathcal{G}_i$ are of the same type $\mathfrak{e}_i$.

Definition 4.2. A cell c of $\mathcal{G}$ is called a *source*, if $I(c) = \emptyset$; and c is called a *source for* $\mathcal{G}_i$, if $I_i(c) = \emptyset$, for $i = 1, 2$. $\diamond$

Let $\Lambda_{\mathcal{G}_i}$ be the lattice of balanced equivalence relations on $\mathcal{G}_i$ for $i = 1, 2$. We discuss the conditions under which $\Lambda_{\mathcal{G}}$ can be 'recovered' from $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$. As we will see later, this strongly depends on whether the edge types $\mathfrak{e}_1, \mathfrak{e}_2$ are equal or distinct.

Definition 4.3. Let $\bowtie$ be an equivalence relation on $\mathcal{G}$. For $i = 1, 2$, define the *restriction* $\bowtie_i$ of $\bowtie$ on $\mathcal{G}_i$ by

$$c \bowtie_i d \quad \Leftrightarrow \quad c, d \in \mathcal{C}_i \;\wedge\; c \bowtie d. \tag{4.9}$$

That is, $[c]_{\bowtie_i} = [c]_{\bowtie} \cap \mathcal{C}_i$, for all $c \in \mathcal{C}_i$, $i = 1, 2$. $\diamond$

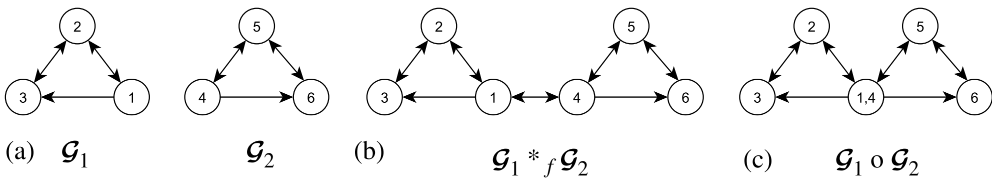


Figure 4. (*a*) $\mathcal{G}_1$ and $\mathcal{G}_2$; (*b*) $\mathcal{G}_1 *_f \mathcal{G}_2$; (*c*) $\mathcal{G}_1 \circ \mathcal{G}_2$.

Definition 4.4. Given two equivalence relations $\bowtie_1, \bowtie_2$ on $\mathcal{G}_1, \mathcal{G}_2$, respectively, define the *join extension* $\bowtie_{1,2} := \bowtie_1 \dot{\vee} \bowtie_2$ to $\mathcal{G}$ by

$$c \bowtie_{1,2} d \quad \Leftrightarrow \quad \big(c, d \in \mathcal{C}_1 \ \wedge \ c \bowtie_1 d\big) \ \vee \ \big(c, d \in \mathcal{C}_2 \ \wedge \ c \bowtie_2 d\big) \vee \ c = d. \tag{4.10}$$

That is, $[c]_{\bowtie_{1,2}} = [c]_{\bowtie_i}$, for all $c \in \mathcal{C}_i$, $i = 1, 2$. In the case $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$, $[\theta]_{\bowtie_{1,2}} = [\theta]_{\bowtie_1} \cup [\theta]_{\bowtie_2}$. $\diamond$

Let $\boldsymbol{E}_{\mathcal{G}}$ be the lattice of all equivalence relations on the cells of $\mathcal{G}$. Then, we have the following composition of operations on lattices

$$\begin{aligned} \mathcal{R} : \quad & \boldsymbol{E}_{\mathcal{G}} \overset{\text{res.}}{\to} \boldsymbol{E}_{\mathcal{G}_1} \times \boldsymbol{E}_{\mathcal{G}_2} \overset{\text{join ext.}}{\longrightarrow} \boldsymbol{E}_{\mathcal{G}} \\ & \bowtie \ \mapsto \ (\bowtie_1, \bowtie_2) \ \mapsto \ \bowtie_1 \dot{\vee} \bowtie_2, \end{aligned} \tag{4.11}$$

where $\bowtie_1, \bowtie_2$ are restrictions of $\bowtie$ on $\mathcal{G}_1, \mathcal{G}_2$. Clearly, $\mathcal{R}(\bowtie)$ is a refinement of $\bowtie$.

In general, the property of being balanced may not be preserved under $\mathcal{R}$; that is, the restriction of a balanced equivalence relation need not to be balanced again; and the join extension of two balanced equivalence relations may be non-balanced for $\mathcal{G}$ (see example 4.5). This depends on whether the considered equivalence relations are 'compatible' with the network operation (see definition 4.9 for the f-join and definition 4.25 for the coalescence).

Example 4.5.

(i) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be given in figure 4(*a*) such that $\mathfrak{e}_1 = \mathfrak{e}_2$. Let $\tilde{\mathcal{C}}_1 = \{1\}$, $\tilde{\mathcal{C}}_2 = \{4\}$ and $f : \tilde{\mathcal{C}}_1 \to P(\tilde{\mathcal{C}}_2)$ be defined by $f(1) = \{4\}$. Then, the f-join $\mathcal{G}_1 *_f \mathcal{G}_2$ is given by figure 4(*b*). Let $1 = 4 = \theta$ be the common cell of $\mathcal{G}_1$ and $\mathcal{G}_2$. Then, the coalescence $\mathcal{G}_1 \circ \mathcal{G}_2$ is given by figure 4(*c*). It can be verified that $\bowtie = \{\{1, 2, 5, 6\}, \{3, 4\}\}$ is balanced on $\mathcal{G}_1 *_f \mathcal{G}_2$, but its restriction $\bowtie_1 = \{\{1, 2\}, \{3\}\}$ is not balanced on $\mathcal{G}_1$. Also, the equivalence relation $\tilde{\bowtie} = \{\{\theta, 3, 5\}, \{2, 6\}\}$ is balanced on $\mathcal{G}_1 \circ \mathcal{G}_2$, but its restrictions $\tilde{\bowtie}_1 = \{\{\theta, 3\}, \{2\}\}$ and $\tilde{\bowtie}_2 = \{\{\theta, 5\}, \{6\}\}$ are both non-balanced.

(ii) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be given in figure 5(*a*). Let $f : \mathcal{C}_1 \to P(\mathcal{C}_2)$ be defined by $f(1) = \{3, 4\}$, $f(2) = \{4\}$. Then, the f-join $\mathcal{G}_1 *_f \mathcal{G}_2$ is given by figure 5(*b*). Identify the cell $1 \in \mathcal{C}_1$ with the cell $3 \in \mathcal{C}_2$, which is denoted by θ. Then, the coalescence $\mathcal{G}_1 \circ \mathcal{G}_2$ is given by figure 5(*c*).

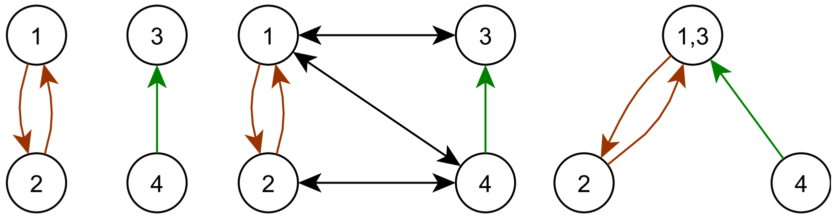
Consider the equivalence relations $\bowtie_1 = \{\{1, 2\}\}$ on $\mathcal{G}_1$ and $\bowtie_2 = \{\{3\}, \{4\}\}$ on $\mathcal{G}_2$. It can be verified that $\bowtie_1$ and $\bowtie_2$ are balanced on $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, but their join extension on $\mathcal{G}_1 *_f \mathcal{G}_2$

$$\bowtie_1 \dot{\vee} \bowtie_2 = \{\{1, 2\}, \{3\}, \{4\}\},$$

and their join extension on $\mathcal{G}_1 \circ \mathcal{G}_2$

$$\bowtie_1 \dot{\vee} \bowtie_2 = \{\{\theta, 2\}, \{4\}\}$$

are both non-balanced. $\diamond$



(a) $\mathcal{G}_1$ $\mathcal{G}_2$ (b) $\mathcal{G}_1 *_f \mathcal{G}_2$ (c) $\mathcal{G}_1 \circ \mathcal{G}_2$

Figure 5. *(a)* $\mathcal{G}_1$ and $\mathcal{G}_2$; *(b)* $\mathcal{G}_1 *_f \mathcal{G}_2$; *(c)* $\mathcal{G}_1 \circ \mathcal{G}_2$.

We distinguish different kinds of equivalence relations $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$ on $\mathcal{G}$.

Definition 4.6.

(i) Let $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$. An equivalence relation $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$ is called *bipartite*, if there exists a $\bowtie$-class $[\alpha]_{\bowtie}$ such that $[\alpha]_{\bowtie} \cap \mathcal{C}_1 \neq \emptyset$ and $[\alpha]_{\bowtie} \cap \mathcal{C}_2 \neq \emptyset$. Otherwise, $\bowtie$ is called *non-bipartite*. A bipartite equivalence relation $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$ is called *pairing bipartite*, if $\#([\alpha]_{\bowtie} \cap \mathcal{C}_1) = \#([\alpha]_{\bowtie} \cap \mathcal{C}_2) = 1$ for all nontrivial $\bowtie$-classes $[\alpha]_{\bowtie}$. Otherwise, $\bowtie$ is called *non-pairing bipartite*.

(ii) Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$. An equivalence relation $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$ is called *bipartite*, if there exists a $\bowtie$-class $[\alpha]_{\bowtie} \neq [\theta]_{\bowtie}$ such that $[\alpha]_{\bowtie} \cap \mathcal{C}_1 \neq \emptyset$ and $[\alpha]_{\bowtie} \cap \mathcal{C}_2 \neq \emptyset$. Otherwise, $\bowtie$ is called *non-bipartite*. A bipartite equivalence relation $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$ is called *pairing bipartite*, if $[\theta]_{\bowtie} = \{\theta\}$ and $\#([\alpha]_{\bowtie} \cap \mathcal{C}_1) = \#([\alpha]_{\bowtie} \cap \mathcal{C}_2) = 1$ for all nontrivial $\bowtie$-classes $[\alpha]_{\bowtie}$. Otherwise, $\bowtie$ is called *non-pairing bipartite*. $\diamond$

Example 4.7. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the networks discussed in example 4.5(i). Consider $\mathcal{G}_1 *_f \mathcal{G}_2$ and $\mathcal{G}_1 \circ \mathcal{G}_2$ given by figures 4(*b*) and (*c*). Then, we have the following balanced equivalence relations on $\mathcal{G}_1 *_f \mathcal{G}_2$

$$\triangleright\!\circ\!\triangleleft = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\},$$
$$\triangleright\!\bullet\!\triangleleft = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\},$$
$$\triangleright\!\diamond\!\triangleleft = \{\{2, 3, 5, 6\}, \{1, 4\}\},$$

where $\triangleright\!\circ\!\triangleleft$ is non-bipartite, $\triangleright\!\bullet\!\triangleleft$ is pairing bipartite and $\triangleright\!\diamond\!\triangleleft$ is non-pairing bipartite. On $\mathcal{G}_1 \circ \mathcal{G}_2$, we have the following balanced equivalence relations

$$\triangleright\!\tilde{\circ}\!\triangleleft = \{\{2, 3\}, \{\theta\}, \{5, 6\}\},$$
$$\triangleright\!\tilde{\bullet}\!\triangleleft = \{\{\theta\}, \{2, 6\}, \{3, 5\}\},$$
$$\triangleright\!\tilde{\diamond}\!\triangleleft = \{\{2, \theta, 6\}, \{3, 5\}\},$$

where $\triangleright\!\tilde{\circ}\!\triangleleft$ is non-bipartite, $\triangleright\!\tilde{\bullet}\!\triangleleft$ is pairing bipartite and $\triangleright\!\tilde{\diamond}\!\triangleleft$ is non-pairing bipartite. $\diamond$

The following lemma is practical in distinguishing these different kinds of equivalence relations on $\mathcal{G}$.

Lemma 4.8. *Let $\mathcal{G}$ be the network obtained by applying a binary operation defined in section 3. Let $\mathcal{R}$ be defined by (4.11) and $\bowtie \in \boldsymbol{E}_{\mathcal{G}}$. Then,*

(i) $\bowtie$ is non-bipartite if and only if $\mathcal{R}(\bowtie) = \bowtie$;

(ii) $\bowtie$ is pairing bipartite if and only if $\mathcal{R}(\bowtie) \neq \bowtie$ and $\mathcal{R}(\bowtie)$ is trivial;

(iii) $\bowtie$ is non-pairing bipartite if and only if $\mathcal{R}(\bowtie) \neq \bowtie$ and $\mathcal{R}(\bowtie)$ is nontrivial.

Proof.

(i) Let $\bowtie$ be such that $\mathcal{R}(\bowtie) = \bowtie$. Assume to the contrary that $\bowtie$ is bipartite. In case $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$, this implies that there exist $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ such that $x \bowtie y$. Since $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$, we have either $x, y \in \mathcal{C}_1$ with $x \bowtie_1 y$ or $x, y \in \mathcal{C}_2$ with $x \bowtie_2 y$, which gives a contradiction to the fact that $x \in \mathcal{C}_1$, $y \in \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. In case $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$, we can assume additionally that $x \neq \theta$ and $y \neq \theta$. Thus, this gives the same contradiction, since $\mathcal{C}_1 \cap \mathcal{C}_2 = \{\theta\}$. On the other hand, if $\bowtie$ is non-bipartite. Then, by definition, we have $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2 = \mathcal{R}(\bowtie)$.

(ii) Let $\bowtie$ be such that $\mathcal{R}(\bowtie) \neq \bowtie$ and $\mathcal{R}(\bowtie)$ is trivial. Let $[\alpha]_{\bowtie}$ be a nontrivial $\bowtie$-class. Since $\bowtie$ is bipartite by (i), we have $[\alpha]_{\bowtie} = [a]_{\bowtie_1} \cup [b]_{\bowtie_2}$ for some $a \in \mathcal{C}_1$, $b \in \mathcal{C}_2$. In case $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$, we have that $[a]_{\bowtie_1} = [a]_{\mathcal{R}(\bowtie)}$ and $[b]_{\bowtie_2} = [b]_{\mathcal{R}(\bowtie)}$ are singletons. In case $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$, we have additionally that $[\theta]_{\bowtie} = [\theta]_{\mathcal{R}(\bowtie)} = \{\theta\}$. Thus, $\bowtie$ is pairing bipartite. On the other hand, $\mathcal{R}(\bowtie)$ is composed of $([\alpha]_{\bowtie} \cap \mathcal{C}_i)$ as equivalence classes, for $\alpha \in \mathcal{C}_i$, $i = 1, 2$, where in case $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$, $[\theta]_{\mathcal{R}(\bowtie)} = [\theta]_{\bowtie} = \{\theta\}$. Thus, the statement follows.

