General equilibrium with uncertain delivery

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Abstract. We present a general equilibrium model of trade *ex ante* with differential information. Agents choose plans of state-contingent *lists of bundles*, that give them the right to receive, in each state of nature, one of the bundles in the corresponding list. Being unable to verify that the state of nature is s and not t, an agent has to accept the delivery of any bundle in the list for delivery in state s or in the list for delivery in state t. In equilibrium, the price of a list coincides with the price of the cheapest bundle that belongs to the list, and it is always this cheapest bundle that is delivered. This property leads to a system of linear inequalities which are *deliverability* constraints on the choice set. We establish existence of equilibrium under the assumption that each state of nature can be verified by at least one agent.

Keywords: General equilibrium, Differential information, Verifiability, Uncertain delivery, Lists of bundles, Rational expectations.

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1 Introduction

In chapter 7 of his "Theory of Value", Debreu (1959) showed how to extend the general equilibrium model to the case of trade under uncertainty with public state verification. All that is needed is to consider a generalized notion of commodity that also includes in its description the state of nature on which its delivery is contingent (Arrow, 1953). The model becomes equivalent to the model without uncertainty (Arrow and Debreu, 1954; McKenzie, 1959): prices of the contingent commodities are announced, and agents choose the consumption plan that they prefer (specifying a consumption bundle for each of the possible states of nature), among those that satisfy their budget restriction; after trade agreements are made, the state of nature is publicly announced and agents receive the consumption bundle that corresponds to the announced state.

We are interested in studying the implications of differential information, in the form of private and incomplete state verification. While keeping the basic structure of the model, we assume that each agent is only able to verify (in a court of law, for contracts to be enforced) that the state of nature belongs to a set of his/her information partition.

The consequence of incomplete verification is that if an agent has bought different bundles for delivery in two states and is not able to verify whether the true state is one or the other, then he/she has to accept delivery of any of the two bundles. This is a natural generalization of the classical model, in which state verification is complete.[3]

To study this economic setting, we consider that objects of choice are plans of lists of bundles such that the agents have the right to receive one of the bundles in the list that corresponds to the state of nature that occurs (they have to accept any of the alternatives in the list).[4] Contracts in which lists are traded are pervasive.

A plane ticket gives you the right to travel if the plane is available at the date of departure, and, if the plane is not available, the right to stay in a hotel and travel on the next plane. But you cannot verify whether the plane is available or not. If, at the date of traveling, the airline announces that the plane is not available, you may have no alternative other than to accept staying in a hotel and traveling on the next day.

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[3] A closely related line of research, initiated by Radner (1968), is based on the idea that the consequence of incomplete information is that an agent must consume the same in states of nature that he/she cannot distinguish.

[4] This concept builds on Arrow's (1953) notion of contingent goods. A contingent bundle is obviously a contingent list of bundles with a single element.

Some car insurance contracts give you the right to use another car temporarily, in case of accident or malfunction. But the substitute car is left undefined in the contract. It is only stipulated that the car should belong to a certain class. It may be red or yellow, have radio or not, etc.

Consider an agent who cannot verify whether the state of nature is s or t, but has, however, bought x^s for delivery in state s and x^t for delivery in state t. Then: if state s occurs, the agent can receive x^s or x^t. When receiving x^t in state s, the agent cannot prove in a court of law that the contract has been violated (state t could be the actual state and x^t the contracted delivery). For the same reason: if state t occurs, the possible deliveries are also x^s or x^t. Observe that the set of alternatives that may be delivered, $\{x^s, x^t\}$, is the same in the set of states that the agent cannot distinguish, $\{s, t\}$.

Something that is constant across states of nature that agent i is unable to distinguish is said to be "measurable with respect to private information", or P_i-measurable.[5] We could restrict our attention to P_i-measurable plans of lists, because, as exemplified above, any non-measurable choice can be converted into a measurable one that is equivalent. Buying a non-measurable consumption plan (x^s for state s and x^t for state t), agent i obtains a P_i-measurable plan of lists ($x^s \vee x^t$ in state s and $x^s \vee x^t$ in state t). It is important to understand that this P_i-measurability property of lists is not a restriction on trade, but the consequence of incomplete state verification on the enforceability of trade agreements.

We have introduced this model of general equilibrium with private and incomplete state verification in two previous papers (2008, 2009). All trade is agreed *ex ante*, that is, before private information is received. Prices are announced, and agents choose the plan of contingent lists that they prefer, among those that belong to their budget set. After receiving their private information, agents are able to verify to which set of their information partition belongs the true state of nature. Then, each agent receives a bundle that belongs to the contingent list that corresponds to the actual state of nature, or to a state of nature that belongs to the same set of his/her information partition (the agent cannot prove that the contract has been violated). And, of course, the deliveries to all the agents in the economy must constitute a feasible allocation.

When buying a list, which of the alternatives should an agent expect to receive? In those preliminary explorations of this framework, we have studied the case of extreme pessimism and

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[5] Technically, with P_i denoting the information partition of agent i, a function that is constant in elements of the σ-algebra generated by P_i is designated as "P_i-measurable".

the case of continuous expectations.[6] In this paper, we study the case of rational expectations: agents know the model of the economy, and form their expectations accordingly (Muth, 1961). This solution concept is more consistent, but also more challenging to investigate.[7]

We find that, in equilibrium: (1) the price of a plan of contingent lists (specifying a list of possible bundles for delivery in each state of nature) is equal to the price of the cheapest consumption plan (specifying a bundle for delivery in each state of nature) that satisfies the requirements of the plan of contingent lists; and (2) this cheapest consumption plan is actually the alternative that is selected for delivery (in each state, the bundle that is selected for delivery is the cheapest according to prices for delivery in this state).

Rational agents expect, then, to receive the cheapest possible alternative in each state of nature.[8] Observing the prices of all the contingent commodities and of all the lists, they can predict which bundle is going to be selected for delivery in each state of nature. In case of a tie, agents expect to receive the alternative that they prefer (a similar assumption is made in the mechanism design literature: in case of indifference, agents are truthful).

Being able to anticipate the consumption plan that results from buying each plan of contingent lists, agents can, instead of choosing a plan of lists, choose the resulting consumption plan. Deliverable consumption plans are those that satisfy a system of linear inequalities. Consider an agent who does not distinguish between states s and t. For a consumption plan, (x^s, x^t), to be deliverable, it must be such that $p^s \cdot x^s \leq p^s \cdot x^t$ and $p^t \cdot x^t \leq p^t \cdot x^s$. If these deliverability conditions are not satisfied, then the agent will not receive x^s in state s and x^t in state t (because these would not be the cheapest alternatives in the corresponding states). An agent with rational expectations chooses among plans which are deliverable in this sense (denoted $x \in C_i(p)$).

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[6] If agents expect to receive the worst possible bundle in a list, there exists an equilibrium in which these expectations are fulfilled. This is a *prudent expectations equilibrium* (2009). Agents act very defensively, selecting alternatives with the same utility for delivery in states that they cannot distinguish. They insure themselves completely against being deceived. Even if they are deceived, it implies no utility loss. A more general notion is that of a *subjective expectations equilibrium* (2008), where the agent's beliefs about the probabilities of delivery of the different alternatives in a list are a continuous function of the prices that they observe (perfectly or imperfectly) and of the alternatives specified in the list.

[7] A different line of research, associated with the notion of rational expectations, focuses on the revelation of information by prices (Radner, 1979; Allen, 1981). But with trade taking place *ex ante*, an agent cannot infer the information of the other agents because, at the moment of trade, the other agents still haven't received their information. From the deliveries made at date 1, agents could be able to infer the true state of nature. But we assume that the information obtained through these inferences cannot be used (in a court of law, for example) to enforce contracts.

[8] Prices differ across states, thus, the cheapest bundle may also differ (which implies that the consumption plan may not be P_i-measurable).

This deliverable choice set depends, therefore, on prices and on each agent's private information. The choice set of each agent is the intersection of the budget set and the deliverable set, $B_i(p) \cap C_i(p)$. If the correspondence from prices to the choice set were continuous, equilibrium existence would be guaranteed. In a bounded economy, $B_i(p) \cap C_i(p)$ is upper hemicontinuous. But $C_i(p)$ is not lower hemicontinuous.[9] This property fails when prices in some state are null or when prices in states s and t, with $t \in P_i(s)$, are collinear.[10]

We give an example of non-existence of equilibrium caused by null prices. In the presence of differential information, prices for delivery (of any commodity) in some state may be null, even if state-contingent preferences are strictly monotonic. In such a state, resources are abundant, but no agent can verify that this state has occurred. As a result, no agent is willing to pay a positive price for delivery contingent on the occurrence of this state.

To establish existence of equilibrium, we assume that any state of nature can be verified by at least one agent. In the model of Radner (1968), if free disposal is not allowed, the same hypothesis is necessary to guarantee the existence of equilibrium with non-negative prices.

This paper is a contribution to the theory of general equilibrium with differential information. We improve on the model of Radner (1968), essentially by considering that objects of choice are plans of contingent lists instead of plans of contingent bundles. The seminal work of Radner (1968) has been complemented by many developments.[11] It is an open question whether these can be extended to the model presented here.

Central to the literature on general equilibrium with differential information is the pioneering work of Prescott and Townsend (1984a, 1984b). In their setup, an allocation is a vector of lotteries over consumption plans that satisfies a set of incentive compatibility constraints (so that agents do not gain by pretending to be of any other type).[12] They showed that, under general conditions, Pareto optimal allocations exist and, in the case of trade ex-ante, can be

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[9] The intersection of continuous correspondences may not be continuous, anyway (Aliprantis and Border, 2007).

[10] With agents having preferences that are P_i-measurable, collinearity does not prevent existence of equilibrium. In this case, it can be shown that (having convex preferences) agents choose the same bundle for delivery in both states, implying that the deliverability restrictions are satisfied in equality.

[11] Such as the private core notion (Yannelis, 1991), core-convergence results (Einy, Moreno and Shitovitz, 2001), study of incentive compatibility (Krasa and Yannelis, 1994; Glycopantis, Muir and Yannelis, 2003), extension to infinite commodity spaces (Podczeck and Yannelis, 2008) and infinite state spaces (Hervés-Beloso, Martins-da-Rocha and Monteiro, 2009), etc. For a comprehensive view, see the volume edited by Glycopantis and Yannelis (2005).

[12] Instead of considering that objects of choice are lotteries, one may allow trade to be contingent on sunspots (Kehoe, Levine and Prescott, 2002).

decentralized through a price system (with the intermediation of a profit maximizing firm).[13] A difficulty with this approach lies in justifying the restriction of agents' choices to the incentive compatible set. It would be more acceptable to restrict the firm to offer incentive compatible trades. But, in this case, prices (which are linear in probabilities) become non-linear in consumption (Jerez, 2005). Our approach is more compatible with anonymous trading, as there is no restriction to an incentive compatible choice set, and prices are linear in consumption.[14]

A further drawback common to these works is the consideration of exclusive contracts: each agent can select only one contract, and trade at the *ex post* stage is prohibited. We may allow agents to select more than one contract, but we also restrict trade to be made *ex ante*, that is, before agents receive their information. The inclusion of spot markets that open after agents receive their information is left for future research.

The important case of non-exclusive contracts was studied by Bisin and Gottardi (1999).[15] Our context is related to their "*Hidden Information Economy*",[16] but there are some substantial differences (besides the reopening of markets): (i) they consider uncertainty only about endowments, and no private information on the aggregate endowment, while we consider uncertainty about endowments and preferences and allow agents to have private information about the aggregate endowment; and (ii) in their model, the outcome of trade depends on the set of messages sent by the agents, while in our model, individual trade is not contingent on the messages sent by the others.

A fundamental difference with respect to our work is that, in this literature inspired on mechanism design, if an agent can conceal the fact that he/she is of type s and announces type t, the trade that corresponds to type t is carried out. In our model, the onus of the proof is inverted: being unable to prove that his/her type is s and not t (we consider states of nature instead of types but the approaches are similar), the agent has to accept either the outcome associated with s or the outcome associated with t.

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[13] For a cooperative solution, see the survey on the incentive compatible core by Forges, Minelli and Vohra (2002) and the study of core-equivalence by Forges, Heifetz and Minelli (2001).

[14] See the comments by Rustichini and Siconolfi (2008), recognizing that, in the standard view of competitive markets, individual trade is anonymous and the price system alone decentralizes efficient allocations.

[15] They follow Dubey, Geanakoplos and Shubik (2005) in thinking of assets of pools with payoffs being equilibrating variables. See also Minelli and Polemarchakis (2000).

[16] Other studies on economies with differential information study the case of hidden action (Bennardo and Chiappori, 2003; Bisin and Guaitoli, 2004; Jerez, 2005) or adverse selection (Bisin and Gottardi, 2006; Rustichini and Siconolfi, 2008). Recently, Zame (2007) developed a comprehensive model in which the set of firms and the contracts that appear are also determined endogenously at equilibrium. Our scope is more limited: we study the case of pure exchange with hidden information.

The paper is organized as follows: in section 2 we present the basic setup, explain the consequences of incomplete information and describe prices and preferences over lists; in section 3, we define and characterize equilibrium; in section 4 we establish existence; and section 5 concludes with some remarks. In appendix, we: (1) collect all the proofs, (2) give an example of non-existence of equilibrium, and (3) study continuity of the deliverability correspondence.

2 The economy

2.1 Basic setup

The economy extends over two time periods, date 0 and date 1. There is a finite number of agents, $\mathcal{I} = \{1, ..., I\}$, who trade (at date 0) a finite number of commodities, $\mathcal{L} = \{1, ..., L\}$, under uncertainty about which of a finite number of possible states of nature, $\Omega = \{1, ..., S\}$, will occur (at date 1). The state of nature determines the endowments and preferences of the agents.

At date 0, agents know the probabilities of occurrence of each state, $\mu = (\mu^1, ..., \mu^S) \in \Delta^S$. At date 1, if state s occurs, each agent is only able to verify (and prove in a court of law, for contracts to be enforced) that the state of nature belongs to the corresponding set of his/her information partition, $P_i(s)$.[17]

Knowing their state-dependent endowments, $e_i : \Omega \to \mathbb{R}^L_{++}$, agents make (at date 0) contingent trade agreements with the objective of obtaining a consumption plan, $x_i : \Omega \to \mathbb{R}^L_+$, that maximizes their expected utility, $U_i(x_i) = \sum_{s \in \Omega} \mu^s u_i^s(x_i^s)$. Afterwards (at date 1), agents receive their endowments and their private information about the state of nature, and trade agreements are carried out.[18]

In the case of public state verification (Debreu, 1959), agents trade contingent commodities.[19] At date 0, taking as given the price system, $p \in \Delta^{SL}$, agents select the consumption plan that maximizes their expected utility, among those that belong to their budget set, $B_i(p) =$

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[17] Public state verification corresponds to the particular case in which $P_i(s) = \{s\}$, $\forall (i, s) \in \mathcal{I} \times \Omega$.

[18] We restrict our study to the case of P_i-measurable endowments ($t \in P_i(s) \Rightarrow e_i^t = e_i^s$) and preferences ($t \in P_i(s) \Rightarrow u_i^t = u_i^s$).

[19] Besides being defined by their physical properties and by their location in space and time, contingent goods are also defined by the state of nature in which they are made available. Instead of talking about consumption of good A in state 1 and consumption of good B in state 2, we talk about consuming good $A1$ and good $B2$.

$\left\{x_i \in \mathbb{R}_+^{SL} : \sum_{s\in\Omega} p^s \cdot x_i^s \leq \sum_{s\in\Omega} p^s \cdot e_i^s\right\}$. At date 1, the state of nature is publicly announced, and the corresponding trade agreements take place (if state s occurs, each agent i delivers his/her endowment, e_i^s, and receives the consumption bundle he/she is entitled to, x_i^s).

What happens if agents receive different information? What happens if, instead of being publicly verifiable, the state of nature is only privately and incompletely verifiable by each of the agents?

To answer this question, we will consider the model of an economy with uncertain delivery, in which agents select plans of lists instead of consumption plans. At date 0, taking as given the prices of plans of lists, $\tilde{p}$, each agent i chooses the plan of lists that he/she prefers, $\tilde{x}_i$, among those that belong to his/her budget set, $\tilde{B}_i(\tilde{p})$. The plan of lists specifies a set of possible consumption bundles, $\tilde{x}_i^s$, for delivery in each state of nature, s. At date 1, if state s occurs, agent i delivers his/her endowment, e_i^s, and receives one of the alternatives in the list $\tilde{x}_i^s$ (truthful delivery) or an alternative in another list, $\tilde{x}_i^t$, contracted for delivery in an indistinguishable state of nature, $t \in P_i(s)$ (concealed violation).

In the remainder of this section, we motivate and explain our modeling choices, describe the structure of price systems that are compatible with absence of arbitrage, and make some assumptions on the agents' preferences over lists.

2.2 The solution of Radner

In a seminal contribution, Radner (1968) postulated in states of nature that an agent does not distinguish, s and t such that $t \in P_i(s)$, the same bundle would be consumed. By simply restricting the consumption set to $\mathbb{R}_+^{SL} \cap P_i$ (meaning that if $t \in P_i(s)$, then $x_i^t = x_i^s$), the classical model could be reinterpreted to cover the case of private information.

Before presenting a critique of this solution, and an alternative concept, we stress that, in our economy: having made a contract for the contingent delivery of commodities, an agent needs to prove that an event has occurred to enforce delivery. The meaning of the information partition, P_i, is that, if state s occurs, agent i can prove that the state of nature belongs to $P_i(s)$, and can use this and only this information to enforce delivery.[20]

The main objection to the model of Radner (1968) is that agents should not be restricted to

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[20] An event is a set of states of nature. In state s, agent i can prove that the event E has occurred if and only if $P_i(s) \subseteq E$.

consume the same bundle in states of nature that they do not distinguish (even if contracts are only contingent upon events that they can observe). The example that follows shows that such restriction is too strong.

Consider an economy with two agents. Agent A is endowed with two units of '*sugar*', in all states of nature, $\Omega = \{s_1, s_2\}$, while agent B has uncertain endowments: two units of '*tea*' in state s_1 and two units of '*coffee*' in state s_2:

$$e_A^{s_1} = e_A^{s_2} = (2,0,0), \;\; e_B^{s_1} = (0,2,0) \;\text{ and }\; e_B^{s_2} = (0,0,2).$$

The preferences of the agents are the same, and do not depend on the state of nature. The goods '*tea*' and '*coffee*' are perfect substitutes, that agents like to drink with '*sugar*':

$$u_A^{s_1} = u_A^{s_2} = u_B^{s_1} = u_B^{s_2} = \sqrt{(x_{tea} + x_{cof})x_{sug}}.$$

Agent A cannot distinguish the two states, which are equiprobable:

$$P_A = \{s_1, s_2\} \;\text{ and }\; P_B = \{\{s_1\}, \{s_2\}\}.$$

With the restriction of consuming the same in indistinguished states of nature, there is *no trade*. To see this, observe that agent A would like to consume some '*tea*' in state s_1. But this would imply equal consumption in state s_2, and there is no '*tea*' in state s_2 (only '*coffee*'...).

In a real-life situation, the two agents could make the following agreement (valid for both states of nature): agent A would deliver one unit of '*sugar*' in exchange for one unit of '*tea*' or one unit of '*coffee*'. Agent A would get the right to receive a '*tea or coffee*', or, to put it another way, would get the right to consume $(1,1,0)$ or $(1,0,1)$. Both agents would end up consuming $(1,1,0)$ in state s_1 and $(1,0,1)$ in state s_2. This *contract for uncertain delivery* allows the agents to attain an optimal outcome.[21]

Agent A is buying what we call a *list of bundles*: a derivative good that gives him/her the right to receive one of the bundles in the list. This suggests that, to improve upon the solution of Radner (1968), we should allow agents to trade lists of bundles. Then, some questions arise:

(1) What are the consequences of private state verification?

(2) What is the price of a list of bundles?

(3) What is the utility of a list of bundles?

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[21] For other examples and a more detailed explanation, see our previous work (2008, 2009).

2.3 The consequences of private state verification

We do not restrict agents to select P_i-measurable consumption plans. Agents are allowed to buy different rights for delivery in states that they do not distinguish. But, if an agent buys different rights for delivery in two states and is not able to verify whether the true state is one or the other, then the agent has to accept delivery of any of the two.

Consider an agent who cannot prove in a court of law whether the true state is s or t, but that, nevertheless, has contracted for the delivery of bundle x^s in state s and bundle x^t in state t. When receiving bundle x^t in state s (or bundle x^s in state t), the agent cannot prove that the contract is being violated. Then: if state s occurs, the agent can receive x^s or x^t; and if state t occurs, the agent can also receive the same bundles, x^s or x^t. Notice that the set of alternatives that may be delivered, $\{x^s, x^t\}$, is the same in states that the agent cannot distinguish, $\{s, t\}$.

The same reasoning applies to lists. Suppose that agent i has contracted for the delivery of some alternative in the list $\tilde{x}_i^s$ if state s occurs and of some alternative in the list $\tilde{x}_i^t$ if state t occurs. In state s, agent i may receive $x_i^s \notin \tilde{x}_i^s$ (the bundle that is delivered in state s, x_i^s, may not belong to the list that was contracted for delivery in state s, $\tilde{x}_i^s$). Delivery may be of an alternative of a list contracted for delivery in a state $t \in P_i(s)$ (this would be a concealed violation of the contract). The adequate condition to describe enforceability is: $x_i^s \in \bigcup_{t \in P_i(s)} \tilde{x}_i^t$.

Notice that if agent i buys the same lists for delivery in the states that he cannot distinguish, then $\bigcup_{t \in P_i(s)} \tilde{x}_i^t = \tilde{x}_i^s$, and the enforceability condition becomes $x_i^s \in \tilde{x}_i^s$. Buying the same bundle for delivery in states that are not distinguished is a sufficient condition for the contract to be enforceable (but not a necessary condition).

Formally, for each agent $i \in \mathcal{I}$:

(i) a contingent list for delivery in state s is a finite, non-empty, subset of $\mathbb{R}_+^L$, denoted $\tilde{x}_i^s \in \mathbb{F}(\mathbb{R}_+^L)$;[22]

(ii) a plan of lists is a vector of contingent lists, $\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S$, specifying a list for delivery in each of the possible states of nature;[23]

(iii) a P_i-measurable plan of lists is a vector of contingent lists such that $t \in P_i(s) \Rightarrow \tilde{x}_i^t = \tilde{x}_i^s$,

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[22] Everywhere below, $\mathbb{F}(\cdot)$ denotes the set of finite and non-empty subsets.

[23] It is equivalent to consider that objects of choice are: plans of lists of consumption bundles (there is one list for each state, and an alternative in a list is a consumption bundle); or, alternatively, lists of plans of consumption bundles (there is a single list, and an alternative in the list is a consumption plan).

denoted $\tilde{x}_i \in (\mathbb{F}(\mathbb{R}^L_+))^S \cap P_i$.

We define a transformation, M_i, to describe the consequences of incomplete information. If agent i buys a plan of lists $\tilde{x}_i$, the set of consumption bundles that he/she may receive in state s is $M^s_i(\tilde{x}_i)$, defined as:

$$M^s_i : (\mathbb{F}(\mathbb{R}^L_+))^S \longrightarrow \mathbb{F}(\mathbb{R}^L_+) \ ;$$

$$M^s_i(\tilde{x}_i) = \bigcup_{t \in P_i(s)} \tilde{x}^t_i \ .$$

When agent i buys a plan of lists, $\tilde{x}_i$, that is not P_i-measurable, he/she will obtain a consumption plan, x_i, that belongs to a P_i-measurable (by construction) plan of lists, $M_i(\tilde{x}_i) = [M^1_i(\tilde{x}_i), ..., M^S_i(\tilde{x}_i)]$. With $x_i \in M_i(\tilde{x}_i)$, either we have a truthful delivery or a concealed violation of the contract.