(iii) It follows from (i)–(ii). □

4.1. *Synchrony for f-join of networks*

Throughout this subsection, let $\mathcal{G}$ denote the f-join of $\mathcal{G}_1$ and $\mathcal{G}_2$. We are interested in classifying balanced equivalence relations of $\mathcal{G}$ using balanced equivalence relations of $\mathcal{G}_1$ and $\mathcal{G}_2$. It turns out that in case of different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$, the lattice $\Lambda_{\mathcal{G}}$ can be completely characterized using $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$, together with interior symmetry of $\mathcal{G}$ (see theorem 4.22); however, in case of identical edge type $\mathfrak{e}_1 = \mathfrak{e}_2$, only those balanced equivalence relations that are 'compatible' with the f-join operation can be classified (see theorem 4.17).

Definition 4.9.

(i) Let $\bowtie_i$ be an equivalence relation on $\mathcal{G}_i$ for $i = 1, 2$. We say that $\bowtie_1$ and $\bowtie_2$ are *f-related* if for every $c_1, c_2 \in \mathcal{C}_1$ such that $c_1 \bowtie_1 c_2$, we have

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c_2)), \qquad \forall \beta \in \mathcal{C}_2,$$

where $f(c_i) = \emptyset$ if $c_i \notin \tilde{\mathcal{C}}_1$ for $i = 1, 2$. Similarly, we say that $\bowtie_1$ and $\bowtie_2$ are *f^{-1}-related*, if for every $d_1, d_2 \in \mathcal{C}_2$ such that $d_1 \bowtie_2 d_2$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_1}, f^{-1}(d_1)) = \mathfrak{m}([\alpha]_{\bowtie_1}, f^{-1}(d_2)), \qquad \forall \alpha \in \mathcal{C}_1.$$

(ii) Let $\bowtie$ be an equivalence relation on $\mathcal{G}$ and $\bowtie_1, \bowtie_2$ be the restriction of $\bowtie$ on $\mathcal{G}_1, \mathcal{G}_2$ respectively. We say that $\bowtie$ is *f-compatible* (respectively *f^{-1}-compatible*), if $\bowtie_1$ and $\bowtie_2$ are f-related (respectively f^{-1}-related). ◇

For convenience, denote by

$$\Lambda^f_{\mathcal{G}} = \{\bowtie \in \Lambda_{\mathcal{G}} : \bowtie \text{ is } f\text{-compatible and } f^{-1}\text{-compatible}\}.$$

Remark 4.10.

(i) If $\mathcal{G} = \mathcal{G}_1 * \mathcal{G}_2$ is the join of $\mathcal{G}_1$ and $\mathcal{G}_2$, then $\Lambda^f_{\mathcal{G}} = \Lambda_{\mathcal{G}}$.

(ii) If $\mathcal{G} = \mathcal{G}_1 *_p \mathcal{G}_2$ is the partial join of $\mathcal{G}_1$ and $\mathcal{G}_2$, then

$$\Lambda^f_{\mathcal{G}} = \{\bowtie \in \Lambda_{\mathcal{G}} \ : \ [x]_{\bowtie_i} \subset \tilde{\mathcal{C}}_i \text{ or } [x]_{\bowtie_i} \subset \mathcal{C}_i \setminus \tilde{\mathcal{C}}_i, \ \forall x \in \mathcal{C}_i, \ i = 1, 2\}.$$

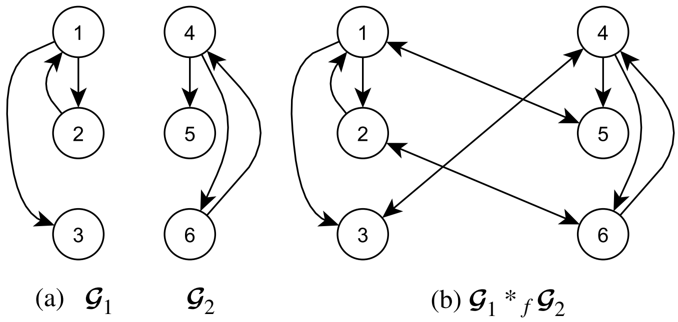


Figure 6. (*a*) Two regular networks $\mathcal{G}_1, \mathcal{G}_2$; (*b*) $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$.

(iii) If $\mathcal{G} = \mathcal{G}_1 *_{pp} \mathcal{G}_2$ is the point-wise partial join of $\mathcal{G}_1$ and $\mathcal{G}_2$, then

$$\Lambda_{\mathcal{G}}^f = \{\bowtie \in \Lambda_{\mathcal{G}} \,:\, [x]_{\bowtie_i} \subset \tilde{\mathcal{C}}_i \wedge f([x]_{\bowtie_i}) = [f(x)]_{\bowtie_j} \text{ or } [x]_{\bowtie_i} \subset \mathcal{C}_i \setminus \tilde{\mathcal{C}}_i,\ \forall x \in \mathcal{C}_i,\ i, j = 1, 2, i \neq j\}.$$

(iv) In general, $\Lambda_{\mathcal{G}}^f \subsetneq \Lambda_{\mathcal{G}}$, even if $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ for two regular networks $\mathcal{G}_1, \mathcal{G}_2$. Consider $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G} = \mathcal{G}_1 *_{pp} \mathcal{G}_2$ given by figure 6, for $f(1) = \{5\}$, $f(2) = \{6\}$ and $f(3) = \{4\}$. Then, $\bowtie = \{\{1, 3, 5\}, \{2, 4\}, \{6\}\}$ is balanced but not f-compatible, since

$$\mathfrak{m}([4]_{\bowtie_2}, f(1)) = 0 \neq 1 = \mathfrak{m}([4]_{\bowtie_2}, f(3)).$$

Indeed, as we will see later, every non-compatible relation $\bowtie \in \Lambda_{\mathcal{G}} \setminus \Lambda_{\mathcal{G}}^f$ is non-pairing bipartite (see lemma 4.11 and lemma 4.12).

(v) If $\mathcal{G} = \mathcal{G}_1 *_p \mathcal{G}_2$ for two regular networks $\mathcal{G}_1, \mathcal{G}_2$ of valency v_1, v_2, then a necessary condition for $\mathcal{G}$ to support (non-pairing) bipartite balanced equivalence relations is that $v_1 = v_2$ or $v_1 + n_2 = v_2 + n_1$, where $n_i = \#\tilde{\mathcal{C}}_i$ is the number of cells in $\tilde{\mathcal{C}}_i$, for $i = 1, 2$. If $\mathcal{G} = \mathcal{G}_1 *_{pp} \mathcal{G}_2$, then it is necessary that $v_1 = v_2$. ◇

Lemma 4.11. *Let $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ and $\mathfrak{e}_i$ be the edge type of $\mathcal{G}_i$ for $i = 1, 2$. Let $\bowtie \in \Lambda_{\mathcal{G}}$ and $\bowtie_i$ be the restrictions of $\bowtie$ on $\mathcal{G}_i$, for $i = 1, 2$. Then, $\bowtie_i \in \Lambda_{\mathcal{G}_i}$ if one of the following conditions holds:*

(i) $\mathfrak{e}_1 \neq \mathfrak{e}_2$*;*
(ii) $\mathfrak{e}_1 = \mathfrak{e}_2$ *and* $\bowtie$ *is non-bipartite;*
(iii) $\mathfrak{e}_1 = \mathfrak{e}_2$ *and* $\bowtie$ *is pairing bipartite;*
(iv) $\mathfrak{e}_1 = \mathfrak{e}_2$ *and* $\bowtie$ *is non-pairing bipartite such that* $\bowtie \in \Lambda_{\mathcal{G}}^f$.

Proof. Since the statement is symmetric with respect to $\mathcal{G}_1, \mathcal{G}_2$ (see lemma 3.4), we present the proof only for $\bowtie_1$.

(i) Let $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Then, $\mathcal{G}$ has three edge types $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_f$. Let $x, y \in \mathcal{C}_1$ be such that $x \bowtie_1 y$. Then, $x \bowtie y$ and thus

$$\mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}}(x)) = \mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}}(y)), \qquad \forall \alpha \in \mathcal{C},\ \mathfrak{e} \in \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_f\}.$$

Thus,

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}_1}(x)) = \mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}_1}(y)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(y)), \forall \alpha \in \mathcal{C}_1.$$

Consequently, $\bowtie_1$ is balanced.

(ii) Assume that $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie$ is non-bipartite. Then, $[\alpha]_{\bowtie} = [\alpha]_{\bowtie_1}$ for $\alpha \in \mathcal{C}_1$. Let $c_1, c_2 \in \mathcal{C}_1$ such that $c_1 \bowtie_1 c_2$. Then, we have

$$[\alpha]_{\bowtie_1} \cap I_1(c_i) = [\alpha]_{\bowtie} \cap I(c_i), \qquad \forall \alpha \in \mathcal{C}_1, \quad i = 1, 2,$$

since $I(c_i) = I_1(c_i) \cup f(c_i)$ and $[\alpha]_{\bowtie_1} \cap f(c_i) = \emptyset$, for $i = 1, 2$. Thus, it follows from $c_1 \bowtie c_2$ that

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c_1)) = \mathfrak{m}([\alpha]_{\bowtie}, I(c_1)) = \mathfrak{m}([\alpha]_{\bowtie}, I(c_2)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c_1)).$$

Thus, $\bowtie_1$ is balanced.

(iii) Let $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie$ be pairing bipartite. Then, $\bowtie_1$ is the trivial equivalence relation on $\mathcal{G}_1$, thus balanced.

(iv) Let $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie \in \Lambda_{\mathcal{G}}^f$ be non-pairing bipartite. Let $c_1, c_2 \in \mathcal{C}_1$ such that $c_1 \bowtie_1 c_2$. Then, $c_1 \bowtie c_2$ and we have

$$\mathfrak{m}([\alpha]_{\bowtie}, I(c_1)) = \mathfrak{m}([\alpha]_{\bowtie}, I(c_2)), \qquad \forall \alpha \in \mathcal{C}.$$

Let $x \in \mathcal{C}_1$ and consider its $\bowtie$-equivalence class $[x]_{\bowtie}$. In the case $[x]_{\bowtie} \subset \mathcal{C}_1$, we have

$$\mathfrak{m}([x]_{\bowtie_1}, I_1(c_1)) = \mathfrak{m}([x]_{\bowtie}, I(c_1)) = \mathfrak{m}([x]_{\bowtie}, I(c_2)) = \mathfrak{m}([x]_{\bowtie_1}, I_1(c_2)).$$

Otherwise, write $[x]_{\bowtie} = [x]_{\bowtie_1} \cup [y]_{\bowtie_2}$ for some $y \in \mathcal{C}_2$. Then, we have

$$\mathfrak{m}([x]_{\bowtie}, I(c_i)) = \mathfrak{m}([x]_{\bowtie_1}, I_1(c_i)) + \mathfrak{m}([y]_{\bowtie_2}, f(c_i)), \qquad i = 1, 2.$$

Since $\bowtie$ is f-compatible, $\bowtie_1, \bowtie_2$ are f-related, thus

$$\mathfrak{m}([y]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([y]_{\bowtie_2}, f(c_2)).$$

It follows that

$$\mathfrak{m}([x]_{\bowtie_1}, I_1(c_1)) = \mathfrak{m}([x]_{\bowtie_1}, I_1(c_2)).$$

Consequently, $\bowtie_1$ is balanced. □

Lemma 4.12. *Let $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ and $\mathfrak{e}_i$ be the edge type of $\mathcal{G}_i$ for $i = 1, 2$. Let $\bowtie \in \Lambda_{\mathcal{G}}$ and $\bowtie_i$ be the restrictions of $\bowtie$ on $\mathcal{G}_i$, for $i = 1, 2$. Then, $\bowtie_1, \bowtie_2$ are f- and f^{-1}-related, if one of the conditions from (i)–(iv) in lemma 4.11 holds.*

Proof. By lemma 4.11, the restrictions $\bowtie_1, \bowtie_2$ of $\bowtie$ are balanced on $\mathcal{G}_1, \mathcal{G}_2$, respectively. Since the statement is symmetric with respect to $\mathcal{G}_1, \mathcal{G}_2$, we only show that $\bowtie_1$ and $\bowtie_2$ are f-related.

(i) Let $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Let $c_1 \bowtie_1 c_2$ for some $c_1, c_2 \in \mathcal{C}_1$. Since $\bowtie$ is balanced and $c_1 \bowtie c_2$, we have

$$\mathfrak{m}([\beta]_{\bowtie}, I^{\mathfrak{e}_f}(c_1)) = \mathfrak{m}([\beta]_{\bowtie}, I^{\mathfrak{e}_f}(c_2)), \qquad \forall \beta \in \mathcal{C}_2.$$

On the other hand, for every $c \in \mathcal{C}_1$, we have $I^{\mathfrak{e}_f}(c) = f(c) \subseteq \mathcal{C}_2$ and $[\beta]_{\bowtie} \cap \mathcal{C}_2 = [\beta]_{\bowtie_2}$. Thus,

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c_2)), \qquad \forall \beta \in \mathcal{C}_2.$$

It follows that $\bowtie_1$ and $\bowtie_2$ are f-related.

(ii) Let $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie$ be non-bipartite. Let $c_1 \bowtie_1 c_2$ for some $c_1, c_2 \in \mathcal{C}_1$. Then, since $\bowtie$ is balanced and $c_1 \bowtie c_2$, we have

$$\mathfrak{m}([\beta]_{\bowtie}, I(c_1)) = \mathfrak{m}([\beta]_{\bowtie}, I(c_2)), \qquad \forall \beta \in \mathcal{C}_2. \tag{4.12}$$

Note that for every cell $c \in \mathcal{C}_1$, we have $I(c) = I_1(c) \cup f(c)$. Let $\beta \in \mathcal{C}_2$ be such that $[\beta]_{\bowtie} \subset \mathcal{C}_2$. Then, $[\beta]_{\bowtie} = [\beta]_{\bowtie_2}$ and so we have

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([\beta]_{\bowtie}, I(c_1)) = \mathfrak{m}([\beta]_{\bowtie}, I(c_2)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c_2)).$$

Let $\beta \in \mathcal{C}_2$ be such that $[\beta]_{\bowtie} \not\subset \mathcal{C}_2$. Then, $[\beta]_{\bowtie} = [\alpha]_{\bowtie_1} \cup [\beta]_{\bowtie_2}$ for some $\alpha \in \mathcal{C}_1$. Thus,

$$\mathfrak{m}([\beta]_{\bowtie}, I(c)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)) + \mathfrak{m}([\beta]_{\bowtie_2}, f(c)), \qquad \forall c \in \mathcal{C}_1. \tag{4.13}$$

Since $\bowtie_1$ is balanced and $c_1 \bowtie_1 c_2$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c_1)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c_2)). \tag{4.14}$$

It follows from (4.12)–(4.14) that

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c_2)).$$

Therefore, $\bowtie_1$ and $\bowtie_2$ are f-related.

(iii) Let $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie$ be pairing bipartite. Then, $\bowtie_i$ is the trivial equivalence relation on $\mathcal{G}_i$, for $i = 1, 2$. Thus, $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related.

(iv) Let $\mathfrak{e}_1 = \mathfrak{e}_2$ and $\bowtie$ be non-pairing bipartite such that $\bowtie \in \Lambda_{\mathcal{G}}^f$. Then, by definition of $\Lambda_{\mathcal{G}}^f$, $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related. $\square$

Lemma 4.13. *Let $\bowtie_i \in \Lambda_{\mathcal{G}_i}$ be a balanced equivalence relation for $i = 1, 2$. If $\bowtie_1$ and $\bowtie_2$ are both f- and f^{-1}-related, then the join extension $\bowtie_1 \dot{\vee} \bowtie_2$ is balanced on $\mathcal{G}_1 *_f \mathcal{G}_2$.*

Proof. Let $\mathfrak{e}_i$ be the edge type of $\mathcal{G}_i$, for $i = 1, 2$. Denote by $\bowtie_{1,2} = \bowtie_1 \dot{\vee} \bowtie_2$. Let $x, y \in \mathcal{C}$ be such that $x \bowtie_{1,2} y$. Thus, either $x, y \in \mathcal{C}_1$ with $x \bowtie_1 y$ or $x, y \in \mathcal{C}_2$ with $x \bowtie_2 y$. Without loss of generality, assume $x, y \in \mathcal{C}_1$ and $x \bowtie_1 y$.

If $\mathfrak{e}_1 \neq \mathfrak{e}_2$ then, $\mathcal{G}_1 *_f \mathcal{G}_2$ has three distinct edge types $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_f$, and the input sets of every cell $c \in \mathcal{C}_1$ with respect to these edge types are given by

$$I^{\mathfrak{e}_1}(c) = I_1(c), \;\; I^{\mathfrak{e}_2}(c) = \emptyset, \;\; I^{\mathfrak{e}_f}(c) = f(c).$$

Thus, for every equivalence class $[\alpha]_{\bowtie_{1,2}}$ for some $\alpha \in \mathcal{C}$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}}(c)) = \begin{cases} \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)), & \text{if } \alpha \in \mathcal{C}_1, \mathfrak{e} = \mathfrak{e}_1 \\ \mathfrak{m}([\alpha]_{\bowtie_2}, f(c)), & \text{if } \alpha \in f(c), \mathfrak{e} = \mathfrak{e}_f \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

If $\mathfrak{e}_1 = \mathfrak{e}_2$ then, $\mathcal{G}_1 *_f \mathcal{G}_2$ has only one edge type $\mathfrak{e} = \mathfrak{e}_1 = \mathfrak{e}_2$, and the input set of every cell $c \in \mathcal{C}_1$ is $I(c) = I_1(c) \cup f(c)$. Thus, for every equivalence class $[\alpha]_{\bowtie_{1,2}}$ for some $\alpha \in \mathcal{C}$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(c)) = \begin{cases} \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)) & \text{if } \alpha \in \mathcal{C}_1 \\ \mathfrak{m}([\alpha]_{\bowtie_2}, f(c)) & \text{if } \alpha \in \mathcal{C}_2. \end{cases} \tag{4.16}$$

Since $\bowtie_1$ is balanced and $x \bowtie_1 y$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(y)), \qquad \forall \alpha \in \mathcal{C}_1. \tag{4.17}$$

Also, since $\bowtie_1$ and $\bowtie_2$ are f-related, we have

$$\mathfrak{m}([\alpha]_{\bowtie_2}, f(x)) = \mathfrak{m}([\alpha]_{\bowtie_2}, f(y)), \qquad \forall \alpha \in \mathcal{C}_2. \tag{4.18}$$

Therefore, if $\mathfrak{e}_1 \neq \mathfrak{e}_2$, by (4.15), (4.17) and (4.18), we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}}(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}}(y)), \quad \forall \alpha \in \mathcal{C},\ \mathfrak{e} \in \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_f\}. \tag{4.19}$$

If $\mathfrak{e}_1 = \mathfrak{e}_2$, by (4.16), (4.17) and (4.18), we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(y)), \quad \forall \alpha \in \mathcal{C},\ \alpha \in \mathcal{C}.$$

The same argument applies for the case $x, y \in \mathcal{C}_2$ with $x \bowtie_2 y$, since $\bowtie_1, \bowtie_2$ are f^{-1}-related. Consequently, $\bowtie_{1,2}$ is balanced. $\square$

4.1.1. Synchrony for $\mathcal{G}_1 *_f \mathcal{G}_2$ with the same edge type $\mathfrak{e}_1 = \mathfrak{e}_2$. To state the main result for non-pairing-bipartite balanced relations on $\mathcal{G}$, we need to introduce the 'quotient' of the multimap f on a quotient network $\mathcal{G}_{\bowtie}$, for some non-bipartite relation $\bowtie$.

Definition 4.14. Let $\bowtie \in \Lambda_{\mathcal{G}}^{f}$ be non-bipartite and $\mathcal{G}_{\bowtie}$ be the quotient network of $\bowtie$. Let $\tilde{\mathcal{C}}_{i,\bowtie} = \{[c]_{\bowtie} : c \in \tilde{\mathcal{C}}_i\}$, where $\tilde{\mathcal{C}}_i$ is given by the definition of $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$, for $i = 1, 2$. Define a multimap $\bar{f}$ by

$$\begin{aligned} \bar{f} : \tilde{\mathcal{C}}_{1,\bowtie} &\to P(\tilde{\mathcal{C}}_{2,\bowtie}) \\ [c]_{\bowtie} &\mapsto \{[d]_{\bowtie} : d \in f(c)\}, \end{aligned} \tag{4.20}$$

and call it the *induced multimap* by f on $\mathcal{G}_{\bowtie}$. $\diamond$

Note that the induced map $\bar{f}$ is well-defined. Indeed, since $\bowtie$ is non-bipartite, we have $[c]_{\bowtie} = [c]_{\bowtie_1}$ if $c \in \mathcal{C}_1$ and $[d]_{\bowtie} = [d]_{\bowtie_2}$ if $d \in \mathcal{C}_2$. Let $[c']_{\bowtie} = [c]_{\bowtie} \in \tilde{\mathcal{C}}_{1,\bowtie}$ for some $c' \neq c$. Then, $c, c' \in \mathcal{C}_1$ and $c \bowtie_1 c'$. Since $\bowtie$ is f-compatible, the restrictions $\bowtie_1, \bowtie_2$ on $\mathcal{G}_1, \mathcal{G}_2$ are f-related, that is,

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c')), \qquad \forall \beta \in \mathcal{C}_2.$$

It follows from $[\beta]_{\bowtie_2} = [\beta]_{\bowtie}$ for all $\beta \in \mathcal{C}_2$ that $\bar{f}([c']_{\bowtie}) = \bar{f}([c]_{\bowtie})$.

Lemma 4.15. *Let $\bowtie \in \Lambda_{\mathcal{G}}^{f}$ be non-bipartite and $\bar{f}$ be defined by (4.20) on the quotient network $\mathcal{G}_{\bowtie}$. Let $\Lambda_{\mathcal{G}}^{\bowtie}$ be given by (2.2) and $\bowtie \in \Lambda_{\mathcal{G}}^{\bowtie}$. If $\bowtie \in \Lambda_{\mathcal{G}}^{f}$, then $\bowtie_r \in \Lambda_{\mathcal{G}_{\bowtie}}^{\bar{f}}$.*

Proof. For convenience, write $\bar{c} = [c]_{\bowtie}$ for $\bowtie$-equivalence classes on $\mathcal{G}$. Then, for all $c \in \mathcal{C}_1$, the sets $\bar{f}(\bar{c})$ and $f(c)$ are isomorphic as multiset. Also, by definition of quotient network, we have

$$\mathfrak{m}([\bar{\beta}]_{\bowtie_{r,2}}, \bar{f}(\bar{c})) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c)), \qquad \forall c \in \mathcal{C}_1, \quad \beta \in \mathcal{C}_2. \tag{4.21}$$

Let $\bar{c}_1, \bar{c}_2 \in \mathcal{C}_{1,\bowtie}$ be such that $\bar{c}_1 \bowtie_r \bar{c}_2$. Then, $c_1 \bowtie_1 c_2$. Thus, since $\bowtie$ is f-compatible, we have

$$\mathfrak{m}([\beta]_{\bowtie_2}, f(c_1)) = \mathfrak{m}([\beta]_{\bowtie_2}, f(c_2)), \qquad \forall \beta \in \mathcal{C}_2.$$

It then follows from (4.21) that $\bowtie_r$ is $\bar{f}$-compatible. In analog, $\bowtie_r$ is also $\bar{f}^{-1}$-compatible. $\square$

Additionally, we need the concept of *f-symmetric* pairing-bipartite relations.

Definition 4.16. Let $\bowtie$ be a pairing-bipartite equivalence relation on $\mathcal{G}$. Let $\{c_i, d_i\}$ be nontrivial $\bowtie$-classes for $i = 1, \ldots, m$. We say that $\bowtie$ is *f-symmetric*, if

$$d_j \in f(c_i) \quad \Rightarrow \quad d_i \in f(c_j), \quad \forall i, j \in \{1, \ldots, m\}. \qquad \diamond$$

Theorem 4.17. *Let $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ be the f-join of two networks $\mathcal{G}_1, \mathcal{G}_2$ with the same edge type $\mathfrak{e}_1 = \mathfrak{e}_2$. Then, we have*

(1) $\bowtie \in \Lambda_{\mathcal{G}}$ is non-bipartite if and only if $\bowtie = \bowtie_1 \dot\vee \bowtie_2$ for some $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, $i = 1, 2$, where $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related;
(2) $\bowtie \in \Lambda_{\mathcal{G}}$ is pairing bipartite and f-symmetric if and only if $\bowtie = \bowtie_\sigma$ for some interior symmetry σ of $\mathcal{G}$, where σ is a product of disjoint transpositions $\tau_i = (c_i, d_i)$ for $c_i \in \mathcal{C}_1$, $d_i \in \mathcal{C}_2$;
(3) $\bowtie \in \Lambda^f_{\mathcal{G}}$ is non-pairing bipartite if and only if $\bowtie$ is the lifting of a pairing-bipartite equivalence relation $\bar\bowtie \in \Lambda^{\bar f}_{\mathcal{G}_{\mathcal{R}(\bowtie)}}$ on the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}$ induced by $\mathcal{R}(\bowtie)$, where $\mathcal{R}$ is defined by (4.11) and $\bar f$ is defined by (4.20).

Proof.

(1) Let $\bowtie \in \Lambda_{\mathcal{G}}$ be non-bipartite. By lemma 4.8, $\bowtie = \bowtie_1 \dot\vee \bowtie_2$, where $\bowtie_i$ is the restriction of $\bowtie$ on $\mathcal{G}_i$, for $i = 1, 2$. By lemma 4.11, $\bowtie_i$ is balanced on $\mathcal{G}_i$, for $i = 1, 2$. By lemma 4.12, $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related.
On the other hand, assume that $\bowtie = \bowtie_1 \dot\vee \bowtie_2$ for some $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, $i = 1, 2$ such that $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related. Then, $\bowtie$ is non-bipartite, by definition. Moreover, by lemma 4.13, $\bowtie$ is balanced.

(2) Let $\bowtie \in \Lambda_{\mathcal{G}}$ be pairing bipartite and f-symmetric. Let $\{c_i, d_i\}$ be nontrivial $\bowtie$-classes for $i = 1, \ldots, m$. For convenience, index the cells of $\mathcal{G}$ by $x_1, \ldots, x_n$ such that $c_i = x_{2i-1}$, $d_i = x_{2i}$ for $i = 1, \ldots, m$. Define $\mathcal{S} = \{x_1, x_2, \ldots, x_{2m-1}, x_{2m}\}$ and $\sigma = (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$. Then, $\bowtie = \bowtie_\sigma$. We show that σ is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$.
Let $A := (a_{ij})_{n\times n}$ be the adjacency matrix of $\mathcal{G}$. Then,

$$a_{ij} = \mathfrak{m}(x_j, I(x_i)), \qquad \forall\, 1 \leqslant i, j \leqslant n.$$

Consider $c_i = x_{2i-1}$, $d_i = x_{2i}$ and $c_j = x_{2j-1}$, $d_j = x_{2j}$ for some $i, j \in \{1, \ldots, m\}$. Then, we have

$$a_{2i-1,2j-1} + a_{2i-1,2j} = \mathfrak{m}(c_j, I(c_i)) + \mathfrak{m}(d_j, I(c_i)) = \mathfrak{m}([c_j]_{\bowtie}, I(c_i))$$

and

$$a_{2i,2j-1} + a_{2i,2j} = \mathfrak{m}(c_j, I(d_i)) + \mathfrak{m}(d_j, I(d_i)) = \mathfrak{m}([c_j]_{\bowtie}, I(d_i)).$$

Since $c_i \bowtie d_i$ and $\bowtie$ is balanced, we have

$$a_{2i-1,2j-1} + a_{2i-1,2j} = a_{2i,2j-1} + a_{2i,2j}. \tag{4.22}$$

Since $\bowtie$ is f-symmetric, we have $d_j \in f(c_i)$ if and only if $d_i \in f(c_j)$ for $i, j \in \{1, \ldots, m\}$ and thus

$$a_{2i-1,2j} = \mathfrak{m}(d_j, I(c_i)) = \mathfrak{m}(d_i, I(c_j)) = \mathfrak{m}(c_j, I(d_i)) = a_{2i,2j-1}.$$

Thus, it follows from (4.22) that

$$a_{2i-1,2j-1} = a_{2i,2j}.$$

Moreover, for $k > 2m$, we have

$$a_{2i-1,k} = \mathfrak{m}(x_k, I(c_i)) = \mathfrak{m}([x_k]_{\bowtie}, I(c_i)) = \mathfrak{m}([x_k]_{\bowtie}, I(d_i)) = \mathfrak{m}(x_k, I(d_i)) = a_{2i,k}.$$

Therefore, σ is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$.

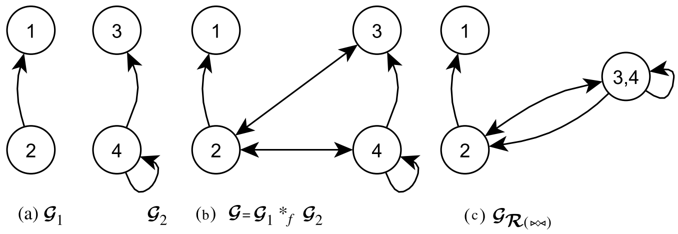


Figure 7. *(a)* $\mathcal{G}_1$ and $\mathcal{G}_2$; *(b)* $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$; *(c)* the quotient network of $\{\{1\}, \{2\}, \{3, 4\}\}$.

On the other hand, if $\bowtie = \bowtie_\sigma$ for an interior symmetry, then $\bowtie$ is balanced on $\mathcal{G}$. We show that $\bowtie$ is f-symmetric. Let d_j, c_i be such that $d_j \in f(c_i)$. Thus,

$$\mathfrak{m}(d_j, I(c_i)) = \mathfrak{m}(c_i, I(d_j)) = 1.$$

Since σ is an interior symmetry, we have

$$a_{2j-1,2i} = a_{2j,2i-1} \quad \Rightarrow \quad \mathfrak{m}(d_i, I(c_j)) = \mathfrak{m}(c_i, I(d_j)).$$

Therefore, $\mathfrak{m}(d_i, I(c_j)) = \mathfrak{m}(c_i, I(d_j)) = 1$, which implies that $d_i \in f(c_j)$.