In the model of Radner (1968), the consequence of incomplete information is a restriction of the choice set to P_i-measurable plans of consumption bundles. Here the consequences are less stringent. An agent can enforce delivery of P_i-measurable plans of lists (which include all P_i-measurable consumption plans), and this does not imply P_i-measurability of the resulting consumption plan.

2.4 Prices of lists

In economies with uncertain delivery, prices are defined on the space of plans of lists:

$$\tilde{p} : (\mathbb{F}(\mathbb{R}^L_+))^S \longrightarrow \mathbb{R}_+.$$

In this subsection, we find properties of price systems that are necessary for the absence of arbitrage opportunities.

2.4.1 Arbitrage

Arbitrage is a trade that involves a gain and no possibility of a loss. It consists of buying a plan of lists, $\tilde{x}_j$, and selling another, $\tilde{y}_j$, such that: (i) some income is retained; (ii) the net delivery is surely non-negative.

A perfectly informed arbitrageur that buys the plan of lists $\tilde{x}_j$ and sells the plan of lists $\tilde{y}_j$ will receive some $x_j \in \tilde{x}_j$ and, to keep its contract, must deliver some $y_j \in \tilde{y}_j$. An arbitrageur that buys $\tilde{x}_j$ can only sell $\tilde{y}_j$ such that $\forall x_j \in \tilde{x}_j, \exists y_j \in \tilde{y}_j : \ x_j - y_j \geq 0$ (for the net delivery to be surely non-negative).

An incompletely informed arbitrageur that buys $\tilde{x}_j$ and sells $\tilde{y}_j$ may, in state s, receive an element of $\tilde{x}^t_j$ and be forced to deliver an element of $\tilde{y}^t_j$, with $t \in P_j(s)$.[24] In this case, the incompletely informed arbitrageur obtains a net delivery planned for another state, $t \in P_i(s)$. But, since all net deliveries are non-negative, he/she is sure of receiving a non-negative net delivery. Incomplete information does not reduce the opportunities of arbitrage.

Definition 1.

An arbitrage opportunity consists of a pair of plans of lists, $(\tilde{x}_j, \tilde{y}_j)$, such that:

(i) $\tilde{p}(\tilde{x}_j) < \tilde{p}(\tilde{y}_j)$;

(ii) $\forall x_j \in \tilde{x}_j, \exists y_j \in \tilde{y}_j : \ x_j - y_j \geq 0.$

If there exists an arbitrage opportunity, then: in the first period, an arbitrageur buys the plan of lists $\tilde{x}_j$ and sells the plan of lists $\tilde{y}_j$ (retaining some rent); in the second period, the arbitrageur receives $x^t_j \in \tilde{x}^t_j$, with $t \in P_j(s)$, and delivers $y^t_j \in \tilde{y}^t_j$, with $y^t_j \leq x^t_j$.

2.4.2 No-arbitrage prices

Buying two or more lists yields the same possible deliveries as buying a single list with more alternatives. Suppose that an agent buys (separately) a list that delivers '*tea*' or '*coffee*' and a list that delivers '*toast*' or '*cookie*'. The agent will receive one of four alternatives: '*tea and toast*', '*tea and cookie*', '*coffee and toast*' or '*coffee and cookie*'. More generally, an agent that buys the plans of lists $\tilde{x}$ and $\tilde{y}$ should receive an alternative in the plan of lists $\tilde{z}$, defined as:

$$\tilde{z} = \tilde{x} \oplus \tilde{y} = \{z \in \mathbb{R}^{SL}_+ : \ \exists (x, y) \in (\tilde{x}, \tilde{y}) \ s.t. \ z = x + y\}.$$

A necessary condition for absence of arbitrage opportunities is that prices be additive, in the sense made precise below. All the proofs are collected in Appendix 1.

Proposition 1.

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[24] Another option would be to consider that he/she could receive an element of $\tilde{x}^t_j$, with $t \in P_j(s)$, and be forced to deliver an element of $\tilde{y}^{t'}_j$, with $t' \in P_j(s)$. This would reduce the arbitrage opportunities of an incompletely informed arbitrageur.

Absence of arbitrage opportunities implies that: $\forall \tilde{x}, \tilde{y} \in (\mathbb{F}(\mathbb{R}^L_+))^S,\ \tilde{p}(\tilde{x} \oplus \tilde{y}) = \tilde{p}(\tilde{x}) + \tilde{p}(\tilde{y})$.

With prices being additive in this sense, it costs the same to buy two or more lists or to buy a single list with the same possible deliveries. If the list that guarantees delivery of '*coffee*' or '*tea*' costs 3, and the list that guarantees delivery of a '*toast*' or a '*cookie*' costs 5, then the list that guarantees delivery of '*coffee and toast*' or '*coffee and cookie*' or '*tea and toast*' or '*tea and cookie*' must cost 8.

It is useful to define $\tilde{p}^s$ as the price of a list for delivery contingent on the occurrence of state s:

$$\tilde{p}^s :\ \mathbb{F}(\mathbb{R}^L_+) \longrightarrow \mathbb{R}_+;$$

$$\tilde{p}^s(\tilde{x}^s) = \tilde{p}(0, ..., \tilde{x}^s, ..., 0).$$

Observe that a plan of contingent lists, $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^S)$, is also the sum of the contingent lists: $\tilde{x} = \tilde{x}^1 \oplus ... \oplus \tilde{x}^S$. By Proposition 1, no arbitrage implies that the price of a plan of contingent lists is equal to the sum of the prices of the contingent lists:

$$\tilde{p}(\tilde{x}) = \sum_{s \in \Omega} \tilde{p}^s(\tilde{x}^s).$$

The classical assumptions on the price systems imply that it costs the same to buy a single bundle or to buy its constituents in separate (prices are linear):

(i) $\forall x, y \in \mathbb{R}^{SL}_+ :\ p(x + y) = p(x) + p(y)$;

(ii) $\forall x \in \mathbb{R}^{SL}_+, \lambda \in \mathbb{R} :\ p(\lambda x) = \lambda p(x)$.

The classical assumption (i) is a particular case of Proposition 1 (they are equivalent for lists with a single element). We only assume (*ii*) for singleton plans of lists, that is, plans of bundles.

Assumption 1.

Given any plan of lists with a single element, $x \in \mathbb{R}^{SL}_+$*, and any positive scalar,* $\lambda \geq 0$*:*

$$\tilde{p}(\lambda x) = \lambda \tilde{p}(x).$$

As a consequence of Proposition 1 and Assumption 1, the restriction of any price system, $\tilde{p}$, to the space of consumption plans can be represented by a vector of prices of the SL contingent commodities, $p \in \Delta^{SL}$, such that the price of a bundle is the inner product between the vector of prices and the vector of quantities:

$$\forall x \in \mathbb{R}_+^{SL},\ \tilde{p}(x) = p \cdot x,\ \text{ with } p \in \Delta^{SL} = \left\{ p \in \mathbb{R}_+^{SL} :\ \sum_{s \in \Omega} \sum_{l \in \mathcal{L}} p^{sl} = 1 \right\}.$$[25]

With P_i-measurable endowments, $e_i \in \mathbb{R}_+^{SL} \cap P_i$, the budget set of agent i becomes:

$$\tilde{B}_i(\tilde{p}) = \{\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S :\ \tilde{p}(\tilde{x}_i) \leq p \cdot e_i\}.$$

If a list, $\tilde{x}$, is cheaper than a list that contains it, $\tilde{y} \supset \tilde{x}$, there exists an arbitrage opportunity. An arbitrageur could buy $\tilde{x}$ and sell $\tilde{y} \supset \tilde{x}$, retaining some rent. In state s, he/she could use the goods received, $x^s \in \tilde{x}^s$, to keep the contract for delivery of $\tilde{y}^s$ (because $x^s \in \tilde{y}^s$).

Proposition 2.

Absence of arbitrage opportunities implies that: $\tilde{x} \subseteq \tilde{y} \ \Rightarrow \ \tilde{p}(\tilde{y}) \leq \tilde{p}(\tilde{x})$.

A corollary is that if a list, $\tilde{x}$, is more expensive than one of its alternatives, $x \in \tilde{x}$, then there exists an *arbitrage opportunity*. An arbitrageur would buy the bundle x and sell the list $\tilde{x}$, receiving x and delivering the same $x \in \tilde{x}$. The arbitrageur would get a null delivery, $x - x$, retaining some rent.

Corollary 1.

Absence of arbitrage opportunities implies that: $\forall x \in \tilde{x} :\ \tilde{p}(\tilde{x}) \leq p \cdot x$.

Another corollary is that the plan of lists $M_i(\tilde{x}_i)$, which describes what agent i may receive when he buys the plan of lists $\tilde{x}_i$, cannot be more expensive than $\tilde{x}_i$.

Corollary 2.

Absence of arbitrage opportunities implies that: $\tilde{p}(M_i(\tilde{x}_i)) \leq \tilde{p}(\tilde{x}_i)$.

We restrict our study to the set of price systems for which there are no arbitrage opportunities, denoted $\mathcal{P}$.

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[25] We denote by p the restriction of the $\tilde{p}$ to singleton plans of lists, that is, to consumption plans.

2.5 Preferences over lists

When buying a plan of lists, agents form expectations about the resulting consumption plan. These expectations, together with preferences over consumption plans, induce preferences over plans of lists.

We start by making standard assumptions on preferences over consumption plans. After studying the relationship between lists and deliveries, it will become clear how to obtain preferences over plans of lists from preferences over consumption plans.

Assumption 2.

Preferences over consumption plans are represented by an expected utility function, $U_i(x_i) = \sum_{s \in \Omega} \mu^s u_i^s(x_i^s)$*, where each state-dependent utility function,* $u_i^s : \mathbb{R}_+^L \to \mathbb{R}$*, is continuous, concave and strictly increasing.*[26]

In economies with uncertain delivery, agents choose a plan of lists, and therefore we need an objective function defined over plans of lists. Preferences over plans of lists depend on prices of lists, $\tilde{p} \in \mathcal{P}$, because rational agents see prices as a signal of the alternative that will be delivered:[27]

$$\tilde{U}_i : \ (\mathbb{F}(\mathbb{R}_+^L))^S \times \mathcal{P} \longrightarrow \mathbb{R}.$$

When an agent buys the plan of lists $\tilde{x}_i$, the possible deliveries are $M_i(\tilde{x}_i)$. Therefore, we assume that the agent attributes the same utility to the plans of lists $\tilde{x}_i$ and $M_i(\tilde{x}_i)$.[28] Knowing the utility of the P_i-measurable plans of lists, we can obtain, using only this assumption, the utility of all the plans of lists that are not P_i-measurable.

Assumption 3.

$$\forall (\tilde{x}_i, \tilde{p}) \in (\mathbb{F}(\mathbb{R}_+^L))^S \times \Delta^{SL} : \ \tilde{U}_i(\tilde{x}_i, \tilde{p}) = \tilde{U}_i(M_i(\tilde{x}_i), \tilde{p}).$$

Under this assumption, agents would not mind being restricted to select P_i-measurable plans of lists. They are never worse off by selecting $M_i(\tilde{x}_i)$ instead of $\tilde{x}_i$ (utility is the same and the price is not higher, by Corollary 2).

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[26] By strictly increasing, we mean that $x^s \geq y^s$ and $x^s \neq y^s$ implies that $u_i^s(x^s) > u_i^s(y^s)$.

[27] The precise relationship between prices, lists and deliveries will be established later (Proposition 3).

[28] Recall that if $\tilde{x}_i$ is P_i-measurable, then $\tilde{x}_i = M_i(\tilde{x}_i)$.

Finally, we make an assumption of *no satiation.* Agents select a list in the frontier of their budget sets.

Assumption 4.

Let $\tilde{x}_i \in \underset{\tilde{z}_i \in \tilde{B}_i(\tilde{p})}{\operatorname{argmax}} \tilde{U}_i(\tilde{z}_i, \tilde{p})$*. Then:* $\tilde{p}(\tilde{x}_i) = \tilde{p}(e_i)$*.*

We will find that, in equilibrium, agents always receive the cheapest of the possible deliveries (Proposition 3). Therefore, we will also assume that they attribute to a plan of lists the utility of the cheapest consumption plan that may be delivered.

2.6 Deliveries and expectations

In this subsection, we study the relationship between the plans of lists selected by the agents, $\tilde{x} = \{\tilde{x}_i\}_{i \in \mathcal{I}}$, and the resulting deliveries, $x = \{x_i\}_{i \in \mathcal{I}}$. These must constitute a feasible allocation, that is, $\sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} e_i$.

An individually optimal plan of lists, $\tilde{x}_i$, is in the frontier of the budget set (Assumption 4), and costs the same as the plan of possible deliveries, $M_i(\tilde{x}_i)$ (Assumption 3 and Corollary 2). Buying such a plan of lists, which of the possible consumption plans, $x_i \in M_i(\tilde{x}_i)$, should agent i expect to receive?

We show that if the resulting allocation, $x = \{x_i\}_{i \in \mathcal{I}}$, is feasible:

(1) the price of the plan of lists that is chosen, $\tilde{x}_i$, is equal to the price of the consumption plan that is delivered, x_i;

(2) the consumption plan that is delivered, x_i, is among the cheapest in $M_i(\tilde{x}_i)$.

Proposition 3.

Consider a price system with no arbitrage opportunities, $\tilde{p} \in \mathcal{P}$*, plans of lists that are in the frontier of the budget sets and that cost the same as the plan of possible deliveries,* $\tilde{x} = \{\tilde{x}_i\}_{i \in \mathcal{I}}$ *with* $\tilde{p}(\tilde{x}_i) = \tilde{p}(M_i(\tilde{x}_i)) = p \cdot e_i$, $\forall i \in \mathcal{I}$*, and a feasible allocation,* $x = \{x_i\}_{i \in \mathcal{I}}$*, that is compatible with the plans of possible deliveries,* $x_i \in M_i(\tilde{x}_i)$, $\forall i \in \mathcal{I}$*. Then, for each* $i \in \mathcal{I}$*:*

(1) $\tilde{p}(\tilde{x}_i) = \min_{z \in M_i(\tilde{x}_i)} \{p \cdot z\}$*;*

(2) $x_i \in \underset{z \in M_i(\tilde{x}_i)}{\operatorname{argmin}} \{p \cdot z\}$*.*

Knowledge of Proposition 3 induces rational agents to expect to receive the cheapest of the consumption plans in $M_i(\tilde{x}_i)$. We assume that in case of a tie for the cheapest bundle, agents expect to receive the consumption plan with the highest utility among the cheapest. This is in the spirit of the mechanism design literature, where incentive compatibility conditions only need to be satisfied in equality (in case of indifference, the agent is assumed to select the action that is preferred by the principal).[29]

Consider the cheapest consumption plans, at prices p, in the list $M_i(\tilde{x}_i)$, denoted $\tilde{Y}_i(\tilde{x}_i, p)$:

$$\tilde{Y}_i : \ (\mathbb{F}(\mathbb{R}^L_+))^S \times \Delta^{SL} \longrightarrow (\mathbb{F}(\mathbb{R}^L_+))^S;$$

$$\tilde{Y}_i(\tilde{x}_i, p) = \left\{ x_i \in \mathbb{R}^{SL}_+ : \ x_i \in \underset{z \in M_i(\tilde{x}_i)}{\operatorname{argmin}} \{p \cdot z\} \right\}.$$

The expected utility of a plan of lists, $\tilde{x}_i$, at prices p, is defined as:

$$V_i : \ (\mathbb{F}(\mathbb{R}^L_+))^S \times \Delta^{SL} \longrightarrow \mathbb{R};$$

$$V_i(\tilde{x}_i, p) = \max_{x_i \in \tilde{Y}_i(\tilde{x}_i, p)} U_i(x_i).$$

Observe that the preferences of rational agents over plans of lists only depend on $p \in \Delta^{SL}$, and not on its extension to lists, $\tilde{p} \in \mathcal{P}$.

Assumption 5.

$$\tilde{U}_i(\tilde{x}_i, \tilde{p}) = V_i(\tilde{x}_i, p).$$

The problem of agent i can, therefore, be written as:

$$\max_{\tilde{x}_i \in \tilde{B}_i(\tilde{p})} V_i(\tilde{x}_i, p).$$

---

[29] If an agent did not expect to receive the plan with the highest utility (among the cheapest in $M_i(\tilde{x}_i)$), he/she would prefer to modify the list very slightly, in order to have a single cheapest bundle. We could never have an individually optimal choice of a list $x \vee y$ with $p \cdot x = p \cdot y$, $U(x) > U(y)$ and $U(x) > \tilde{U}(x \vee y)$. The agent would prefer the list $(1-\epsilon)x \vee y$, that has an expected delivery of $(1-\epsilon)x$ (the strictly cheaper alternative), which has almost the same utility as x, implying that $\tilde{U}[(1-\epsilon)x \vee y] = U[(1-\epsilon)x] > \tilde{U}(x \vee y)$.

3 Equilibrium

3.1 Concept

Recall that we are considering an economy with a finite number of agents, $\mathcal{I} = \{1, ..., I\}$, who trade (at date 0) a finite number of commodities, $\mathcal{L} = \{1, ..., L\}$, under uncertainty about which of a finite number of possible states of nature, $\Omega = \{1, ..., S\}$, will occur (at date 1).

At date 0, taking as given the prices of plans of lists, $\tilde{p} \in \mathcal{P}$, agent i trades his/her contingent endowments, $e_i \in \mathbb{R}^{SL}_{++} \cap P_i$, for a plan of contingent lists, $\tilde{x}_i = (\tilde{x}_i^1, \tilde{x}_i^2, ..., \tilde{x}_i^S) \in (\mathbb{F}(\mathbb{R}^L_+))^S$, that specifies the bundles that may be delivered in each state of nature. At date 1, if state s occurs, agent i receives a bundle $x_i^s \in M_i^s(\tilde{x}_i) = \bigcup_{t \in P_i(s)} \tilde{x}_i^t$.

When buying a plan of lists, $\tilde{x}_i$, agent i expects to receive the cheapest consumption plan in $M_i(\tilde{x}_i)$. Therefore, the expected utility of the plan of lists, $\tilde{U}_i(\tilde{x}_i, \tilde{p})$, is the expected utility of this cheapest consumption plan, $V_i(\tilde{x}_i, p)$.[30]

An equilibrium is a situation in which, given prices $\tilde{p}^* \in \mathcal{P}$, each agent i chooses an individually optimal plan of lists, $\tilde{x}_i^*$, and receives a consumption plan, $x_i^* \in M_i(\tilde{x}_i^*)$ that has the anticipated utility, $V_i(\tilde{x}_i, p)$, with the allocation, $\{x_i^*\}_{i \in \mathcal{I}}$, being feasible.

Definition 2.

An equilibrium of an economy with uncertain delivery, $(\tilde{x}^*, x^*, \tilde{p}^*)$, is composed by: plans of lists, $\tilde{x}^* = (\tilde{x}_1^*, ..., \tilde{x}_I^*)$; an allocation, $x^* = (x_1^*, ..., x_I^*)$; and a price system, $\tilde{p}^* \in \mathcal{P}$. These are such that, for every agent $i \in \mathcal{I}$:

(1) The plan of lists, $\tilde{x}_i^* \in (\mathbb{F}(\mathbb{R}^L_+))^S$, maximizes expected utility, $V_i(\tilde{x}_i^*, p^*)$, in the agent's budget set, $\tilde{B}_i(\tilde{p}^*) = \left\{\tilde{x}_i \in (\mathbb{F}(\mathbb{R}^L_+))^S : \tilde{p}^*(\tilde{x}_i) \leq p^* \cdot e_i\right\}$;

(2) In each state of nature, $s \in \Omega$, the bundle that is delivered is an alternative that the agent has to accept, $x_i^{s*} \in \bigcup_{t \in P_i(s)} \tilde{x}_i^{t*}$, that is, $x_i^* \in M_i(\tilde{x}_i^*)$;

(3) The utility of the list is correctly anticipated, $U_i(x_i^*) = V_i(\tilde{x}_i^*, p^*)$;

(4) The allocation, $x^* \in (\mathbb{R}^{SL}_+)^I$, is feasible, $\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} e_i$.

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[30] Recall that p denotes the restriction of $\tilde{p}$ to consumption plans (plans of singleton lists).

3.2 Fundamental equilibria

The same equilibrium allocation, $\{x_i^*\}_{i\in\mathcal{I}}$, may be associated with many optimal choices of plans of lists, $\{\tilde{x}_i^*\}_{i\in\mathcal{I}}$, and many equilibrium price systems, $\tilde{p}^* \in \mathcal{P}$. In this subsection, we propose a refinement to eliminate this kind of multiplicity of equilibria.

Plans of lists that are not chosen in equilibrium may be strictly cheaper than the cheapest consumption plan that they would deliver. In this case, we can raise the price of these lists to equal the price of the cheapest consumption plan, remaining in equilibrium (if these plans of lists were not chosen by the agents before the price raise, then they remain not being chosen after their price goes up).

A price system is *fundamental* if and only if the price of a plan of lists, $\tilde{x}_i$, is equal to the price of the cheapest consumption plan that belongs to the plan of lists.

Definition 3.

The price system $\tilde{p}$ *is fundamental if and only if:* $\forall \tilde{z} \in (\mathbb{F}(\mathbb{R}_+^L))^S,\ \tilde{p}(\tilde{z}) = \min\limits_{z\in\tilde{z}}\{p\cdot z\}$.

If $\tilde{p}$ is fundamental: $\tilde{p}^s(\tilde{x}_i^s) = \min_{z_i^s\in\tilde{x}_i^s}\{p^s\cdot z_i^s\}$ and $\tilde{p}(\tilde{x}_i) = \sum_{s\in\Omega}\min_{z_i^s\in\tilde{x}_i^s}\{p^s\cdot z_i^s\}$. Thus, the budget restriction can be written as: $\tilde{B}_i(p) = \left\{\tilde{x}_i \in (\mathbb{F}(\mathbb{R}_+^L))^S : \sum\limits_{s\in\Omega}\min\limits_{z_i^s\in\tilde{x}_i^s}\{p^s\cdot z_i^s\} \leq p\cdot e_i\right\}$.

The refinement of the equilibrium set that we propose consists in restricting the price system to be fundamental and to remove the irrelevant alternatives in the lists (those that do not affect the price of the list and that are never delivered), setting $\tilde{x}^* = M(x^*)$. We designate such equilibria as *fundamental equilibria.*

Definition 4.

The pair (x^*, p^*) *is a fundamental equilibrium if and only if* $(M(x^*), x^*, \tilde{p}^*)$ *is an equilibrium, with* $\tilde{p}^*$ *being a fundamental price system defined by* $\tilde{p}^*(\tilde{z}) = \min\limits_{z\in\tilde{z}}\{p^*\cdot z\},\ \forall\tilde{z}\in(\mathbb{F}(\mathbb{R}_+^L))^S$.

For every equilibrium of the economy with uncertain delivery, $(\tilde{x}^*, x^*, \tilde{p}^*)$, there exists a fundamental equilibrium, (x^*, p^*), that is equivalent in the sense that: (i) the allocation is the same; and (ii) the prices of the consumption plans coincide, $p^*\cdot z = \tilde{p}^*(z),\ \forall z\in\mathbb{R}_+^{SL}$.

Proposition 4.

Let $(\tilde{x}^*, x^*, \tilde{p}^*)$ *be an equilibrium of the economy with uncertain delivery. Recall that the restriction of* $\tilde{p}^* \in \mathcal{P}$ *to singleton lists is denoted by* $p^* \in \Delta^{SL}$*. Let* $\tilde{q}^*(\tilde{z}) = \min_{z \in \tilde{z}} \{p^* \cdot z\}$*. Then:*

(i) $\forall \tilde{y}^*$ *s.t.* $x^* \subseteq \tilde{y} \subseteq M(\tilde{x}^*) :$ $(\tilde{y}^*, x^*, \tilde{q}^*)$ *is also an equilibrium.*

(ii) (x^*, p^*) *is a fundamental equilibrium.*[31]

3.3 Deliverable consumption plans

Suppose that agent i buys a plan of singleton lists for delivery in two possible states of nature, $\tilde{x} = (x^s, x^t)$. If the agent can distinguish states s and t, then $M(\tilde{x}) = (x^s, x^t)$, and thus delivery of x^s in state s and x^t in state t is guaranteed. But if the agent cannot distinguish the two states, then we have $M(\tilde{x}) = (x^s \vee x^t, x^s \vee x^t)$. As a result: the agent receives, in state s, the cheapest of the two alternatives according to p^s; and, in state t, the cheapest according to p^t.