(3) Let $\bowtie \in \Lambda^f_{\mathcal{G}}$ be non-pairing bipartite. Then, by lemma 4.13, $\mathcal{R}(\bowtie) = \bowtie_1 \dot{\vee} \bowtie_2$ is a balanced refinement of $\bowtie$. Write $\mathcal{R}(\bowtie) = \rhd\!\lhd$. Consider the quotient network $\mathcal{G}_{\rhd\lhd}$. By proposition 2.9, $\bowtie$ is the lifting of $\bar{\bowtie} = \bowtie_r$, the restriction of $\bowtie$ to $\mathcal{G}_{\rhd\lhd}$. Moreover, by lemma 4.15, $\bowtie_r \in \Lambda^{\bar{f}}_{\mathcal{G}_{\rhd\lhd}}$. On the other hand, let $\bowtie$ be the lifting of a balanced relation $\bar{\bowtie} \in \Lambda^{\bar{f}}_{\mathcal{G}_{\rhd\lhd}}$ on $\mathcal{G}_{\rhd\lhd}$. By proposition 2.9 and lemma 4.15, we have $\Lambda^f_{\mathcal{G}} \simeq \Lambda^{\bar{f}}_{\mathcal{G}_{\rhd\lhd}}$ are isomorphic as lattices. Thus, $\bowtie \in \Lambda^f_{\mathcal{G}}$. Since $\bar{\bowtie}$ is bipartite, $\bowtie$ is bipartite. Moreover, since $\mathcal{G}_{\rhd\lhd} \neq \mathcal{G}$, we have $\rhd\!\lhd$ is nontrivial. Thus, by lemma 4.8, $\bowtie$ is non-pairing bipartite. □

We illustrate the three cases in theorem 4.17 by the following example.

Example 4.18. Let $\mathcal{G}_1, \mathcal{G}_2$ be given in figure 7(*a*) with the same edge types $\mathfrak{e}_1 = \mathfrak{e}_2$. Let $\tilde{\mathcal{C}}_1 = \{2\}$ and $\tilde{\mathcal{C}}_2 = \{3, 4\}$. Define $f : \tilde{\mathcal{C}}_1 \to P(\tilde{\mathcal{C}}_2)$ by $f(2) = \{3, 4\}$. Then, $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ is as shown in figure 7(*b*). It can be verified that

$$\rhd\!\lhd = \{\{1\}, \{2\}, \{3, 4\}\}, \qquad \rhd\!\bullet\!\lhd = \{\{1\}, \{2, 3\}, \{4\}\}, \qquad \rhd\!\diamond\!\lhd = \{\{1\}, \{2, 3, 4\}\},$$

are balanced on $\mathcal{G}$, which are non-bipartite, pairing bipartite and non-pairing bipartite, respectively. One sees that $\rhd\!\lhd = \{\{1\}, \{2\}\} \dot{\vee} \{\{3, 4\}\} = \rhd\!\lhd_1 \dot{\vee} \rhd\!\lhd_2$ as indicated by the *case 1* of theorem 4.17; and $\rhd\!\bullet\!\lhd$ corresponds to the interior symmetry $\sigma = (2\ 3)$ of $\mathcal{G}$, which is the *case 2* of theorem 4.17. For $\rhd\!\diamond\!\lhd$, we have

$$\mathcal{R}(\rhd\!\diamond\!\lhd) = \{\{1\}, \{2\}, \{3, 4\}\},$$

which is a balanced refinement of $\rhd\!\diamond\!\lhd$. Let $\bar{1} = \{1\}$, $\bar{2} = \{2\}$, $\bar{3} = \{3, 4\}$. Then, the quotient network $\mathcal{G}_{\mathcal{R}(\rhd\diamond\lhd)}$ is given by figure 7(*c*). The multimap $\bar{f}$ on $\mathcal{G}_{\mathcal{R}(\rhd\diamond\lhd)}$ is defined by $\bar{f}(\bar{2}) = \bar{3}$. Clearly, $\rhd\!\bar{\diamond}\!\lhd = \{\{\bar{1}\}, \{\bar{2}, \bar{3}\}\}$ is $\bar{f}$-compatible and $\bar{f}^{-1}$-compatible. Moreover, it is a pairing-bipartite balanced equivalence relation on $\mathcal{G}_{\mathcal{R}(\rhd\diamond\lhd)}$ such that $\rhd\!\diamond\!\lhd$ is the lifting of $\rhd\!\bar{\diamond}\!\lhd$ to $\mathcal{G}$. This is *case 3* of theorem 4.17. ◇

Remark 4.19. We note that in example 4.18, the quotient network $\mathcal{G}_{\mathcal{R}(\rhd\diamond\lhd)}$ is not of form of an f-join of networks. In general, for $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$, if $\bowtie \in \Lambda^f_{\mathcal{G}}$ is a non-pairing-bipartite relation such that there exists a $\bowtie_i$-class $[x]_{\bowtie_i}$ with $\#([x]_{\bowtie_i} \cap \tilde{\mathcal{C}}_i) = k > 1$ for some $i \in \{1, 2\}$, then $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}_{1_{\bowtie_1}} *_{\bar{f}} \mathcal{G}_{2_{\bowtie_2}}$. This is true because in the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)}$ there will be k

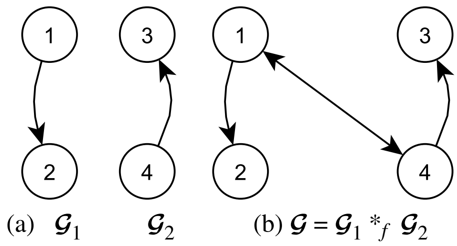


Figure 8. (*a*) $\mathcal{G}_1$ and $\mathcal{G}_2$; (*b*) $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$.

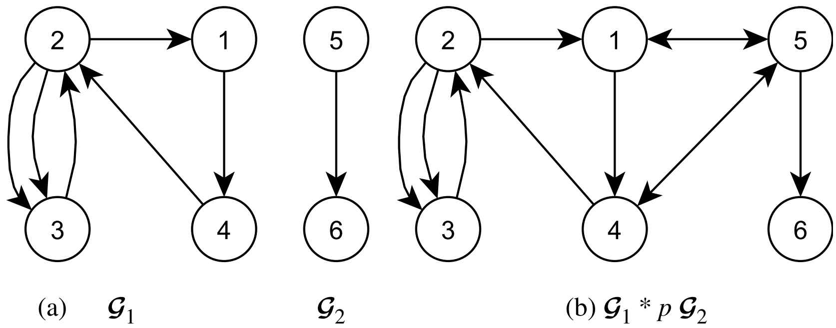


Figure 9. (*a*) $\mathcal{G}_1$ and $\mathcal{G}_2$; (*b*) $\mathcal{G}_1 *_p \mathcal{G}_2$.

edges from the cell $\bar{x} = [x]_{\bowtie_i}$ of the quotient network $\mathcal{G}_{i_{\bowtie_i}}$ to cell $\bar{y}$ with $\bar{y} \in f([x]_{\bowtie_i})$ of the quotient network $\mathcal{G}_{j_{\bowtie_j}}$, for $i, j \in \{1, 2\}$ with $i \neq j$. $\diamond$

The following example shows that the 'f-symmetric' requirement of *case 2* in theorem 4.17 is necessary.

Example 4.20. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be given by figure 8(*a*) for $\mathfrak{e}_1 = \mathfrak{e}_2$. Let $\tilde{\mathcal{C}}_1 = \{1\}, \tilde{\mathcal{C}}_2 = \{4\}$ and $f : \tilde{\mathcal{C}}_1 \to P(\tilde{\mathcal{C}}_2)$ be defined by $f(1) = \{4\}$. Then, $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ is as shown in figure 8(*b*). Consider two pairing-bipartite equivalence relations on $\mathcal{G}$ given by

$$\rhd\!\!\lhd = \{\{1, 4\}, \{2, 3\}\}, \qquad \rhd\!\!\bullet\!\!\lhd = \{\{1, 3\}, \{2, 4\}\},$$

which are both balanced. However, $\rhd\!\!\lhd$ is f-symmetric, while $\rhd\!\!\bullet\!\!\lhd$ is not f-symmetric. It can be directly verified that (1 4)(2 3) is an interior symmetry of $\mathcal{G}$, while (1 3)(2 4) is not an interior symmetry of $\mathcal{G}$, since $a_{12} = 0 \neq 1 = a_{34}$, for $A = [a_{ij}]_{4\times 4}$ being the adjacency matrix of $\mathcal{G}$. $\diamond$

We close this subsection with an example of non-pairing bipartite $\bowtie \in \Lambda_{\mathcal{G}} \setminus \Lambda_{\mathcal{G}}^f$ which is 'irreducible'; that is, it does not admit any nontrivial balanced refinement on $\mathcal{G}$.

Example 4.21. Let $\mathcal{G} = \mathcal{G}_1 *_p \mathcal{G}_2$ be the partial join of $\mathcal{G}_1$ and $\mathcal{G}_2$ given in figure 9, for $\tilde{\mathcal{C}}_1 = \{1, 4\}$ and $\tilde{\mathcal{C}}_2 = \{5\}$, where the edge types $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are considered to be the same. Then, $\bowtie = \{\{1, 2, 4\}, \{3, 5\}, \{6\}\}$ is balanced on $\mathcal{G}$. Since $\{1, 2, 4\} \not\subset \tilde{\mathcal{C}}_1$ nor $\{1, 2, 4\} \not\subset \mathcal{C} \setminus \tilde{\mathcal{C}}_1$, it follows from (ii) in remark 4.10 that $\bowtie \notin \Lambda_{\mathcal{G}}^f$.

Assume that $\rhd\!\!\lhd$ is a nontrivial balanced refinement of $\bowtie$. Suppose that $1 \rhd\!\!\lhd 2$. Then, there exists a bijection between $I(1) = \{2, 5\}$ and $I(2) = \{3, 4\}$ that preserves $\rhd\!\!\lhd$. Since 2 and 3 are not $\bowtie$-equivalenct, they cannot be $\rhd\!\circ\!\lhd$-equivalent. Thus, we have $2 \rhd\!\circ\!\lhd 4$ and $3 \rhd\!\circ\!\lhd 5$. A similar analysis leads to the following implication relations

$$
\begin{aligned}
1 \rhd\!\circ\!\lhd 2 \quad &\Rightarrow \quad 2 \rhd\!\circ\!\lhd 4 \ \wedge \ 3 \rhd\!\circ\!\lhd 5, \\
2 \rhd\!\circ\!\lhd 4 \quad &\Rightarrow \quad 1 \rhd\!\circ\!\lhd 4 \ \wedge \ 3 \rhd\!\circ\!\lhd 5, \\
1 \rhd\!\circ\!\lhd 4 \quad &\Rightarrow \quad 1 \rhd\!\circ\!\lhd 2 \quad \Rightarrow \quad 2 \rhd\!\circ\!\lhd 4 \ \wedge \ 3 \rhd\!\circ\!\lhd 5, \\
3 \rhd\!\circ\!\lhd 5 \quad &\Rightarrow \quad 1 \rhd\!\circ\!\lhd 2 \rhd\!\circ\!\lhd 4.
\end{aligned}
$$

Thus, any nontrivial balanced refinement $\rhd\!\circ\!\lhd$ of $\bowtie$ is in fact equal to $\bowtie$. Therefore, $\bowtie$ cannot be 'recovered' using balanced equivalence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$. $\diamond$

4.1.2. Synchrony for $\mathcal{G}_1 *_f \mathcal{G}_2$ with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$. In the case of different edge types, we can obtain a complete classification result for $\Lambda_{\mathcal{G}}$, under much simpler conditions. The reason is that the different edge types largely confine the possibility of $\mathcal{G}$ supporting bipartite balanced relations. Since balanced equivalent cells are necessarily input equivalent and two cells having different input edges cannot be input equivalent, the only possibility for $a \in \mathcal{C}_1$ and $b \in \mathcal{C}_2$ to be equivalent is that they are sources in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively; that is, $\#I_1(a) = \#I_2(b) = 0$. In contrast, in case of the same edge type, this condition is weakened to $\#I_1(a) = \#I_2(b)$.

Theorem 4.22. *Let $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ be the f-join of two networks $\mathcal{G}_1, \mathcal{G}_2$ with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Then, we have*

(1) $\bowtie \in \Lambda_{\mathcal{G}}$ is non-bipartite if and only if $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$ for some $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, $i = 1, 2$, where $\bowtie_1$ and $\bowtie_2$ are f- and f^{-1}-related;

(2) $\bowtie \in \Lambda_{\mathcal{G}}$ is pairing bipartite if and only if $\bowtie = \bowtie_\sigma$ for some interior symmetry σ of $\mathcal{G}$, where σ is a product of disjoint transpositions $\tau_i = (c_i, d_i)$ for $c_i \in \mathcal{C}_1$, $d_i \in \mathcal{C}_2$;

(3) $\bowtie \in \Lambda_{\mathcal{G}}$ is non-pairing bipartite if and only if $\bowtie$ is the lifting of a pairing-bipartite equivalence relation $\tilde{\bowtie} \in \Lambda_{\mathcal{G}_{\mathcal{R}(\bowtie)}}$ on the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}$ induced by $\mathcal{R}(\bowtie)$, where $\mathcal{R}$ is defined by (4.11).

Proof. The proof is analogous to that of theorem 4.17. $\square$

Remark 4.23.

(i) If $\mathcal{G} = \mathcal{G}_1 * \mathcal{G}_2$ is the join of $\mathcal{G}_1$ and $\mathcal{G}_2$ with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$, then every balanced equivalence relation $\bowtie \in \Lambda_{\mathcal{G}}$ is non-bipartite. Indeed, assume otherwise that $a \bowtie b$ for some $a \in \mathcal{C}_1$ and $b \in \mathcal{C}_2$. Then, a (respectively b) is a source of $\mathcal{G}_1$ (respectively $\mathcal{G}_2$, which implies that $I(a) = I^{\mathfrak{e}_f}(a) = \mathcal{C}_2$ and $I(b) = I^{\mathfrak{e}_f}(b) = \mathcal{C}_1$. Since $\bowtie$ is balanced, there exists a bijection $\beta : I(a) = \mathcal{C}_2 \to I(b) = \mathcal{C}_1$ such that $d \bowtie \beta(d) := c$ for all $d \in \mathcal{C}_2$. Every cell $c \in \mathcal{C}_1$ and $d \in \mathcal{C}_2$ are sources of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Consequently, $\mathcal{G}_1$ and $\mathcal{G}_2$ consist of isolated cells without edges, a contradiction to the definition of coupled cell networks.