The bundle that is delivered in state s cannot be more expensive (according to prices for delivery in state s) than any of the bundles promised for delivery in states $t \in P_i(s)$:

$$\forall t \in P_i(s),\ p^s \cdot x^s \leq p^s \cdot x^t.$$

The set of deliverable consumption plans depends on prices. This dependence is described by the deliverability correspondence, defined below.

Definition 5.

$$C_i : \Delta^{SL} \twoheadrightarrow \mathbb{R}_+^{SL};$$

$$C_i(p) = \left\{ x_i \in \mathbb{R}_+^{SL} :\ \forall s \in \Omega,\ p^s \cdot x_i^s = \min_{t \in P_i(s)} \{p^s \cdot x_i^t\} \right\}.$$

A consumption plan is deliverable, $x_i \in C_i(p)$, if and only if there exists a P_i-measurable plan of lists of which x_i is the cheapest alternative. It is enough to check whether or not x_i is the cheapest alternative in the list $M_i(x_i)$.

We can formulate the problem of the agent, equivalently, as a choice over lists in $\tilde{B}_i(p)$ or as a choice over consumption plans in $B_i(p) \cap C_i(p)$, with $B_i(p) = \left\{ x_i \in \mathbb{R}_+^{SL} :\ p \cdot x_i \leq p \cdot e_i \right\}$. The following propositions make this precise.

---

[31] Notice that the restriction of $\tilde{q}^*$ to singleton lists, q^*, coincides with p^*.

Proposition 5.

Let $\tilde{p}$ be a fundamental price system. Then:

$$\tilde{x}_i \in \operatorname*{argmax}_{\tilde{x}_i \in \tilde{B}_i(\tilde{p})} V_i(\tilde{x}_i, p) \;\Rightarrow\; \exists x_i \in M_i(\tilde{x}_i) \text{ s.t. } U_i(x_i) = V_i(\tilde{x}_i, p) \text{ and } x_i \in \operatorname*{argmax}_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i).$$

Proposition 6.

Let $\tilde{p}$ be a fundamental price system. Then:

$$x_i \in \operatorname*{argmax}_{x_i \in B_i(p) \cap C_i(p)} U_i(x_i) \;\Rightarrow\; M_i(x_i) \in \operatorname*{argmax}_{\tilde{x}_i \in \tilde{B}_i(\tilde{p})} V_i(\tilde{x}_i, p).$$

This equivalence allows us to reformulate the notion of fundamental equilibrium.

Proposition 7.

The pair $(x^*, p^*) \in (\mathbb{R}^{SL}_+)^I \times \Delta^{SL}$ is a fundamental equilibrium if and only if:

(1) Each agent's choice is optimal, $x_i^* \in \operatorname*{argmax}_{x_i \in B_i(p^*) \cap C_i(p^*)} U_i(x_i)$*;*

(2) The allocation, x^*, is feasible. That is, $\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} e_i$.

This alternative definition of equilibrium does not use preferences over lists, $\tilde{U}_i$, nor prices over lists, $\tilde{p}$. Therefore, we can compare it with the equilibrium with public state verification (Arrow-Debreu-McKenzie under uncertainty) and the equilibrium with differential information proposed by Radner (1968). Everything boils down to the choice sets. In Arrow-Debreu-McKenzie: $X_i^{AD} = \mathbb{R}^{SL}_+$; in Radner (1968): $X_i^R = \mathbb{R}^{SL}_+ \cap P_i$; in an economy with uncertain delivery: $X_i(p) = \mathbb{R}^{SL}_+ \cap C_i(p)$. It should be clear that, for all $p \in \Delta^{SL}$, $X_i^{AD} \subseteq X_i(p) \subseteq X_i^R$.

4 Existence of equilibrium

If the correspondence from prices to the deliverable budget set, $B_i(p) \cap C_i(p)$, were continuous, establishing existence of equilibrium would be straightforward. But, as we illustrate in Appendix 3, $C_i(p)$ is not lower hemicontinuous (this property fails when prices in some state are null, or when prices in two indistinguished states are collinear).

4.1 A sequence of economies

In order to establish existence of equilibrium, we construct a sequence of economies. In these economies, the choice set is not constrained to satisfy the deliverability conditions. But violating these constraints implies an utility penalty. The penalty is a function of the difference between the cheapest consumption plan and the consumption plan that is delivered.[32]

In the economy $\mathcal{E}^j$, if state s occurs, the utility penalty imposed on agent i is:

$$Z_i^{sj}(x_i, p) = j \max_{t \in P_i(s)} \{p^s \cdot x_i^s - p^s \cdot x_i^t\}.$$

Since $s \in P_i(s)$, the maximum is at least zero, thus penalties are never negative. Penalties are scaled up along the sequence of economies, and this is actually the only difference among the economies in the sequence.

In the economy $\mathcal{E}^j$, the utility functions of the agents are:

$$U_i^j(x_i, p) = U_i(x_i) - j \sum_{s \in \Omega} \mu^s \max_{t \in P_i(s)} \{p^s \cdot x_i^s - p^s \cdot x_i^t\}.$$

For any $j \in \mathbb{N}$, the utility functions are continuous in consumption plans and prices, $(x_i, p) \in \mathbb{R}_+^{SL} \times \Delta^{SL}$. The maximum of linear functions is a convex function, and multiplying a convex function by a negative constant, $-j$, yields a concave function. Hence, the objective function, $U_i^j(x_i, p)$, is concave in the first variable. Observe also that the utility penalty preserves the property of *no satiation.* The plan $x_i + \epsilon\bar{1}$ is always preferred to x_i (the utility penalty is kept constant). The fact that the utility functions depend (continuously) on prices does not interfere with existence of equilibrium.[33]

Lemma 1.

Let $\mathcal{E}^j$ be an Arrow-Debreu-McKenzie economy in which each agent $i \in \mathcal{I}$:

- has strictly positive initial endowments, $e_i \in \mathbb{R}_{++}^{SL}$;

- maximizes the utility function $U_i^j(x_i, p) = U_i(x_i) - j \sum_{s \in \Omega} \mu^s \max_{t \in P_i(s)} \{p^s \cdot x_i^s - p^s \cdot x_i^t\}$, with $U_i(x_i)$ continuous, concave and strictly increasing.

Then, there exists an Arrow-Debreu-McKenzie equilibrium, (x^j, p^j).

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[32] These economies are an artifice to establish existence of equilibrium.

[33] With price dependent preferences, it is known that equilibrium exists (Arrow and Hahn, 1971). In the context of economies with uncertain delivery, see our previous paper (2008).

The sequence of economies, $\{\mathcal{E}^j\}_{j\in\mathbf{N}}$, has a sequence of equilibria, $\{(x^j, p^j)\}_{j\in\mathbf{N}}$, that lies in the compact set $[0, e_T]^I \times \Delta^{SL}$, where $e_T = \sum_{i\in\mathcal{I}} e_i$. There exists a subsequence that converges. In order to show that the limit, (x^*, p^*), is a fundamental equilibrium of the original economy with uncertain delivery, we must prove that the following conditions are satisfied:

(1) Feasibility: $\sum_{i\in\mathcal{I}} x_i^* \leq \sum_{i\in\mathcal{I}} e_i$;

(2) Budget restriction: $\forall i \in \mathcal{I},\ p^* \cdot x_i^* \leq p^* \cdot e_i$;

(3) Deliverability: $\forall i \in \mathcal{I},\ x_i^* \in C_i(p^*)$;

(4) Individual optimality: $\forall i \in \mathcal{I},\ x_i \in B_i(p^*) \cap C_i(p^*) \Rightarrow U_i(x_i^*) \geq U_i(x_i)$.

It is easy to show that the first three conditions are satisfied.

Lemma 2.

Consider a sequence of economies $\{\mathcal{E}^j\}_{j\in\mathbf{N}}$ defined as in Lemma 1, and a corresponding sequence of equilibria, $\{(x^j, p^j)\}_{j\in\mathbf{N}}$.

Then, the sequence of equilibria has an accumulation point, (x^*, p^*), that satisfies:

(1) Feasibility: $\sum_{i\in\mathcal{I}} x_i^* \leq \sum_{i\in\mathcal{I}} e_i = e_T$;

(2) Budget restriction: $\forall i \in \mathcal{I}, p^* \cdot x_i^* \leq p^* \cdot e_i$;

(3) Deliverability: $\forall i \in \mathcal{I}, x_i^* \in C_i(p^*)$.

The difficult part of the proof is to verify condition (4): that the limit, (x^*, p^*), maximizes utility in the deliverable budget set, $B_i(p^*) \cap C_i(p^*)$. The fact that C_i is not lower hemicontinuous (as shown in Appendix 3) could prevent (x^*, p^*) from being optimal. There could be a deliverable consumption plan $y_i \in B_i(p^*) \cap C_i(p^*)$ that is not even nearly deliverable in the economies in the sequence. In spite of having a low utility level for high j (because of the penalty), this plan could be optimal in the original economy, and, in this case, (x^*, p^*) would not be an equilibrium (an example of non-existence of equilibrium is given in Appendix 2).

We start by showing that, if any state of nature, $s \in \Omega$ can be verified by at least one of the agents, then p^* is strictly positive.

Lemma 3. *Consider a sequence of economies $\{\mathcal{E}^j\}_{j\in\mathbf{N}}$ defined as in Lemma 1, a corresponding sequence of equilibria, $\{(x^j, p^j)\}_{j\in\mathbf{N}}$, and an accumulation point, (x^*, p^*).*

Suppose that for every $s \in \Omega$*, there exists* $i \in \mathcal{I}$ *with* $P_i(s) = \{s\}$*. Then,* $p^* \gg 0$*.*

4.2 The existence result

If agents have P_i-measurable endowments and preferences, and if any state can be verified by at least one of the agents, existence of equilibrium is guaranteed.

Theorem 1.

Consider an economy with uncertain delivery, $\mathcal{E} \equiv (e_i, u_i, \mu_i, P_i)_{i \in \mathcal{I}}$*, such that:*

- Preferences are represented by an expected utility function, $U_i(x_i) = \sum_{s \in \Omega} \mu^s u_i^s(x_i^s)$*, where each* $u_i^s : \mathbb{R}_+^L \to \mathbb{R}$ *is continuous, concave and strictly increasing;*

- Contingent preferences are the same in indistinguished states, $t \in P_i(s) \Rightarrow u_i^t(\cdot) = u_i^s(\cdot)$*;*

- Initial endowments are strictly positive and constant across indistinguished states: $e_i \in \mathbb{R}_{++}^{SL} \cap P_i = \left\{ e_i \in \mathbb{R}_{++}^{SL} : \ t \in P_i(s) \Rightarrow e_i^t = e_i^s \right\}$*;*

- For each $s \in \Omega$*, there exists* $i \in \mathcal{I}$ *with* $P_i(s) = \{s\}$*.*

Then, there exists an equilibrium of the economy with uncertain delivery.

The strategy of the proof is to assume (by way of contradiction) that there exists a x_i' in $B_i(p^*) \cap C_i(p^*)$ that is preferred to x_i^*, and then find that there exists a similar x_i which belongs to $B_i(p^j) \cap C_i(p^j)$, for large j. This contradicts that (x^j, p^j) is an equilibrium of $\mathcal{E}^j$, because x_i would also be preferred to x_i^j in the economy $\mathcal{E}^j$.

5 Concluding remarks

We have introduced a new general equilibrium model of trade *ex ante* with differential information. The agents make contingent trade contracts before receiving their private information, and then use their private information to enforce contracts.

It is assumed that agents cannot use the observed prices, in a court of law, to prove that some event has occurred. If agents could use prices, then the outcome of the economy with differential information would coincide with the outcome of an economy in which all the information is shared (Radner, 1979; Allen, 1981). In our scenario, differential information matters.