(ii) If $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ for two regular networks $\mathcal{G}_1$ and $\mathcal{G}_2$ with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$, then every balanced equivalence relation $\bowtie \in \Lambda_{\mathcal{G}}$ is non-bipartite. This follows from the same argument used in (i). $\diamond$

Remark 4.24. Recall that in remark 3.2, we listed all possible combinations (E1)–(E4) of edge types in $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$. The main result of balanced equivalence relations on $\mathcal{G}$ in case of (E1) and (E2) is stated in theorem 4.17 and theorem 4.22, respectively. Note that the proof is based on properties of input sets of individual cells. In case of (E3), it shares with (E1) a similar character of input sets given by

$$I(x) = I^{\mathfrak{e}_i}(x) = I^{\mathfrak{e}_f}(x) = I_1(x) \cup f(x), \qquad \forall\, x \in \mathcal{C}_i,$$

where (E3) refers to the case $\mathfrak{e}_i = \mathfrak{e}_f \neq \mathfrak{e}_j$, for $i, j \in \{1, 2\}$ and $i \neq j$. In case of (E4), it shares with (E2) a common property of input sets given by

$$\begin{cases} I(x) = I^{\mathfrak{e}_1}(x) \cup I^{\mathfrak{e}_f}(x) = I_1(x) \cup f(x), & \forall\, x \in \mathcal{C}_1 \\ I(y) = I^{\mathfrak{e}_2}(y) \cup I^{\mathfrak{e}_f}(y) = I_2(y) \cup f^{-1}(y), & \forall\, y \in \mathcal{C}_2. \end{cases}$$

Therefore, the main result for the case (E1) given by theorem 4.17 also holds for (E3), while the main result for the case (E2) given by theorem 4.22 remains valid for (E4). ◇

4.2. Synchrony for coalescence of networks

Throughout this subsection, $\mathcal{G}$ stands for the coalescence of $\mathcal{G}_1$ and $\mathcal{G}_2$ and θ denotes the common cell. We give a characterization of balanced equivalence relations on $\mathcal{G}$ using balanced equivalence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$, together with interior symmetry of $\mathcal{G}$. Analogously to the f-join, we treat the cases of same edge type and different edge types separately (see theorem 4.30 and theorem 4.32 for the main result).

Definition 4.25. An equivalence relation $\bowtie$ on $\mathcal{G}$ is called *θ-compatible*, if $I_i(\theta) = \emptyset$ whenever there exists $c \in \mathcal{C}_j \setminus \{\theta\}$ such that $c \bowtie \theta$, for $i, j \in \{1, 2\}$ and $i \neq j$. ◇

Remark 4.26.

(i) If θ is a source for both $\mathcal{G}_1$ and $\mathcal{G}_2$, then every equivalence relation is θ-compatible. If θ is a source only for $\mathcal{G}_i$, then the θ-compatible equivalence relations are precisely those such that $[\theta]_{\bowtie} \subset \mathcal{C}_j$, for $i, j \in \{1, 2\}$ and $i \neq j$. On the other hand, equivalence relations $\bowtie$ such that $[\theta]_{\bowtie} = \{\theta\}$ are always θ-compatible.

(ii) If $\mathfrak{e}_1 \neq \mathfrak{e}_2$ then every balanced equivalence relation in $\Lambda_{\mathcal{G}}$ is θ-compatible. In fact, if there exists $a \in \mathcal{C}_i \setminus \{\theta\}$ such that $a \bowtie \theta$, then since $\bowtie$ is balanced and $I^{\mathfrak{e}_j}(a) = \emptyset$, we have $I^{\mathfrak{e}_j}(\theta) = \emptyset$, for $i, j \in \{1, 2\}$ and $i \neq j$. ◇

Lemma 4.27. *Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ with the common cell θ and $\mathfrak{e}_i$ be the edge type of $\mathcal{G}_i$ for $i = 1, 2$. Let $\bowtie \in \Lambda_{\mathcal{G}}$ and $\bowtie_i$ be the restrictions of $\bowtie$ on $\mathcal{G}_i$, for $i = 1, 2$. Then, $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, if $\bowtie$ is θ-compatible.*

Proof. We present only the proof for $\bowtie_1$, since the proof for $\bowtie_2$ is analogous. Consider $x, y \in \mathcal{C}_1$ such that $x \bowtie_1 y$. Then, $x \bowtie y$. Since $\bowtie$ is balanced, we have

$$\mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}_1}(x)) = \mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}_1}(y)), \qquad \forall\, \alpha \in \mathcal{C}_1. \tag{4.23}$$

In the case $\mathfrak{e}_1 \neq \mathfrak{e}_2$, we have $[\alpha]_{\bowtie} \cap I^{\mathfrak{e}_1}(c) \subset [\alpha]_{\bowtie_1}$ and $I_1(c) = I^{\mathfrak{e}_1}(c)$, for all $c \in \mathcal{C}_1$. Thus,

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)) = \mathfrak{m}([\alpha]_{\bowtie}, I^{\mathfrak{e}_1}(c)), \qquad \forall\, \alpha,\ c \in \mathcal{C}_1.$$

It follows from (4.23) that

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(y)), \qquad \forall\, \alpha \in \mathcal{C}_1.$$

That is, $\bowtie_1$ is balanced.

In the case $\mathfrak{e}_1 = \mathfrak{e}_2$, we have $[\alpha]_{\bowtie} \cap I(c) \subset [\alpha]_{\bowtie_1}$ and $I_1(c) = I(c)$, for $c \in \mathcal{C}_1 \setminus \{\theta\}$. Thus,

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)) = \mathfrak{m}([\alpha]_{\bowtie}, I(c)), \qquad \forall \alpha \in \mathcal{C}_1,\ c \in \mathcal{C}_1 \setminus \{\theta\}. \tag{4.24}$$

It follows from (4.23) that for $x \neq \theta$ and $y \neq \theta$, we have

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(y)), \qquad \forall \alpha \in \mathcal{C}_1.$$

If $[\theta]_{\bowtie} \cap \mathcal{C}_1 = \{\theta\}$, then this implies that $\bowtie_1$ is balanced. Otherwise, assume $x \bowtie \theta$ for some $x \in \mathcal{C}_1$. Then, since $\bowtie$ is assumed to be θ-compatible, we have $I_2(\theta) = \emptyset$. Thus, $I(\theta) = I_1(\theta)$ and consequently,

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(\theta)) = \mathfrak{m}([\alpha]_{\bowtie}, I(\theta)), \qquad \forall \alpha \in \mathcal{C}_1.$$

It follows then from (4.23) and (4.24) that

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(\theta)), \qquad \forall \alpha \in \mathcal{C}_1.$$

Therefore, $\bowtie_1$ is balanced. □

Lemma 4.28. *Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ with the common cell θ and $\mathfrak{e}_i$ be the edge type of $\mathcal{G}_i$ for $i = 1, 2$. Let $\bowtie_i \in \Lambda_{\mathcal{G}_i}$ be a balanced equivalence relation for $i = 1, 2$. If the join extension $\bowtie_1 \dot\vee \bowtie_2$ is θ-compatible then it is a balanced relation in $\Lambda_{\mathcal{G}}$.*

Proof. Denote by $\bowtie_{1,2} := \bowtie_1 \dot\vee \bowtie_2$. Let $x, y \in \mathcal{C}$ be such that $x \bowtie_{1,2} y$. Then, either $x, y \in \mathcal{C}_1$ with $x \bowtie_1 y$ or $x, y \in \mathcal{C}_2$ with $x \bowtie_2 y$. Without loss of generality, we assume that $x, y \in \mathcal{C}_1$ and $x \bowtie_1 y$. Since $\bowtie_1$ is balanced, we have

$$\mathfrak{m}([\alpha]_{\bowtie_1}, I_1(x)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(y)), \qquad \forall \alpha \in \mathcal{C}_1. \tag{4.25}$$

(i) Assume $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Note that $[\alpha]_{\bowtie_{1,2}} \cap I^{\mathfrak{e}_1}(c) \subset [\alpha]_{\bowtie_1}$ and $I^{\mathfrak{e}_1}(c) = I_1(c)$ for all $c \in \mathcal{C}_1$. Thus, we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(c)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)), \qquad \forall \alpha \in \mathcal{C}_1,\ c \in \mathcal{C}_1. \tag{4.26}$$

It follows from (4.25) and (4.26) that

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(y)), \qquad \forall \alpha \in \mathcal{C}_1. \tag{4.27}$$

We note that, for all $c \in \mathcal{C}_1$ we have $I^{\mathfrak{e}_1}(c) \cap \mathcal{C}_2 \setminus \{\theta\} = \emptyset$. Thus, for $\alpha \in \mathcal{C}_2 \setminus \{\theta\}$ such that $\theta \notin [\alpha]_{\bowtie_{1,2}}$ the equality in (4.27) holds as both multiplicities are zero. If $\theta \in [\alpha]_{\bowtie_{1,2}}$, we have $[\theta]_{\bowtie_{1,2}} = [\theta]_{\bowtie_1} \cup [\theta]_{\bowtie_2}$, thus $\mathfrak{m}([\theta]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(c)) = \mathfrak{m}([\theta]_{\bowtie_1}, I_1(c)) + 0$ and so (4.27) holds.

Thus, in summary, we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_1}(y)), \qquad \forall \alpha \in \mathcal{C}.$$

If $x \neq \theta$ and $y \neq \theta$, then $I^{\mathfrak{e}_2}(x) = I^{\mathfrak{e}_2}(y) = \emptyset$, which implies that

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_2}(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I^{\mathfrak{e}_2}(y)) = 0, \qquad \forall \alpha \in \mathcal{C}. \tag{4.28}$$

If $x \neq \theta$ and $y = \theta$, then since $\bowtie_{1,2}$ is θ-compatible, we have $I^{\mathfrak{e}_2}(\theta) = \emptyset$. Thus, (4.28) holds again. The case of $x = \theta$ and $y \neq \theta$ is parallel. Therefore, $\bowtie_{1,2}$ is balanced.

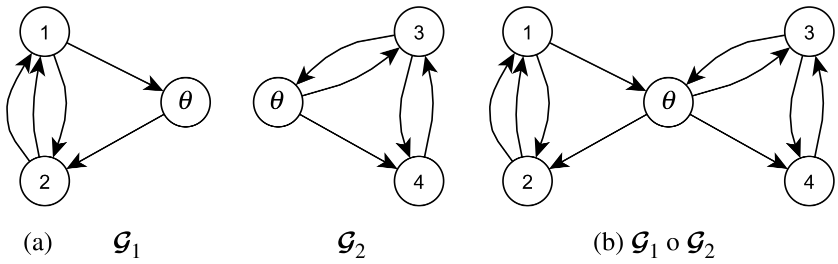


Figure 10. (*a*) Two networks $\mathcal{G}_1$ and $\mathcal{G}_2$; (*b*) $\mathcal{G}_1 \circ \mathcal{G}_2$.

(ii) Assume that $\mathfrak{e}_1 = \mathfrak{e}_2$. Then, $[\alpha]_{\bowtie_{1,2}} \cap I^{\mathfrak{e}_1}(c) \subset [\alpha]_{\bowtie_1}$ for all $c \in \mathcal{C}_1$ and $I(c) = I_1(c)$, for all $c \in \mathcal{C}_1 \setminus \{\theta\}$. Thus, we have

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(c)) = \mathfrak{m}([\alpha]_{\bowtie_1}, I_1(c)), \qquad \forall\, \alpha \in \mathcal{C}_1,\ c \in \mathcal{C}_1 \setminus \{\theta\}.$$

If $x \neq \theta$ and $y \neq \theta$, then it follows from (4.25) that

$$\mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(x)) = \mathfrak{m}([\alpha]_{\bowtie_{1,2}}, I(y)), \qquad \forall\, \alpha \in \mathcal{C}_1. \tag{4.29}$$

Note that (4.29) also holds for $\alpha \in \mathcal{C}_2 \setminus \{\theta\}$ with $\alpha \in [\theta]_{\bowtie_{1,2}}$. Further, if $\alpha \in \mathcal{C}_2 \setminus \{\theta\}$ is such that $\alpha \notin [\theta]_{\bowtie_{1,2}}$, then $[\alpha]_{\bowtie_{1,2}} \cap I^{\mathfrak{e}_1}(c) = \emptyset$, $\forall c \in \mathcal{C}_1 \setminus \{\theta\}$. Thus, (4.29) holds for all $\alpha \in \mathcal{C}$.

If $x \neq \theta$ and $y = \theta$, then since $\bowtie_{1,2}$ is θ-compatible, we have $I(\theta) = I_1(\theta)$. Thus, the above analysis applies and (4.29) holds in the case $y = \theta$, for all $\alpha \in \mathcal{C}$. The case of $x = \theta$ and $y \neq \theta$ is parallel. Therefore, $\bowtie_{1,2}$ is balanced. $\square$

The following example shows the 'θ-compatibility' necessary for the statement of lemma 4.27 and lemma 4.28.

Example 4.29.

(i) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be given in figure 10(*a*) such that $\mathfrak{e}_1 = \mathfrak{e}_2$. Let θ be the common cell. Then, the coalescence of $\mathcal{G}_1$ and $\mathcal{G}_2$ is given by figure 10(*b*). Consider the balanced equivalence relation $\bowtie$ on $\mathcal{G}$ given by

$$\bowtie = \{\{1, 4\}, \{2, \theta, 3\}\},$$

which is not θ-compatible. It can be verified that the restrictions of $\bowtie$

$$\bowtie_1 = \{\{1\}, \{2, \theta\}\}, \qquad \bowtie_2 = \{\{4\}, \{\theta, 3\}\}$$

are both non-balanced.

(ii) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be given in figure 11(a) such that $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Let θ be the common cell. Then, the coalescence of $\mathcal{G}_1$ and $\mathcal{G}_2$ is given by figure 11(*b*). Consider the equivalence relations

$$\bowtie_1 = \{\{1\}, \{2, \theta\}, \qquad \bowtie_2 = \{\{\theta, 3\}, \{4\}\},$$

which are balanced respectively on $\mathcal{G}_1$ and $\mathcal{G}_2$. But $\bowtie_1 \dot{\vee} \bowtie_2 = \{\{1\}, \{2, \theta, 3\}, \{4\}\}$ is not θ-compatible and not balanced. $\diamond$

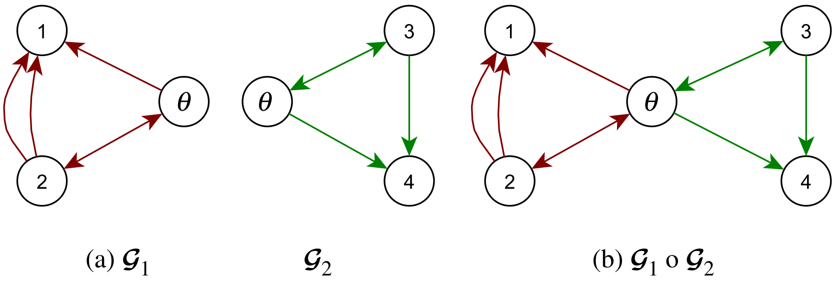


Figure 11. (*a*) Two networks $\mathcal{G}_1$ and $\mathcal{G}_2$; (*b*) $\mathcal{G}_1 \circ \mathcal{G}_2$.

4.2.1. Synchrony for $\mathcal{G}_1 \circ \mathcal{G}_2$ with the same edge type $\mathfrak{e}_1 = \mathfrak{e}_2$. For convenience, we denote $\Lambda_{\mathcal{G}}^{\theta} = \{\bowtie \in \Lambda_{\mathcal{G}} : \bowtie \text{ is } \theta\text{-compatible}\}$.