Another crucial assumption is that the information provided by the agents to enforce contracts is not aggregated by any institution. Otherwise, such an institution could announce the true state of nature, and state verification would become public.

We realize that agents find it useful to trade lists, which are a sort of incomplete contracts (an agent that buys a list has to accept any possible outcome compatible with the list). We have shown that these contracts are enforceable if and only if agents select P_i-measurable plans of lists.

A no-arbitrage argument implies that the price of a plan of lists is equal to the price of the cheapest consumption plan that is compatible with the plan of lists. In equilibrium, it is this cheapest consumption plan that is delivered.

To establish existence of equilibrium, we assumed that every state of nature can be verified by at least one agent. It is easier to accept this hypothesis if the economy has much more agents than states of nature. Without such an assumption, equilibrium may not exist, as shown by the counter-example in Appendix 2.

The consequences of allowing agent to trade after receiving their information remain to be examined. This is left for future research.

Appendix 1: The proofs

Proof of Proposition 1:

Let $\tilde{z}_i = \tilde{x}_i \oplus \tilde{y}_i = \{z_i \in \mathbb{R}_+^{SL} : \exists x_i \in \tilde{x}_i,\ y_i \in \tilde{y}_i,\ z_i = x_i + y_i\}$.

If $\tilde{p}(\tilde{z}_i) < \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i)$, then an arbitrageur can buy $\tilde{z}_i$ and sell both plans of lists $\tilde{x}_i$ and $\tilde{y}_i$. By construction of $\tilde{z}_i$, for each $z_i^s \in \tilde{z}_i^s$, there exist $x_i^s \in \tilde{x}_i^s$ and $y_i^s \in \tilde{y}_i^s$ such that $x_i^s + y_i^s = z_i^s$. When receiving z_i^s, the agent has enough resources to deliver x_i^s and y_i^s, in order to keep the contracts for delivery of $\tilde{x}_i$ and $\tilde{y}_i$. In the process, the arbitrageur retains some rent.

If $\tilde{p}(\tilde{z}_i) > \tilde{p}(\tilde{x}_i) + \tilde{p}(\tilde{y}_i)$, then an arbitrageur can sell $\tilde{z}_i$ and buy both plans of lists $\tilde{x}_i$ and $\tilde{y}_i$. Receiving $x_i^s \in \tilde{x}_i^s$ and $y_i^s \in \tilde{y}_i^s$, the agent delivers $z_i^s = x_i^s + y_i^s$, keeping the contract for delivery of $\tilde{z}_i$. Again, the arbitrageur retains some rent. **QED**

Proof of Proposition 2:

If $\tilde{p}(\tilde{x}) < \tilde{p}(\tilde{y})$, an arbitrageur that buys $\tilde{x}$ and sells $\tilde{y}$ retains some rent.

In each state of nature, s, the arbitrageur can use exactly what is received, $x^s \in \tilde{x}^s$, to keep the contract for delivery of $\tilde{y}^s$, because $x^s \in \tilde{y}^s$. **QED**

Proof of Proposition 3:

Since $p \in \mathcal{P}$ and $x_i \in M_i(\tilde{x}_i)$, we know that $\tilde{p}(\tilde{x}_i) = \tilde{p}(M_i(\tilde{x}_i)) \leq p \cdot x_i,\ \forall i \in \mathcal{I}$.

Suppose that $\exists i \in \mathcal{I} :\ \tilde{p}(\tilde{x}_i) < p \cdot x_i$. Summing across agents: $\sum_{i \in \mathcal{I}} \tilde{p}(\tilde{x}_i) < \sum_{i \in \mathcal{I}} p \cdot x_i$.

But $\sum_{i \in \mathcal{I}} \tilde{p}(\tilde{x}_i) = \sum_{i \in \mathcal{I}} p \cdot e_i$, therefore, the allocation is not feasible:

$$\sum_{i \in \mathcal{I}} p \cdot e_i < \sum_{i \in \mathcal{I}} p \cdot x_i \ \Rightarrow\ \exists (s,l) \in \Omega \times \mathcal{L},\ \sum_{i \in \mathcal{I}} e_i^{sl} < \sum_{i \in \mathcal{I}} x_i^{sl}.$$

Contradiction. Then: $\tilde{p}(\tilde{x}_i) = \tilde{p}(M_i(\tilde{x}_i)) = p \cdot x_i$.

Using Corollary 1, we finish the proof:

$$\begin{cases} \tilde{p}(\tilde{x}_i) \leq \min\limits_{z \in M_i(\tilde{x}_i)} \{p \cdot z\} \\ \tilde{p}(\tilde{x}_i) = p \cdot x_i \geq \min\limits_{z \in M_i(\tilde{x}_i)} \{p \cdot z\} \end{cases} \Rightarrow \tilde{p}(\tilde{x}_i) = p \cdot x_i = \min_{z \in M_i(\tilde{x}_i)} \{p \cdot z\}.$$

QED

Proof of Proposition 4:

If $(\tilde{x}^*, x^*, \tilde{p}^*)$ is an equilibrium, then $(M(\tilde{x}^*), x^*, \tilde{p}^*)$ is also an equilibrium, because the price of $M(\tilde{x}^*)$ is not higher (Corollary 2) and the utility is the same (Assumption 3). If $\tilde{x}_i$ solves the problem of the agent, then $M_i(\tilde{x}_i)$ also does.

If $M_i(\tilde{x}^*)$ solves the problem of agent i under prices $\tilde{p}^*$, then it also solves the problem of the agent under prices $\tilde{q}^*$, because: (i) the price of $M_i(\tilde{x}^*)$ remains the same, $\tilde{q}^*(M_i(\tilde{x}^*)) = \tilde{p}^*(M_i(\tilde{x}^*))$; (ii) prices of other lists do not decrease, $\forall z \in (\mathbb{F}(\mathbb{R}_+^L))^S :\ \tilde{q}^*(\tilde{z}) \geq \tilde{p}^*(\tilde{z})$; and (iii) preferences remain the same, $q^* = p^* \Rightarrow \tilde{U}_i(\cdot, \tilde{q}^*) = \tilde{U}_i(\cdot, \tilde{p}^*)$. The price and the utility of $M_i(x_i^*)$ and $M_i(\tilde{x}_i^*)$ are the same, thus $(M(x^*), x^*, \tilde{q}^*)$ is also an equilibrium of the economy with uncertain delivery.

The same applies to any $\tilde{y}^*$ s.t. $x^* \subseteq \tilde{y} \subseteq M(\tilde{x}^*)$. **QED**

Proof of Proposition 5:

Suppose that there exists $y_i \in B_i(p) \cap C_i(p)$ that is preferred to x_i: $U_i(y_i) > U_i(x_i) = \tilde{U}_i(\tilde{x}_i, \tilde{p})$.

Since $y_i \in C_i(p)$: $\tilde{U}_i(M_i(y_i), \tilde{p}) \geq U_i(y_i) > U_i(x_i) = \tilde{U}_i(\tilde{x}_i, \tilde{p})$.

Since $\tilde{p}$ is a fundamental price system: $\tilde{p}(\tilde{x}_i) = p \cdot x_i$ and $\tilde{p}(M_i(y_i)) = p \cdot y_i$.

If $y_i \in B_i(p)$, then $M_i(y_i) \in \tilde{B}_i(\tilde{p})$. Contradiction. **QED**

Proof of Proposition 6:

We know that $\tilde{U}_i(M_i(x_i), \tilde{p}) = U_i(x_i)$.

Suppose that there exists $\tilde{y}_i \in \tilde{B}_i(\tilde{p})$ that is preferred to $M_i(x_i)$: $\tilde{U}_i(\tilde{y}_i, \tilde{p}) > \tilde{U}_i(M_i(x_i), \tilde{p})$.

Pick the cheapest consumption plan in $\tilde{y}_i$, $y_i = Y_i(\tilde{y}_i, p)$ (one with the highest utility, in case of a tie). Then: $U_i(y_i) = \tilde{U}_i(\tilde{y}_i, \tilde{p}) > \tilde{U}_i(M_i(x_i), \tilde{p}) = U_i(x_i)$.

Since $\tilde{p}$ is a fundamental price system: $\tilde{y}_i \in \tilde{B}_i(\tilde{p}) \Rightarrow y_i \in B_i(p)$. Contradiction. **QED**

Proof of Proposition 7:

The proof follows from Definitions 2 and 4 and from Propositions 5 and 6. **QED**

Proof of Lemma 1:

Restrict the choice set to the compact $[0, T]$, with $T = 2\sum_{i \in \mathcal{I}} e_i$.

Consider correspondences, $\{\psi_i\}_{i \in \mathcal{I}}$, which assign to given prices, p, consumption plans, x'_i, that maximize $U_i^j(x_i, p)$ in the restricted budget set, $\overline{B}_i(p) = B_i(p) \cap [0, T]$:

$$\psi_i : [0, T]^I \times \Delta^{SL} \longrightarrow [0, T];$$

$$x'_i \in \psi_i(x, p) \Leftrightarrow x'_i \in \operatorname*{argmax}_{x_i \in \overline{B}_i(p)} U_i^j(x_i, p).$$

Consider also a correspondence, ψ_p, that assigns to the total demand, $\sum_{i \in \mathcal{I}} x_i$, the prices, p', which maximize the value of excess demand:

$$\psi_p : [0, T]^I \times \Delta^{SL} \longrightarrow \Delta^{SL};$$

$$p' \in \psi_p(x, p) \Leftrightarrow p' \in \operatorname*{argmax}_{p \in \Delta^{SL}} \left\{ p \cdot \sum_{i \in \mathcal{I}} (x_i - e_i) \right\}.$$

The objective functions, $\{U_i^j\}_{i \in \mathcal{I}}$ and $p \cdot \sum_{i \in \mathcal{I}} (x_i - e_i)$, are continuous, and $\overline{B}_i(p)$ is a continuous correspondence. We can, therefore, use Berge's Maximum Theorem to show that each of the correspondences, $\{\psi_i\}_{i \in \mathcal{I}}$ and ψ_p, is upper hemicontinuous with non-empty and compact values. They also have convex values because the objective functions are concave. The product correspondence retains these properties and maps a compact set into itself:

$$\psi \equiv \prod_{i=1}^{I} \psi_i \times \psi_p;$$

$$\psi : [0,T]^I \times \Delta^{SL} \longrightarrow [0,T]^I \times \Delta^{SL};$$

$$(x', p') \in \psi(x,p) \Leftrightarrow x'_i \in \psi_i(x,p),\ \forall i \in \mathcal{I} \text{ and } p' \in \psi_p(x,p).$$

Existence of a fixed-point, (x^*, p^*), follows from Kakutani's Theorem.

It is clear that x_i^* solves the problem of agent i.

The fact that p^* maximizes the value of excess demand implies that:

$$p' \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) \leq p^* \cdot \sum_i (x_i^* - e_i) \leq 0, \text{ for all } p' \in \Delta^{SL}.$$

Making $p' = e^{sl} = (0, ..., 1, ..., 0)$, for each $(s,l) \in \Omega \times \mathcal{L}$, shows that x^* is feasible: $\sum_{i \in \mathcal{I}} (x_i^* - e_i) \leq 0$.

The usual extension from $[0,T]^I$ to $(\mathbb{R}_+^{SL})^I$ applies.

QED

Proof of Lemma 2:

Consider only the subsequence that converges to (x^*, p^*). Ignore the remaining terms of the sequence.

(1) The set of feasible allocations is closed, and the limit allocation, $\{x_i^*\}_{i \in \mathcal{I}}$, is the limit of a sequence of feasible allocations, therefore it is feasible ($\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} e_i$).

(2) the limit allocation is the limit of a sequence of allocations in the sequence of budget sets ($x_i^j \in B_i(p^j), \forall i \in \mathcal{I}$), therefore, it also belongs to the limit budget set ($x_i^* \in B_i(p^*), \forall i \in \mathcal{I}$).

Suppose that x_i^* does not satisfy the budget restriction of agent i. Let $\alpha = 2\|e_T\| + 1$, and select $\epsilon > 0$ such that $p^* \cdot x_i^* - p^* \cdot e_i = \alpha\epsilon$. Choosing a sufficiently high j, we can guarantee that $\|x^* - x^j\| < \epsilon$ and $\|p^* - p^j\| < \epsilon$. With $p^j = p^* + dp$, $x^j = x_i^* + dx_i$, and manipulating:

$$(p^* + dp) \cdot (x_i^* + dx_i) - (p^* + dp) \cdot e_i = p^* \cdot x_i^* - p^* \cdot e_i + p^* \cdot dx_i + dp \cdot x_i^* + dp \cdot dx_i - dp \cdot e_i =$$

$$= \alpha\epsilon + (p^* + dp) \cdot dx_i + dp \cdot (x_i^* - e_i) > \alpha\epsilon - \epsilon - \epsilon \cdot 2\|e_T\| = 0.$$

This means that $x_i^j \notin B_i(p^j)$. Contradiction.

(3) To verify that the limit allocation, x^*, satisfies the deliverability restrictions in the limit economy, suppose that x_i^* violated one of the restrictions by more than $\delta > 0$. then, for sufficiently high j, x_i^j would also violate the same restriction by more than δ. For $t \in P_i(s)$, $\exists j_0 \in \mathbb{N}$:

$$p^{s*} \cdot x_i^{s*} > p^{s*} \cdot x_i^{t*} + \delta \Rightarrow p^{sj} \cdot x_i^{sj} > p^{sj} \cdot x_i^{tj} + \delta, \text{ for all } j > j_0.$$

Utility among feasible allocations is bounded by $U_i(e_T)$, so we can consider a j that is sufficiently high for $j\delta > U_i(e_T) - U_i(e_i)$. It would follow that $U_i^j(x_i^j) < U_i(x_i^j) - j\delta < U_i(x_i^j) - U_i(e_T) + U_i(e_i) < U_i(e_i) = U_i^j(e_i)$. Contradiction. **QED**

Proof of Lemma 3:

Consider a subsequence of equilibria, $\{(x^n, p^n)\}_{n\in\mathbb{N}}$, that converges to (x^*, p^*), and assume (by way of contradiction) that $\exists (s,l) \in \Omega \times \mathcal{L} : \; p^{sl*} = 0$.

Consider agent i with $P_i(s) = \{s\}$ and his/her optimal choices, $x_i^n \in \underset{z\in B_i(p^n)}{\operatorname{argmax}} U_i^n(z)$.

We know that $x_i^n \leq e_T, \forall n \in \mathbb{N}$. Then, $x_i^* \leq e_T$.

Observe that $u_i^s(z^s) > u_i^s(x_i^{s\;n}) \Rightarrow p^{s\;n} \cdot z^s > p^{s\;n} \cdot x_i^{s\;n}$, otherwise z^s would have been chosen instead of $x_i^{s\;n}$ in the plan x_i^n (this is only true because $P_i(s) = \{s\}$).

Define y_i^s by adding some of the free good, (s,l), to x_i^s. By strict monotonicity: $u_i^s(y_i^s) > u_i^s(x_i^s)$. By continuity, there exists some $\alpha \in (0,1)$ such that $u_i^s(\alpha y_i^s) > u_i^s(x_i^s)$. For large n, we also have $u_i^s(\alpha y_i^s) > u_i^s(x_i^{s\;n})$. This implies that $p^{s\;n} \cdot (\alpha y_i^s) > p^{s\;n} \cdot x_i^{s\;n}$, and, in the limit: $p^s \cdot (\alpha y_i^s) \geq p^s \cdot x_i^s$. Therefore, $p^s \cdot y_i^s > p^s \cdot x_i^s$. Contradiction.

QED

Proof of Theorem 1:

Given Lemmas 1 and 2, all that is left to prove is (4), which states that the limit of the sequence of equilibria, (x^*, p^*), is composed by optimal choices in the original economy with uncertain delivery, that is: $x_i \in B_i(p^*) \cap C_i(p^*) \Rightarrow U_i(x_i^*) \geq U_i(x_i), \forall i \in \mathcal{I}$.

By Lemma 3, we are sure that $p^* \gg 0$.

Assume (by way of contradiction) that there exists $y_i \in B_i(p^*) \cap C_i(p^*)$ such that $U_i(y_i) > U_i(x_i^*)$. We will show that this implies that (x^j, p^j) is not an equilibrium of $\mathcal{E}^j$, for high j.

A preliminary remark

Suppose that prices for delivery in s and in $t \in P_i(s)$ are parallel: $p^{*s} = ap^{*t}$. The two deliverability conditions that involve prices p^{*s} and p^{*t} yield equalities:

$$\begin{cases} p^{*s} \cdot y_i^s \leq p^{*s} \cdot y_i^t \\ p^{*t} \cdot y_i^t \leq p^{*t} \cdot y_i^s \end{cases} \Leftrightarrow \begin{cases} ap^{*t} \cdot y_i^s \leq ap^{*t} \cdot y_i^t \\ p^{*t} \cdot y_i^t \leq p^{*t} \cdot y_i^s \end{cases} \Leftrightarrow \begin{cases} p^{*s} \cdot y_i^s = p^{*s} \cdot y_i^t \\ p^{*t} \cdot y_i^t = p^{*t} \cdot y_i^s. \end{cases}$$

The two consumption bundles, y_i^s and y_i^t, cost the same in both states. Since $t \in P_i(s)$, we have $u_i^s = u_i^t$. If $u_i^s(y_i^s) > u_i^s(y_i^t)$, then the agent would be better off selecting y_i^s for consumption in both states. Thus, we must have $u_i^s(y_i^s) = u_i^s(y_i^t)$. Since the utility functions are concave, the agent is not worse off consuming the average bundle in both states. Notice that if the original vector satisfies the deliverability conditions, then this average vector also does.

Construct x_i'' by modifying y_i, considering the average bundle whenever there are parallel prices. Therefore, we have $x_i^{\prime\prime s} = x_i^{\prime\prime t}$ whenever $p^{*s} = ap^{*t}$.

Strategy of the proof

Reformulating, we assume (by way of contradiction) that there exists a $x_i'' \in B_i(p^*) \cap C_i(p^*)$ such that $U_i(x_i'') > U_i(x_i^*)$, with $x_i^{\prime\prime s} = x_i^{\prime\prime t}$ whenever $p^{*s} = ap^{*t}$.

By continuity of U_i, there exists $\delta > 0$ such that $x_i' = (1-\delta)x_i''$ is strictly preferred to x_i^*, belongs to $C_i(p^*)$, is in the interior of $B_i(p^*)$, and is also in the interior of $B_i(p^j)$, for high j:

$$U_i(x_i') > U_i(x_i^*); \quad x_i' \in C_i(p^*); \quad p^* \cdot x_i' < p^* \cdot e_i; \quad p^j \cdot x_i' < p^j \cdot e_i, \forall j \geq j_0.$$

Again, by continuity of U_i, there exists $\epsilon > 0$ such that $d(x_i, x_i') < \epsilon$ implies that $U_i(x_i) > U_i(x_i^*)$, with x_i in the interior of $B_i(p^*)$. For $j \geq j_1 \geq j_0$, $U_i(x_i) > U_i(x_i^j)$ (notice that we are considering U_i and not U_i^j) and x_i is in the interior of $B_i(p^j)$.

Let $j_2 \geq j_1$ be sufficiently high for $d(p^j, p^*) < \epsilon, \forall j \geq j_2$.

Consider, for easiness of exposition and without loss of generality, the following element of the agent's information partition: $P_i(s) = \{1, ..., s\}$. It should be clear that this reasoning extends to any element of P_i. Since $x_i' \in C_i(p^*)$, the deliverability conditions are satisfied:

$$\left\{\begin{array}{l}
p^{*1} \cdot x_i^{\prime 1} \leq p^{*1} \cdot x_i^{\prime 2}; \\
... \\
p^{*1} \cdot x_i^{\prime 1} \leq p^{*1} \cdot x_i^{\prime s}; \\
p^{*2} \cdot x_i^{\prime 2} \leq p^{*2} \cdot x_i^{\prime 1}; \\
... \\
p^{*2} \cdot x_i^{\prime 2} \leq p^{*2} \cdot x_i^{\prime s}; \\
... \\
... \\
p^{*s} \cdot x_i^{\prime s} \leq p^{*s} \cdot x_i^{\prime 1}; \\
... \\
p^{*s} \cdot x_i^{\prime s} \leq p^{*s} \cdot x_i^{\prime s-1}.
\end{array}\right.
\Leftrightarrow
\left\{\begin{array}{l}
p^{*1} \cdot x_i^{\prime 2} - p^{*1} \cdot x_i^{\prime 1} = k^{12} \geq 0; \\
... \\
p^{*1} \cdot x_i^{\prime s} - p^{*1} \cdot x_i^{\prime 1} = k^{1s} \geq 0; \\
p^{*2} \cdot x_i^{\prime 1} - p^{*2} \cdot x_i^{\prime 2} = k^{21} \geq 0; \\
... \\
p^{*2} \cdot x_i^{\prime s} - p^{*2} \cdot x_i^{\prime 2} = k^{2s} \geq 0; \\
... \\
... \\
p^{*s} \cdot x_i^{\prime 1} - p^{*s} \cdot x_i^{\prime s} = k^{s1} \geq 0; \\
... \\
p^{*s} \cdot x_i^{\prime s-1} - p^{*s} \cdot x_i^{\prime s} = k^{s,s-1} \geq 0.
\end{array}\right.$$

We will find an x_i that is a neighbor of x_i' and belongs to $C_i(p^j)$ (contradicting the fact that x_i^j is individually optimal at prices p^j). This proves (4) by contradiction.

Case 1: All inequalities are such that $k^{st} > 0$.

With x_i such that $d(x_i, x_i') < \epsilon$, denote $dx_i = x_i - x_i'$ and $dp^j = p^j - p^*$. Manipulating a deliverability condition:

$$p^{*s} \cdot x_i'^t - p^{*s} \cdot x_i'^s = (p^{js} - dp^{js}) \cdot (x_i^t - dx_i^t) - (p^{js} - dp^{js}) \cdot (x_i^s - dx_i^s) = k^{st} \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s = k^{st} + p^{js} \cdot dx_i^t + dp^s \cdot (x_i^t - dx_i^t) - p^{js} \cdot dx_i^s - dp^s \cdot (x_i^s - dx_i^s) \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s > k^{st} - \epsilon - \epsilon(\|e_T\| + \epsilon) - \epsilon - \epsilon(\|e_T\| + \epsilon) \Leftrightarrow$$

$$\Leftrightarrow p^{js} \cdot x_i^t - p^{js} \cdot x_i^s > k^{st} - 2\epsilon(\|e_T\| + 1 - \epsilon).$$

Let $k^{min} = \min\limits_{t \in P_i(s)} k^{st}$. Choose a smaller $\epsilon > 0$, if necessary, to make $2\epsilon(\|e_T\| + 1 - \epsilon) < k^{min}$. This guarantees that the strict inequalities for x_i' and p^* remain strict for any x_i s.t. $d(x_i, x_i') < \epsilon$, under p^j with $j \geq j_2$.

There is no utility penalty, therefore, $U_i^j(x_i) > U_i^j(x_i^j)$. Contradiction. The consumption plan in the equilibrium sequence, x_i^j, is not a maximizer of U_i^j.

Case 2: For every $t \in P_i(s)$, prices p^{*s} and p^{*t} are not parallel.

The difference relative to *Case 1* lies in checking that the inequalities which are not strict at (x_i', p^*) are still satisfied at (x_i, p^j), for high j. The inequalities that are not strict are those for which $k^{st} = 0$.

Let $\gamma^{st} = \left(1 - \dfrac{p^{*s} \cdot p^{*t}}{\|p^{*s}\|\|p^{*t}\|}\right)\|p^{*s}\|$ and $\gamma^{min} = \min\limits_{t \in P_i(s)} \gamma^{st} > 0$.

Define k^{min} as the minimum among the strictly positive k^{st} and (as in *Case 1*) choose $\epsilon > 0$ such that $2\epsilon(\|e_T\| + 1 - \epsilon) < k^{min}$, to preserve the strict inequalities.