Theorem 4.30. *Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ be the coalescence of two networks $\mathcal{G}_1, \mathcal{G}_2$ with the common cell θ, where the edge types $\mathfrak{e}_1, \mathfrak{e}_2$ are the same. Then, we have*

(1) $\bowtie \in \Lambda_{\mathcal{G}}^{\theta}$ is non-bipartite if and only if $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$ for some $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, $i = 1, 2$ and $\bowtie$ is θ-compatible;

(2) $\bowtie \in \Lambda_{\mathcal{G}}$ is pairing bipartite if and only if $\bowtie = \bowtie_\sigma$ for some interior symmetry σ of $\mathcal{G}$, where σ is a product of disjoint transpositions $\tau_i = (c_i, d_i)$ for $c_i \in \mathcal{C}_1 \setminus \{\theta\}$, $d_i \in \mathcal{C}_2 \setminus \{\theta\}$;

(3) $\bowtie \in \Lambda_{\mathcal{G}}^{\theta}$ is non-pairing bipartite if and only if $\bowtie$ is the lifting of a pairing-bipartite equivalence relation $\bar{\bowtie} \in \Lambda_{\mathcal{G}_{\mathcal{R}(\bowtie)}}^{\bar{\theta}}$ on the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}$, where $\bar{\theta}$ denotes the representative of θ in the quotient network $\Lambda_{\mathcal{G}_{\mathcal{R}(\bowtie)}}$.

Proof. The proof essentially resembles the proof of theorem 4.17.

(1) Let $\bowtie \in \Lambda_{\mathcal{G}}$ be non-bipartite. By lemma 4.8(i), $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$. Also, by lemma 4.27, $\bowtie_i$ is balanced for $i = 1, 2$. On the other hand, if $\bowtie_i \in \Lambda_{\mathcal{G}_i}$ for $i = 1, 2$ and $\bowtie$ is θ-compatible, then by lemma 4.28, $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$ is balanced. Also, by lemma 4.8(i), $\bowtie$ is non-bipartite.

(2) Let $\bowtie \in \Lambda_{\mathcal{G}}$ be pairing bipartite and $[\alpha_i]_{\bowtie}$ be nontrivial classes for $i = 1, \ldots, m$. Then, $[\alpha_i]_{\bowtie} = \{c_i, d_i\}$, for some $c_i \in \mathcal{C}_1 \setminus \{\theta\}$, $d_i \in \mathcal{C}_2 \setminus \{\theta\}$. Note that there are no edges between c_i and d_j for all $i, j \in \{1, \ldots, m\}$. We index the cells of $\mathcal{G}$ by $x_1, \ldots, x_n$ so that $c_i = x_{2i-1}$, $d_i = x_{2i}$ for $i = 1, \ldots, m$. Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix. For all $i, j \in \{1, \ldots, m\}$, we have $a_{2i-1,2j} = a_{2i,2j-1} = 0$ and

$$a_{2i-1,2j-1} = \mathfrak{m}(c_j, I(c_i)) = \mathfrak{m}([c_j]_{\bowtie}, I(c_i)) = \mathfrak{m}([d_j]_{\bowtie}, I(d_i)) = \mathfrak{m}(d_j, I(d_i)) = a_{2i,2j}.$$

For $x > 2m$, we have

$$a_{2i-1,x} = \mathfrak{m}(x, I(c_i)) = \mathfrak{m}([x]_{\bowtie}, I(c_i)) = \mathfrak{m}([x]_{\bowtie}, I(d_i)) = \mathfrak{m}(x, I(d_i)) = a_{2i,x}.$$

Therefore, $\sigma = (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$ is an interior symmetry on $\mathcal{S} = \{c_1, d_1, \ldots, c_m, d_m\}$ and $\bowtie = \bowtie_\sigma$.

(3) Let $\bowtie \in \Lambda_{\mathcal{G}}^{\theta}$ be non-pairing bipartite. Since $\bowtie$ is θ-compatible, by lemma 4.27 and lemma 4.28, we have $\mathcal{R}(\bowtie)$ is balanced. Thus, by proposition 2.9, $\bowtie$ is a lifting of its restriction $\bowtie_r$ on the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)}$. Also, $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}$, since $\mathcal{R}(\bowtie)$ is nontrivial, by lemma 4.8(ii). Moreover, since $[\theta]_{\bowtie} = [\theta]_{\mathcal{R}(\bowtie)} = \bar{\theta}$, we have $\bowtie_r$ is $\bar{\theta}$-compatible and pairing bipartite. □

Analogously to the case of f-join, the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)}$ in *case 3* of theorem 4.30 may not be a coalescence of networks.

Remark 4.31. If $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ for two regular networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of valency v_1 and v_2, respectively, then $\Lambda^{\theta}_{\mathcal{G}} = \Lambda_{\mathcal{G}}$. Indeed, since $\#I(\theta) = v1 + v2$ and $\#I(a) = v_1$, $\#I(b) = v_2$ for all $a \in \mathcal{C}_1 \setminus \{\theta\}$ and $b \in \mathcal{C}_2 \setminus \{\theta\}$, the cell θ is not input equivalent with any other cell in $\mathcal{G}$. Thus, $[\theta]_{\bowtie} = \{\theta\}$ and every balanced relation on $\mathcal{G}$ is θ-compatible (see remark 4.26(i)). $\diamond$

4.2.2. Synchrony for $\mathcal{G}_1 \circ \mathcal{G}_2$ with different edge types $\mathfrak{e}_1 \neq \mathfrak{e}_2$. In the case of different edge types, all balanced equivalence relations can be classified, with simpler conditions. Similar to the f-join, the reason is that a bipartite balanced relation can only be supported by sources of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem 4.32. *Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ be the coalescence of two networks $\mathcal{G}_1, \mathcal{G}_2$ with the common cell θ, where the edge types $\mathfrak{e}_1, \mathfrak{e}_2$ are different. Then, we have*

(1) $\bowtie \in \Lambda_{\mathcal{G}}$ is non-bipartite if and only if $\bowtie = \bowtie_1 \dot{\vee} \bowtie_2$ for some $\bowtie_i \in \Lambda_{\mathcal{G}_i}$, $i = 1, 2$ and $\bowtie$ is θ-compatible;
(2) $\bowtie \in \Lambda_{\mathcal{G}}$ is pairing bipartite if and only if $\bowtie = \bowtie_\sigma$ for some interior symmetry σ of $\mathcal{G}$, where σ is a product of disjoint transpositions $\tau_i = (c_i, d_i)$ for $c_i \in \mathcal{C}_1 \setminus \{\theta\}$, $d_i \in \mathcal{C}_2 \setminus \{\theta\}$;
(3) $\bowtie \in \Lambda_{\mathcal{G}}$ is non-pairing bipartite if and only if $\bowtie$ is the lifting of a pairing bipartite equivalence relation $\bar{\bowtie} \in \Lambda_{\mathcal{G}_{\mathcal{R}(\bowtie)}}$ on the quotient network $\mathcal{G}_{\mathcal{R}(\bowtie)} \neq \mathcal{G}$.

Proof. The proof is analogous to that of theorem 4.30. $\square$

4.3. Evolutionary fitness of synchrony types

In this subsection, we give an example to show how a requirement of evolution of the synchrony can be realized by a specific change of the network structure. Consider two networks $\mathcal{G}_1$ and $\mathcal{G}_2$, whose lattices of balanced equivalence relations are Λ_1 and Λ_2, respectively. Depending on whether Λ_i 'survives' in $\mathcal{G} = \mathcal{G}_1 *_f \mathcal{G}_2$ and whether $\mathcal{G}$ supports a 'novel' synchrony, we give several definitions of evolutionary states of synchrony.

In what follows, Λ denotes the lattice of balanced equivalence relations on $\mathcal{G}$ and $i, j \in \{1, 2\}$ with $i \neq j$.

For $\bowtie \in \Lambda_i$, denote by $\check{\bowtie}$ the join extension of $\bowtie$ with the trivial equivalence relation on $\mathcal{G}_j$. We say that Λ_i *survives to* Λ, denoted by $\Lambda_i \subseteq \Lambda$, if $\check{\bowtie} \in \Lambda$ for all $\bowtie \in \Lambda_i$. Otherwise, we say that Λ_i is *suppressed in* Λ, denoted by $\Lambda_i \not\subset \Lambda$. Note that Λ_1 survives to Λ if and only if

$$f(c_1) = f(c_2), \qquad \forall c_1, c_2 \in \mathcal{C}_1 \quad \text{with} \quad c_1 \bowtie c_2,\ \bowtie \in \Lambda_1. \tag{4.30}$$

The same holds for Λ_2, with f replaced by f^{-1}.

Also, denote by

$$\Lambda_b := \{\bowtie \in \Lambda\ : \bowtie \text{ is bipartite}\},$$

which contains essentially all the 'novel' balanced equivalence relations on $\mathcal{G}$ that can not arise if without communications between $\mathcal{G}_1$ and $\mathcal{G}_2$ through f.

Definition 4.33. Let Λ_i be the lattice of balanced equivalence relations on $\mathcal{G}_i$, for $i = 1, 2$. We say that Λ_1 and Λ_2 *coexist* in $\mathcal{G}$, if $\Lambda_1 \subset \Lambda$, $\Lambda_2 \subset \Lambda$ and $\Lambda_b = \emptyset$; they *cooperate* in $\mathcal{G}$, if $\Lambda_1 \subset \Lambda$, $\Lambda_2 \subset \Lambda$ and $\Lambda_b \neq \emptyset$; they *coevolve*, if $\Lambda_1 \not\subset \Lambda$, $\Lambda_2 \not\subset \Lambda$ and $\Lambda_b \neq \emptyset$; and they *extinct* if $\Lambda_1 \not\subset \Lambda$, $\Lambda_2 \not\subset \Lambda$ and $\Lambda_b = \emptyset$. We say that the synchrony patterns in $\mathcal{G}_i$ *evolve* in $\mathcal{G}$, if $\Lambda_i \subset \Lambda$, $\Lambda_j \not\subset \Lambda$ and $\Lambda_b \neq \emptyset$; and the synchrony patterns in $\mathcal{G}_j$ is *eliminated* in $\mathcal{G}$, if $\Lambda_i \subset \Lambda$, $\Lambda_j \not\subset \Lambda$ and $\Lambda_b = \emptyset$. For a systematic overview, we summarize the conditions in table 1.

Table 1. Definitions of evolutionary states of synchrony patterns, for $i, j \in \{1, 2\}, i \neq j$.

Definition	Λ_i	Λ_j	Λ_b
Coexistence	$\subset \Lambda$	$\subset \Lambda$	$\emptyset$
Cooperation	$\subset \Lambda$	$\subset \Lambda$	$\neq \emptyset$
Coevolution	$\not\subset \Lambda$	$\not\subset \Lambda$	$\neq \emptyset$
Extinction	$\not\subset \Lambda$	$\not\subset \Lambda$	$\emptyset$
Evolution	$\subset \Lambda$	$\not\subset \Lambda$	$\neq \emptyset$
Elimination	$\subset \Lambda$	$\not\subset \Lambda$	$\emptyset$

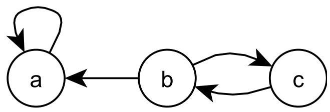


Figure 12. The structure of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Example 4.34. Consider two isomorphic coupled cell networks $\mathcal{G}_1$ and $\mathcal{G}_2$, whose structure is shown in figure 12. Let $\mathcal{G}_1$ be the network given by $a = 1$, $b = 2$, $c = 3$ and $\mathcal{G}_2$ be given by $a = 4$, $b = 5$ and $c = 6$. Then, f-join for different choice of f can realize different evolutionary states of synchrony patterns, namely, the coexistence, cooperation, coevolution, extinction, evolution and elimination (see figure 13).

Indeed, we have

$$\Lambda_1 = \big\{\{\{1\}, \{2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}\big\}, \qquad \Lambda_2 = \big\{\{\{4\}, \{5, 6\}\}, \{\{4\}, \{5\}, \{6\}\}\big\}.$$

Thus, by (4.30), $\Lambda_1 \subset \Lambda$, if $f(2) = f(3)$, which is a condition satisfied by (a), (b), (e) and (f) in figure 13. Similarly, $\Lambda_2 \subset \Lambda$ is satisfied by figures 13(a) and (b), since $f^{-1}(5) = f^{-1}(6)$ holds. To verify the condition related to Λ_b, we note that $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ is a bipartite balanced relation in the case of (b) and (c). In the case of (e), $\{\{1\}, \{2, 3, 4\}, \{5\}, \{6\}\}$ is a bipartite balanced relation. For (a), (d) and (f), it can be directly verified that $\Lambda_b = \emptyset$. $\diamond$

4.4. Reconstruction of $\Lambda_{\mathcal{G}}$ from $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$

Based on the classification results obtained in sections 4.1 and 4.2, we give examples to show how the lattice of balanced equivalence relations on $\mathcal{G}$ can be reconstructed from the lattices of balanced equivalence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\mathcal{G}$ is the join or a coalescence of $\mathcal{G}_1$ and $\mathcal{G}_2$.

4.4.1. Join. Let $\mathcal{G} = \mathcal{G}_1 * \mathcal{G}_2$ be the join of $\mathcal{G}_1$ and $\mathcal{G}_2$ for $\mathfrak{e}_1 = \mathfrak{e}_2$. Denote by $\Lambda_{\mathcal{G}}$ the lattice of balanced equivalence relations on $\mathcal{G}$. Then,

$$\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{G}}^{nb} \cup \Lambda_{\mathcal{G}}^{pb} \cup \Lambda_{\mathcal{G}}^{npb},$$

where the sets on the right hand side stand for the subsets of $\Lambda_{\mathcal{G}}$ composed of non-bipartite, pairing-bipartite and non-pairing bipartite relations, respectively. Since the conditions of f- and f^{-1}-relatedness, as well as the f-symmetry hold automatically for the join of networks, by theorem 4.17, we have

$$\Lambda_{\mathcal{G}}^{nb} = \Lambda_{\mathcal{G}_1} \dot{\vee} \Lambda_{\mathcal{G}_2} := \{\bowtie_1 \dot{\vee} \bowtie_2 : \bowtie_i \in \Lambda_{\mathcal{G}_i}, \ i = 1, 2\} \tag{4.31}$$

and

$$\Lambda_{\mathcal{G}}^{pb} = \{\bowtie_\sigma : \sigma \in \Sigma\}, \tag{4.32}$$

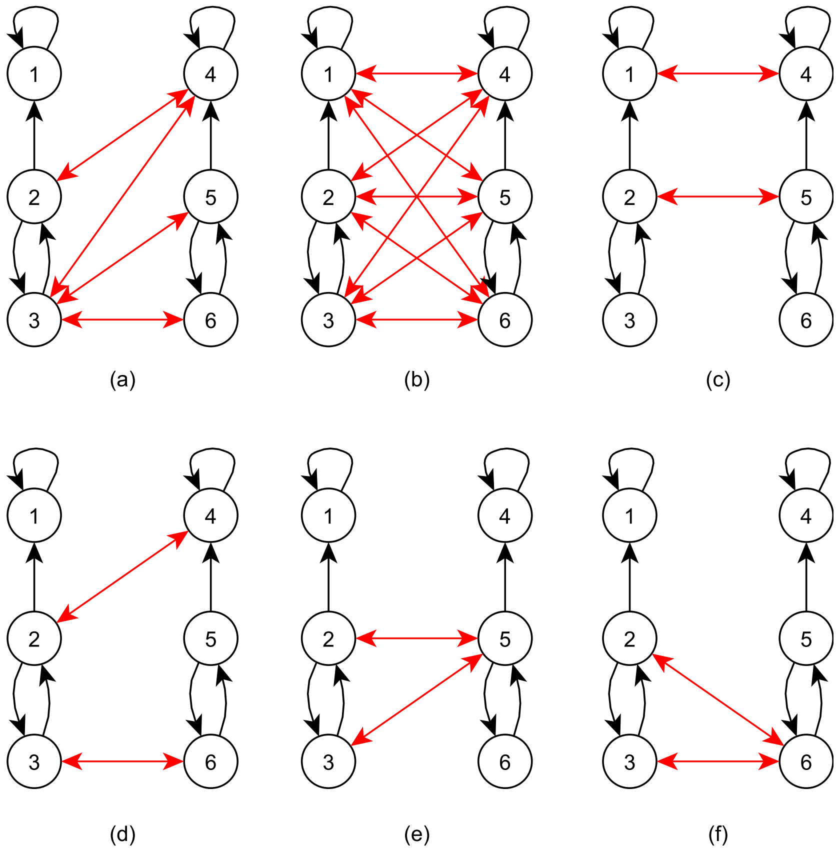


Figure 13. (*a*) Coexistence of Λ_1 and Λ_2; (*b*) cooperation of Λ_1 and Λ_2; (*c*) coevolution of Λ_1 and Λ_2; (*d*) extinction of Λ_1 and Λ_2; (*e*) evolution of Λ_1; (*f*) elimination of Λ_2, where the edge types are all equal and the additional edges are highlighted for emphasis.

where Σ is the subgroup of interior symmetry group of $\mathcal{G}$ which consists of all interior symmetries of $\mathcal{G}$ that can be written as a product of disjoint transpositions $\tau_i = (c_i, d_i)$ for $c_i \in \mathcal{C}_1$, $d_i \in \mathcal{C}_2$. Furthermore, by theorem 4.17 and proposition 2.9,

$$\Lambda_{\mathcal{G}}^{npb} = \bigcup_{\bowtie \in \Lambda_{\mathcal{G}}^{nb}} \Lambda_{\mathcal{G}}^{\bowtie, pb},$$

where $\Lambda_{\mathcal{G}}^{\bowtie, pb} \subset \Lambda_{\mathcal{G}}^{\bowtie}$ stands for the set of lifting of all pairing-bipartite relations in $\Lambda_{\mathcal{G}_{\bowtie}}$. In practice, it is however not necessary to consider all relations from $\Lambda_{\mathcal{G}}^{nb}$ but only a subset of it, to reconstruct $\Lambda_{\mathcal{G}}^{npb}$. Let $\bar{\Lambda}_{\mathcal{G}}^{nb}$ be a minimal subset of $\Lambda_{\mathcal{G}}^{nb}$ such that

$$\Lambda_{\mathcal{G}}^{nb} \setminus \{\bowtie_0\} = \bar{\Lambda}_{\mathcal{G}}^{nb} \dot{\vee} \bar{\Lambda}_{\mathcal{G}}^{nb},$$

where $\bowtie_0$ denotes the trivial equivalence relation. Then, it follows from the fact that $\Lambda_{\mathcal{G}}^{\bowtie_1} \subset \Lambda_{\mathcal{G}}^{\bowtie_2}$ whenever $\bowtie_2 \prec \bowtie_1$, that

$$\Lambda_{\mathcal{G}}^{npb} = \bigcup_{\bowtie \in \bar{\Lambda}_{\mathcal{G}}^{nb}} \Lambda_{\mathcal{G}}^{\bowtie, b}, \tag{4.33}$$

where $\Lambda_{\mathcal{G}}^{\bowtie, b} \subset \Lambda_{\mathcal{G}}^{\bowtie}$ stands for the set of lifting of all bipartite relations in $\Lambda_{\mathcal{G}_{\bowtie}}$. Thus, one can reconstruct $\Lambda_{\mathcal{G}}$ using (4.31)–(4.33).

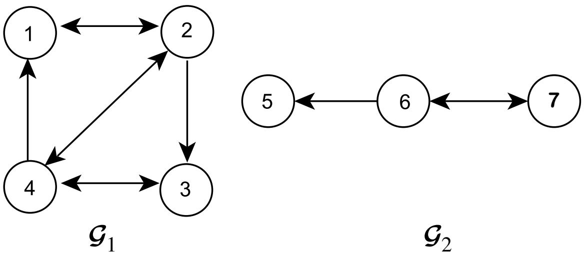


Figure 14. The networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of example 4.35.

Table 2. The summary of lattices of balanced equivalence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$.

$\Lambda_{\mathcal{G}_1}$	$\Lambda_{\mathcal{G}_2}$
$\bowtie_0 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$	$\rhd\!\!\bullet\!\!\lhd_0 = \{\{5\}, \{6\}, \{7\}\}$
$\bowtie_1 = \{\{1, 2\}, \{3\}, \{4\}\}$	$\rhd\!\!\bullet\!\!\lhd_1 = \{\{5, 7\}, \{6\}\}$
$\bowtie_2 = \{\{1, 3\}, \{2\}, \{4\}\}$	$\rhd\!\!\bullet\!\!\lhd_2 = \{\{5\}, \{6, 7\}\}$
$\bowtie_3 = \{\{1\}, \{2\}, \{3, 4\}\}$	$\rhd\!\!\bullet\!\!\lhd_3 = \{\{5, 6, 7\}\}$
$\bowtie_4 = \{\{1, 2, 3\}, \{4\}\}$	
$\bowtie_5 = \{\{1, 3, 4\}, \{2\}\}$	
$\bowtie_6 = \{\{1, 2\}, \{3, 4\}\}$	
$\bowtie_7 = \{\{1, 3\}, \{2, 4\}\}$	
$\bowtie_8 = \{\{1, 2, 3, 4\}\}$	

Table 3. The list of all balanced equivalence relations $\Join_k \in \Lambda_{\mathcal{G}}^{nb}$, for $0 \leqslant k \leqslant 35$.

i	j	$\bowtie_i \dot{\vee} \rhd\!\!\bullet\!\!\lhd_j$	
0	0, 1	$\Join_0 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 7\}, \{6\}\}$
0	2, 3	$\Join_2 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}, \{7\}\}$	$\Join_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6, 7\}\}$
1	0, 1	$\Join_4 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_5 = \{\{1, 2\}, \{3\}, \{4\}, \{5, 7\}, \{6\}\}$
1	2, 3	$\Join_6 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6, 7\}\}$	$\Join_7 = \{\{1, 2\}, \{3\}, \{4\}, \{5, 6, 7\}\}$
2	0, 1	$\Join_8 = \{\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_9 = \{\{1, 3\}, \{2\}, \{4\}, \{5, 7\}, \{6\}\}$
2	2, 3	$\Join_{10} = \{\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6, 7\}\}$	$\Join_{11} = \{\{1, 3\}, \{2\}, \{4\}, \{5, 6, 7\}\}$
3	0, 1	$\Join_{12} = \{\{1\}, \{2\}, \{3, 4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{13} = \{\{1\}, \{2\}, \{3, 4\}, \{5, 7\}, \{6\}\}$
3	2, 3	$\Join_{14} = \{\{1\}, \{2\}, \{3, 4\}, \{5\}, \{6, 7\}\}$	$\Join_{15} = \{\{1\}, \{2\}, \{3, 4\}, \{5, 6, 7\}\}$
4	0, 1	$\Join_{16} = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{17} = \{\{1, 2, 3\}, \{4\}, \{5, 7\}, \{6\}\}$
4	2, 3	$\Join_{18} = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7\}\}$	$\Join_{19} = \{\{1, 2, 3\}, \{4\}, \{5, 6, 7\}\}$
5	0, 1	$\Join_{20} = \{\{1, 3, 4\}, \{2\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{21} = \{\{1, 3, 4\}, \{2\}, \{5, 7\}, \{6\}\}$
5	2, 3	$\Join_{22} = \{\{1, 3, 4\}, \{2\}, \{5\}, \{6, 7\}\}$	$\Join_{23} = \{\{1, 3, 4\}, \{2\}, \{5, 6, 7\}\}$
6	0, 1	$\Join_{24} = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{25} = \{\{1, 2\}, \{3, 4\}, \{5, 7\}, \{6\}\}$
6	2, 3	$\Join_{26} = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6, 7\}\}$	$\Join_{27} = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}\}$
7	0, 1	$\Join_{28} = \{\{1, 3\}, \{2, 4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{29} = \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6\}\}$
7	2, 3	$\Join_{30} = \{\{1, 3\}, \{2, 4\}, \{5\}, \{6, 7\}\}$	$\Join_{31} = \{\{1, 3\}, \{2, 4\}, \{5, 6, 7\}\}$
8	0, 1	$\Join_{32} = \{\{1, 2, 3, 4\}, \{5\}, \{6\}, \{7\}\}$	$\Join_{33} = \{\{1, 2, 3, 4\}, \{5, 7\}, \{6\}\}$
8	2, 3	$\Join_{34} = \{\{1, 2, 3, 4\}, \{5\}, \{6, 7\}\}$	$\Join_{35} = \{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$

Example 4.35. Let $\mathcal{G} = \mathcal{G}_1 * \mathcal{G}_2$ for two networks $\mathcal{G}_1$ and $\mathcal{G}_2$ given by figure 14, where $\mathfrak{e}_1 = \mathfrak{e}_2$. The lattices $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$ are as listed in table 2. By making join extension of $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$, we obtain (see table 3)

$$\Lambda_{\mathcal{G}}^{nb} = \{\bowtie_i \dot{\vee} \rhd\!\!\bullet\!\!\lhd_j \ : \ 0 \leqslant i \leqslant 8,\ 0 \leqslant j \leqslant 3\}.$$

Table 4. The list of all balanced equivalence relations $\bowtie_k \in \Lambda_{\mathcal{G}}^{pb}$, for $36 \leqslant k \leqslant 39$.

$\sigma \in \Sigma$	$\bowtie_\sigma$
(1 5)(3 7)	$\bowtie_{36} = \{\{1,5\},\{2\},\{3,7\},\{4\},\{6\}\}$
(1 5)(3 7)(4 6)	$\bowtie_{37} = \{\{1,5\},\{3,7\},\{4,6\},\{2\}\}$
(1 7)(3 5)	$\bowtie_{38} = \{\{1,7\},\{2\},\{3,5\},\{4\},\{6\}\}$
(1 7)(2 6)(3 5)	$\bowtie_{39} = \{\{1,7\},\{2,6\},\{3,5\},\{4\}\}$

Table 5. The list of all balanced equivalence relations $\bowtie_k \in \Lambda_{\mathcal{G}}^{npb}$, for $40 \leqslant k \leqslant 99$.

$\bowtie \in \bar{\Lambda}_{\mathcal{G}}^{nb}$	$\Lambda_{\mathcal{G}}^{\bowtie,b}$	
$\{\{1,2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$	$\bowtie_{40} = \{\{1,2,5\},\{3,7\},\{4\},\{6\}\}$	$\bowtie_{41} = \{\{1,2,5\},\{3,7\},\{4,6\}\}$
	$\bowtie_{42} = \{\{1,2,5\},\{3,4,7\},\{6\}\}$	$\bowtie_{43} = \{\{1,2,5\},\{3,6,7\},\{4\}\}$
	$\bowtie_{44} = \{\{1,2,5\},\{3,4,6,7\}\}$	$\bowtie_{45} = \{\{1,2,7\},\{3,5\},\{4\},\{6\}\}$
	$\bowtie_{46} = \{\{1,2,7\},\{3,4,5\},\{6\}\}$	$\bowtie_{47} = \{\{1,2,6,7\},\{3,5\},\{4\}\}$
	$\bowtie_{48} = \{\{1,2,6,7\},\{3,4,5\}\}$	$\bowtie_{49} = \{\{1,2,3,5,7\},\{4\},\{6\}\}$
	$\bowtie_{50} = \{\{1,2,3,5,7\},\{4,6\}\}$	$\bowtie_{51} = \{\{1,2,3,5,6,7\},\{4\}\}$
	$\bowtie_{52} = \{\{1,2,3,4,5,7\},\{6\}\}$	
$\{\{1,3\},\{2\},\{4\},\{5\},\{6\},\{7\}\}$	$\bowtie_{53} = \{\{1,3,5,7\},\{2\},\{4\},\{6\}\}$	$\bowtie_{54} = \{\{1,3,5,7\},\{2,6\},\{4\}\}$
	$\bowtie_{55} = \{\{1,3,5,7\},\{2\},\{4,6\}\}$	$\bowtie_{56} = \{\{1,3,5,7\},\{2,4,6\}\}$
	$\bowtie_{57} = \{\{1,2,3,5,7\},\{4\},\{6\}\} = \bowtie_{49}$	$\bowtie_{58} = \{\{1,3,4,5,7\},\{2\},\{6\}\}$
	$\bowtie_{59} = \{\{1,3,4,5,7\},\{2,6\}\}$	$\bowtie_{60} = \{\{1,3,5,6,7\},\{2\},\{4\}\}$
	$\bowtie_{61} = \{\{1,2,3,5,6,7\},\{4\}\} = \bowtie_{51}$	$\bowtie_{62} = \{\{1,2,3,4,5,7\},\{6\}\} = \bowtie_{52}$
	$\bowtie_{63} = \{\{1,3,4,5,6,7\},\{2\}\}$	
$\{\{1\},\{2\},\{3,4\},\{5\},\{6\},\{7\}\}$	$\bowtie_{64} = \{\{1,5\},\{2\},\{3,4,7\},\{6\}\}$	$\bowtie_{65} = \{\{1,5\},\{2\},\{3,4,6,7\}\}$
	$\bowtie_{66} = \{\{1,7\},\{2\},\{3,4,5\},\{6\}\}$	$\bowtie_{67} = \{\{1,7\},\{2,6\},\{3,4,5\}\}$
	$\bowtie_{68} = \{\{1,2,5\},\{3,4,7\},\{6\}\} = \bowtie_{42}$	$\bowtie_{69} = \{\{1,2,7\},\{3,4,5\},\{6\}\} = \bowtie_{46}$
	$\bowtie_{70} = \{\{1,6,7\},\{2\},\{3,4,5\}\}$	$\bowtie_{71} = \{\{1,2,6,7\},\{3,4,5\}\} = \bowtie_{48}$
	$\bowtie_{72} = \{\{1,3,4,5,7\},\{2\},\{6\}\} = \bowtie_{58}$	$\bowtie_{73} = \{\{1,3,4,5,7\},\{2,6\}\} = \bowtie_{59}$
	$\bowtie_{74} = \{\{1,2,3,4,5,7\},\{6\}\} = \bowtie_{62}$	$\bowtie_{75} = \{\{1,3,4,5,6,7\},\{2\}\} = \bowtie_{63}$
$\{\{1,3\},\{2,4\},\{5\},\{6\},\{7\}\}$	$\bowtie_{76} = \{\{1,3,5,7\},\{2,4\},\{6\}\}$	$\bowtie_{77} = \{\{1,3,5,6,7\},\{2,4\}\}$
	$\bowtie_{78} = \{\{1,3,5,7\},\{2,4,6\}\} = \bowtie_{56}$	$\bowtie_{79} = \{\{1,2,3,4,5,7\},\{6\}\} = \bowtie_{52}$
$\{\{1\},\{2\},\{3\},\{4\},\{5,7\},\{6\}\}$	$\bowtie_{80} = \{\{1,3,5,7\}\{2\},\{4\},\{6\}\} = \bowtie_{53}$	$\bowtie_{81} = \{\{1,3,5,7\},\{2,6\},\{4\}\} = \bowtie_{54}$
	$\bowtie_{82} = \{\{1,3,5,7\},\{2\},\{4,6\}\} = \bowtie_{55}$	$\bowtie_{83} = \{\{1,3,5,7\},\{2,4,6\}\} = \bowtie_{56}$
	$\bowtie_{84} = \{\{1,2,3,5,7\},\{4\},\{6\}\} = \bowtie_{49}$	$\bowtie_{85} = \{\{1,3,4,5,7\},\{2\},\{6\}\} = \bowtie_{58}$
	$\bowtie_{86} = \{\{1,3,4,5,7\},\{2,6\}\} = \bowtie_{59}$	$\bowtie_{87} = \{\{1,3,5,6,7\},\{2\},\{4\}\} = \bowtie_{60}$
	$\bowtie_{88} = \{\{1,2,3,5,6,7\},\{4\}\} = \bowtie_{51}$	$\bowtie_{89} = \{\{1,2,3,4,5,7\},\{6\}\} = \bowtie_{52}$
	$\bowtie_{90} = \{\{1,3,4,5,6,7\},\{2\}\} = \bowtie_{63}$	
$\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6,7\}\}$	$\bowtie_{91} = \{\{1,5\},\{2\},\{3,6,7\},\{4\}\}$	$\bowtie_{92} = \{\{1,5\},\{2\},\{3,4,6,7\}\} = \bowtie_{65}$
	$\bowtie_{93} = \{\{1,2,5\},\{3,6,7\},\{4\}\} = \bowtie_{43}$	$\bowtie_{94} = \{\{1,6,7\},\{2\},\{3,5\},\{4\}\}$
	$\bowtie_{95} = \{\{1,6,7\},\{2\},\{3,4,5\}\} = \bowtie_{70}$	$\bowtie_{96} = \{\{1,2,6,7\},\{3,5\},\{4\}\} = \bowtie_{47}$
	$\bowtie_{97} = \{\{1,3,5,6,7\},\{2\},\{4\}\} = \bowtie_{60}$	$\bowtie_{98} = \{\{1,2,3,5,6,7\},\{4\}\} = \bowtie_{51}$
	$\bowtie_{99} = \{\{1,3,4,5,6,7\},\{2\}\} = \bowtie_{52}$	

Using interior symmetries of $\mathcal{G}$ which are products of disjoint transpositions, we obtain $\Lambda_{\mathcal{G}}^{pb}$ (see table 4). To recover $\Lambda_{\mathcal{G}}^{npb}$, we use (4.33) and take $\bar{\Lambda}_{\mathcal{G}}^{nb} = \{\bowtie_1, \bowtie_2, \bowtie_3, \bowtie_7, \rhd\!\bullet\!\lhd_1, \rhd\!\bullet\!\lhd_2\}$. The bipartite balanced equivalence relations in $\Lambda_{\mathcal{G}}^{\bowtie}$ for $\bowtie \in \bar{\Lambda}_{\mathcal{G}}^{nb}$ are listed in table 5.

In summary, we have $\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{G}}^{nb} \cup \Lambda_{\mathcal{G}}^{pb} \cup \Lambda_{\mathcal{G}}^{npb} = \{\bowtie_k : 0 \leqslant k \leqslant 99\}$ (see table 3–5). It was confirmed, using the algorithm in Aguiar and Dias [1], that this list of balanced equivalence relations in $\Lambda_{\mathcal{G}}$ is complete. $\diamond$

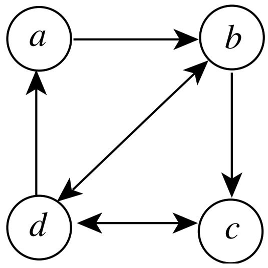


Figure 15. The structure of $\mathcal{G}_1$ and $\mathcal{G}_2$ of example 4.36.

Table 6. The summary of lattices of balanced equivalence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$.

$\Lambda_{\mathcal{G}_1}$	$\Lambda_{\mathcal{G}_2}$
$\rhd\lhd_0 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$	$\rhd\!\bullet\!\lhd_0 = \{\{5\}, \{6\}, \{7\}, \{8\}\}$
$\rhd\lhd_1 = \{\{1\}, \{2\}, \{3, 4\}\}$	$\rhd\!\bullet\!\lhd_1 = \{\{5\}, \{6\}, \{7, 8\}\}$

Table 7. The list of all balanced equivalence relations $\bowtie_k \in \Lambda^{\theta}_{\mathcal{G}} \cap \Lambda^{nb}_{\mathcal{G}}$, for $0 \leqslant k \leqslant 3$.

i	j	$\rhd\lhd_i \dot{\vee} \rhd\!\bullet\!\lhd_j$	
0	0, 1	$\bowtie_0 = \{\{\theta\}, \{2\}, \{3\}, \{4\}, \{6\}, \{7\}, \{8\}\}$	$\bowtie_1 = \{\{\theta\}, \{2\}, \{3\}, \{4\}, \{6\}, \{7, 8\}\}$
1	0, 1	$\bowtie_2 = \{\{\theta\}, \{2\}, \{3, 4\}, \{6\}, \{7\}, \{8\}\}$	$\bowtie_3 = \{\{\theta\}, \{2\}, \{3, 4\}, \{6\}, \{7, 8\}\}$

4.4.2. Coalescence. For the case of the coalescence $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ with $\mathfrak{e}_1 = \mathfrak{e}_2$, we can reconstruct $\Lambda^{\theta}_{\mathcal{G}}$ by following an analogous procedure used in section 4.4.1, based on the results in section 4.2. In general, since $\Lambda^{\theta}_{\mathcal{G}} \subsetneq \Lambda_{\mathcal{G}}$, this procedure may not recover the total lattice $\Lambda_{\mathcal{G}}$ of balanced relations on $\mathcal{G}$. However, as we will see in the following example, in some cases depending on the size of equivalence classes on θ in $\mathcal{G}_1$ and $\mathcal{G}_2$, the total lattice $\Lambda_{\mathcal{G}}$ can be recovered.

Example 4.36. Consider two isomorphic coupled cell networks $\mathcal{G}_1$ and $\mathcal{G}_2$, whose structure is shown in figure 15. Let $\mathcal{G}_1$ be the network given by $a = 1$, $b = 2$, $c = 3$, $d = 4$ and $\mathcal{G}_2$ be given by $a = 5$, $b = 6$, $c = 7$ and $d = 8$. Suppose that they have the same edge types $\mathfrak{e}_1 = \mathfrak{e}_2$. The lattices $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$ are listed in table 6. Consider several different coalescences of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Coalescence 1. Identify $1 \in \mathcal{C}_1$ with $5 \in \mathcal{C}_2$ and denote by $\theta = 1 = 5$. Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ be the coalescence obtained by this identification. Note that $\#[1]_{\rhd\lhd} = 1$ for all balanced relations $\rhd\lhd \in \Lambda_{\mathcal{G}_1}$ and $\#[5]_{\rhd\bullet\lhd} = 1$ for all balanced relations $\rhd\!\bullet\!\lhd \in \Lambda_{\mathcal{G}_2}$.

By making the join extension of $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$, we obtain (see table 7)

$$\Lambda^{\theta}_{\mathcal{G}} \cap \Lambda^{nb}_{\mathcal{G}} = \{\rhd\lhd_i \dot{\vee} \rhd\!\bullet\!\lhd_j \ : \ 0 \leqslant i \leqslant 1,\ 0 \leqslant j \leqslant 1\}.$$

Using the interior symmetries of $\mathcal{G}$, we have $\Lambda^{pb}_{\mathcal{G}}$ (see table 8).

Let $\bar{\Lambda}^{nb}_{\mathcal{G}} = \{\rhd\lhd_3, \rhd\!\bullet\!\lhd_1\}$, from which we obtain the bipartite balanced equivalence relations in $\Lambda^{\rhd\lhd}_{\mathcal{G}}$ for $\rhd\lhd \in \bar{\Lambda}^{nb}_{\mathcal{G}}$, listed in table 9. These are not all the relations in $\Lambda^{npb}_{\mathcal{G}}$. Indeed, there are three more balanced relations which are not θ-compatible, namely, $\bowtie_9 = \{\{\theta, 2, 3, 4, 6, 7, 8\}\}$, $\bowtie_{10} = \bowtie_5 \wedge \bowtie_8 = \{\{\theta, 3, 7\}, \{2, 8\}, \{4, 6\}\}$ and $\bowtie_{11} = \bowtie_4 \vee \bowtie_{10} = \{\{\theta, 3, 7\}, \{2, 4, 6, 8\}\}$. Moreover, note that the relations $\bowtie_5$ and $\bowtie_8$ in table 9 are not θ-compatible. In summary, we have (see tables 7–9)

$$\Lambda^{\theta}_{\mathcal{G}} = \{\bowtie_k : \ k = 0, 1, 2, 3, 4, 6, 7\}$$

Table 8. The list of balanced equivalence relations in $\Lambda_{\mathcal{G}}^{pb}$.

$\sigma \in \Sigma$	$\bowtie_\sigma$
(2 6)(3 7)(4 8)	$\bowtie_4 = \{\{\theta\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$

Table 9. The list of balanced equivalence relations in $\Lambda_{\mathcal{G}}^{npb}$ obtained from $\bar{\Lambda}_{\mathcal{G}}^{nb}$.

$\bowtie \in \bar{\Lambda}_{\mathcal{G}}^{nb}$	$\Lambda_{\mathcal{G}}^{\bowtie,b}$	
$\{\{1\}, \{2\}, \{3, 4\}\}$	$\bowtie_5 = \{\{\theta, 3, 4, 6, 7\}, \{2, 8\}\}$	$\bowtie_6 = \{\{\theta\}, \{2, 6\}, \{3, 4, 7, 8\}\}$
$\{\{5\}, \{6\}, \{7, 8\}\}$	$\bowtie_7 = \{\{\theta\}, \{2, 6\}, \{3, 4, 7, 8\}\} = \bowtie_6$	$\bowtie_8 = \{\{\theta, 2, 3, 7, 8\}, \{4, 6\}\}$

and

$$\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{G}}^{nb} \cup \Lambda_{\mathcal{G}}^{pb} \cup \Lambda_{\mathcal{G}}^{npb} = \{\bowtie_k : 0 \leqslant k \leqslant 11\}.$$

This was confirmed, using the algorithm in Aguiar and Dias [1].

Coalescence 2. Identify $1 \in \mathcal{C}_1$ with $8 \in \mathcal{C}_2$ and denote by $\theta = 1 = 8$. Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ be the coalescence obtained by this identification. Note that $\#[8]_{\triangleright\!\bullet\!\triangleleft} > 1$ for some balanced relations $\triangleright\!\bullet\!\triangleleft \in \Lambda_{\mathcal{G}_2}$. Then,

$$\Lambda_{\mathcal{G}}^{\theta} \cap \Lambda_{\mathcal{G}}^{nb} = \{\bowtie_i \dot{\vee} \triangleright\!\bullet\!\triangleleft_0 \ : \ 0 \leqslant i \leqslant 1 \ \}$$

and using the algorithm in Aguiar and Dias [1], we have

$$\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{G}}^{nb}.$$

Coalescence 3. Identify $4 \in \mathcal{C}_1$ with $8 \in \mathcal{C}_2$ and denote by $\theta = 4 = 8$. Let $\mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2$ be the coalescence obtained by this identification. Note that $\#[4]_{\bowtie} > 1$ for some balanced relations $\bowtie \in \Lambda_{\mathcal{G}_1}$ and $\#[8]_{\triangleright\!\bullet\!\triangleleft} > 1$ for some balanced relations $\triangleright\!\bullet\!\triangleleft \in \Lambda_{\mathcal{G}_2}$. Thus,

$$\Lambda_{\mathcal{G}}^{\theta} \cap \Lambda_{\mathcal{G}}^{nb} = \{\bowtie_0 \dot{\vee} \triangleright\!\bullet\!\triangleleft_0\},$$

and using the algorithm in Aguiar and Dias [1], we have

$$\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{G}}^{nb} \cup \Lambda_{\mathcal{G}}^{pb},$$

with

$$\Lambda_{\mathcal{G}}^{pb} = \{\bowtie_{\sigma_i}, i = 1, 2, 3\},$$

for $\sigma_1 = (1\ 5)$, $\sigma_2 = (1\ 5)(2\ 6)$ and $\sigma_3 = (1\ 5)(2\ 6)(3\ 7)$. $\diamond$

5. Discussion

In this work we examined the evolution of the lattice of synchrony subspaces of the combination of existing networks by network binary operations, through the evolution of the lattice of balanced equivalence relations. We considered the coalescence and different kinds of join of networks.

In practice, our results should help to decide what type of network operation is more preferred under which kind of evolution of patterns of synchrony conditions and/or requirements. For example, in general, for the join operations defined, we have:

(i) In the case of the join $\mathcal{G}_1 * \mathcal{G}_2$, the lattices $\Lambda_{\mathcal{G}_1}$ and $\Lambda_{\mathcal{G}_2}$ always survive, and so, we always have coexistence or cooperation.

(ii) In the case of a partial join $\mathcal{G}_1 *_p \mathcal{G}_2$, we have coexistence or cooperation if and only if for all $[c_i]_{\bowtie}$, with $\bowtie \in \Lambda_{\mathcal{G}_i}$, we have $[c_i]_{\bowtie} \subseteq \tilde{\mathcal{C}}_i$ or $[c_i]_{\bowtie} \cap \tilde{\mathcal{C}}_i = \emptyset$.

(iii) In the case of a point-wise partial join $\mathcal{G}_1 *_{pp} \mathcal{G}_2$, if there is a relation $\bowtie \in \Lambda_{\mathcal{G}_i}$ such that there exists a $\bowtie$-class $[c_i]_{\bowtie}$ satisfying $\#\left([c_i]_{\bowtie} \cap \tilde{\mathcal{C}}_i\right) > 1$, that is, there are $c_1, c_2 \in \tilde{\mathcal{C}}_i$, $i = 1$ or $i = 2$, such that $c_1 \bowtie c_2$ then $\Lambda_{\mathcal{G}_i}$ does not survive, and so, we can only have coevolution or extinction.

A natural extension of our study, that will appear in a future work, is to determine the impact on the lattice of synchrony subspaces caused by structural changes in its topology due to elementary network operations, that is, operations that create a new network from the original one by the addition or deletion of a cell or an edge or by the rewiring of an edge. In these cases, we want to find conditions for synchrony subspaces to persist and, when this happens, find conditions to have the same associated quotient network (the same dynamics). Some partial results where already obtained by Field [11], where he considers the invariants of a network under repatching (rewiring). In the setting of complex networks, research has also been undertaken so far in order to understand how the rewiring of a complex network can affect its synchronizability, Atay [11] and Hagberg and Schult [17].

Acknowledgments

The research of MADA was funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0144/2011 and FCT grant PTDC/MAT/100055/2008. The work of HR was supported by grant DFG LA 525/11-1.

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