Select displacements from x_i' to x_i that are parallel to p^*, choosing x_i such that: $dx_i^s = -\frac{\epsilon}{2}\frac{p^{*s}}{\|p^{*s}\|}$.

Let $\epsilon_2 = \frac{\epsilon\gamma^{min}}{8\|e_T\|}$, and consider $j_3 \geq j_2$ that is high enough for: $d(p^j, p^*) < \min\{\epsilon_2, \epsilon\}, \forall j \geq j_3$.

Consider an inequality that is not strict for p^* and x_i' ($k^{ab} = 0$). Let's verify that this deliverability condition still holds for p^j and x_i:

$$p^{ja} \cdot x_i^b - p^{ja} \cdot x_i^a = (p^{*a} + dp^{ja}) \cdot (x_i'^b + dx_i^b) - (p^{*a} + dp^{ja}) \cdot (x_i'^a + dx_i^a) =$$

$$= p^{*a} \cdot (x_i'^b + dx_i^b) + dp^{ja} \cdot (x_i'^b + dx_i^b) - p^{*a} \cdot (x_i'^a + dx_i^a) - dp^{ja} \cdot (x_i'^a + dx_i^a) =$$

$$= p^{*a} \cdot dx_i^b + dp^{ja} \cdot (x_i'^b + dx_i^b) - p^{*a} \cdot dx_i^a - dp^{ja} \cdot (x_i'^a + dx_i^a) >$$

$$> p^{*a} \cdot dx_i^b - \epsilon_2(\|e_T\| + \epsilon) - p^{*a} \cdot dx_i^a - \epsilon_2(\|e_T\| + \epsilon) =$$

$$= p^{*a} \cdot dx_i^b - p^{*a} \cdot dx_i^a - 2\epsilon_2(\|e_T\| + \epsilon) >$$

$$> -p^{*a} \cdot \tfrac{\epsilon}{2}\tfrac{p^{*b}}{\|p^{*b}\|} + p^{*a} \cdot \tfrac{\epsilon}{2}\tfrac{p^{*a}}{\|p^{*a}\|} - 4\epsilon_2\|e_T\| =$$

$$= \frac{\epsilon}{2}\frac{p^{*a}\cdot p^{*a}}{\|p^{*a}\|\|p^{*a}\|}\|p^{*a}\| - \frac{\epsilon}{2}\frac{p^{*a}\cdot p^{*b}}{\|p^{*a}\|\|p^{*b}\|}\|p^{*a}\| - \frac{\epsilon}{2}\gamma^{min} =$$

$$= \frac{\epsilon}{2}\gamma^{ab} - \frac{\epsilon}{2}\gamma^{min} \geq 0$$

In sum: $p^{ja} \cdot x_i^b - p^{ja} \cdot x_i^a > 0$. The deliverability condition is verified, and thus $U_i^j(x_i) > U_i^j(x_i^j)$. Contradiction.

Case 3: Prices p^{*s} and p^{*t} are parallel.

The same displacement as in *Case 2*, $dx_i^s = -\frac{\epsilon}{2}\frac{p^{*s}}{\|p^{*s}\|}$, is good for the case in which prices p^{*a} and p^{*b} are parallel. In this case: $x_i'^a = x_i'^b$ and also $dx_i^a = dx_i^b$. Hence, $x_i^a = x_i^b$ and the conditions remain satisfied in equality.

All deliverability conditions are satisfied, therefore: $U_i^j(x_i) = U_i(x_i) > U_i(x_i^j) \geq U_i^j(x_i^j)$. The consumption plan x_i^j does not maximize U_i^j, because x_i is preferred. This contradiction proves (4).

QED

Appendix 2: Example of non-existence of equilibrium

Consider an economy in which two agents trade a single good under uncertainty. There are three states of nature, and the endowments depend on the state of nature:

$e_A = (100, 100, 1)$ and $e_B = (1, 100, 100)$.

Agents only observe their endowments:

$P_A = \{\{1, 2\}; \{3\}\}$ and $P_B = \{\{1\}; \{2, 3\}\}$.

The different states occur with objective and publicly known probabilities:

$\mu = (\mu^1, \mu^2, \mu^3) = (0.45, 0.1, 0.45)$.

Risk aversion induces agents to trade *ex ante*, in order to maximize expected utility:

$$U_i(x_i) = \sum_{s=1}^{S} \mu^s \sqrt{x_i^s}.$$

Prices in states 1 and 3 must be strictly positive, or else the demands of agent B and A would be infinite for the corresponding contingent goods.

With strictly positive prices for all the contingent goods, if agents selected different consumption levels in states that they did not distinguish, then they would end up receiving the cheapest of the alternatives, which would be the one with the lowest consumption level. In this case, we must have:

$$x_A = (x_A^{12}, x_A^{12}, x_A^3) \text{ and } x_B = (x_B^1, x_B^{23}, x_B^{23}).$$

Since agents are at the frontier of their budget sets:

$$\begin{cases} (p^1 + p^2)x_A^{12} + p^3 x_A^3 = 100(p^1 + p^2) + p^3; \\ p^1 x_B^1 + (p^2 + p^3)x_B^{23} = p^1 + 100(p^2 + p^3). \end{cases}$$

Adding the two:

$$p^1(x_A^{12} + x_B^1) + p^2(x_A^{12} + x_B^{23}) + p^3(x_A^3 + x_B^{23}) = 101p^1 + 200p^2 + 101p^3.$$

For this to be an equilibrium, the allocation must be feasible:

$$\begin{cases} x_A^{12} + x_B^1 \leq 101; \\ x_A^{12} + x_B^{23} \leq 200; \\ x_A^3 + x_B^{23} \leq 101. \end{cases}$$

With strictly positive prices, the conditions are verified in equality. This implies that the allocation is of the form:

$$\begin{cases} x_A = (x_A^{12}, x_A^{12}, x_A^3) = (x_A^3 + 99, x_A^3 + 99, x_A^3); \\ x_B = (x_B^1, x_B^{23}, x_B^{23}) = (x_B^1, x_B^1 + 99, x_B^1 + 99). \end{cases}$$

The only individually rational allocation of this form corresponds to the initial endowments. There is no trade. But are agents maximizing their utility levels?

$$\begin{cases} x_A = (100, 100, 1); \\ x_B = (1, 100, 100). \end{cases} \Rightarrow \begin{cases} U(x_A) = 0.45 * 10 + 0.1 * 10 + 0.45 * 1 = 5.95; \\ U(x_B) = 0.45 * 1 + 0.1 * 10 + 0.45 * 10 = 5.95. \end{cases}$$

Suppose that $p^1 = p^3$. Agent A can trade consumption in state 1 for consumption in state 3. But consuming less in state 1 implies that delivery in state 2 will also be of this lower quantity. In any case, the agent can select:

$$x_1' = (x_1'^{12}, x_1'^{12}, x_1'^3) = (81, 81, 20).$$

The corresponding utility level is:

$$U(x_1') = 0.45 * 9 + 0.1 * 9 + 0.45 * 4.47 = 6.96.$$

In the case with asymmetric prices ($p^1 \neq p^3$), the same trade is even more favorable for one of the agents. We reached a contradiction, implying that there is no equilibrium with strictly positive prices.

With $p^2 = 0$, an alternative bundle can be big enough to violate feasibility and still be deliverable. The deliverability restriction is not relevant because it is of the form $0 \cdot x^2 \leq 0 \cdot x^s$. Agents can choose a consumption level for state 2 that is big enough to violate feasibility and still desire to increase it. There cannot be a rational expectations equilibrium with $p^2 = 0$.

Appendix 3: The deliverability correspondence

The set of bundles that satisfy the deliverability restrictions depends on the prevailing prices. Consider the correspondence from prices to the set of deliverable bundles:

$$C_i : \Delta^{SL} \longrightarrow \mathbb{R}_+^{SL};$$

$$C_i(p) = \left\{ x \in \mathbb{R}_+^{SL} : \forall s \in \Omega,\ p^s \cdot x^s = \min_{t \in P_i(s)} \{p^s \cdot x^t\} \right\}.$$

If the correspondence $B_i(p) \cap C_i(p)$ were continuous, we could apply Berge's maximum theorem and Kakutani's fixed point theorem to establish existence of equilibrium in economies with uncertain delivery.

Upper hemicontinuity of C_i at p_0 means that, given an arbitrary open set, V, containing $C_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, we have $C_i(p) \subseteq V$.

The correspondence is closed since all the restrictions are inequalities which are not strict. With a compact range, a closed-valued correspondence is upper hemicontinuous if and only if it is closed. Therefore, when restricted to a bounded economy (for example, by the total initial endowments in the economy), C_i is upper hemicontinuous.

Lower hemicontinuity of C_i at p_0 means that given an arbitrary open set, V, intersecting $C_i(p_0)$, there exists $\delta > 0$ such that for all $p \in B(p_0, \delta)$, the image $C_i(p)$ also intersects V.

The correspondence under study, C_i, is not lower hemicontinuous. Lower hemicontinuity fails when prices in are null ($p^s = 0$) or collinear ($p^s = ap^t$).

When prices are null, the deliverability restrictions disappear. It is always true that $0 \cdot x^s \leq 0 \cdot x^t$. But with a small perturbation, the restrictions appear. This is why l.h.c. fails.

When prices are collinear, the failure of l.h.c. is more subtle.

Consider an economy with two goods, A and B, and two states of nature, s and t. Let $p_0 = (p_0^s, p_0^t) = (p_0^{As}, p_0^{Bs}; p_0^{At}, p_0^{Bt}) = (\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4})$. The bundle $x_0 = (1, 0; 0, 1)$ belongs to the deliverable set, since:

$$p_0^s \cdot x_0^s \leq p_0^s \cdot x_0^t \Leftrightarrow \tfrac{1}{4} \leq \tfrac{1}{4}, \text{ and}$$

$$p_0^t \cdot x_0^t \leq p_0^t \cdot x_0^s \Leftrightarrow \tfrac{1}{4} \leq \tfrac{1}{4}.$$

Delivering $(1, 0)$ in state s and $(0, 1)$ in state t does not violate deliverability because both bundles have the same price in both states.

A small perturbation in prices can make $(0, 1)$ cheaper in state s and $(1, 0)$ cheaper in state t. Consider an open ball around x_0 with radius $0 < \epsilon < \frac{1}{10}$. After a perturbation in prices to $p = (\frac{1}{4} + \delta, \frac{1}{4} - \delta, \frac{1}{4} -$

$\delta, \frac{1}{4} + \delta)$, this ball does not intersect the deliverable set.

Suppose that there existed a vector $dx = (\epsilon^{As}, \epsilon^{Bs}, \epsilon^{At}, \epsilon^{Bt})$ such that $x = (1 + \epsilon^{As}, \epsilon^{Bs}; \epsilon^{At}, 1 + \epsilon^{Bt})$ is inside that open ball and belongs to the deliverable set:

$$(1)\ \ (\tfrac{1}{4} + \delta, \tfrac{1}{4} - \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \leq (\tfrac{1}{4} + \delta, \tfrac{1}{4} - \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \Leftrightarrow$$

$$\Leftrightarrow (\tfrac{1}{4} + \delta)(1 + \epsilon^{As}) + (\tfrac{1}{4} - \delta)\epsilon^{Bs} \leq (\tfrac{1}{4} + \delta)\epsilon^{At} + (\tfrac{1}{4} - \delta,)(1 + \epsilon^{Bt}) \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4} + \tfrac{1}{4}\epsilon^{As} + \delta + \delta\epsilon^{As} + \tfrac{1}{4}\epsilon^{Bs} - \delta\epsilon^{Bs} \leq \tfrac{1}{4}\epsilon^{At} + \delta\epsilon^{At} + \tfrac{1}{4} + \tfrac{1}{4}\epsilon^{Bt} - \delta - \delta\epsilon^{Bt} \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{As} + \epsilon^{Bs} - \epsilon^{At} - \epsilon^{Bt}) + \delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta;$$

$$(2)\ \ (\tfrac{1}{4} - \delta, \tfrac{1}{4} + \delta) \cdot (\epsilon^{At}, 1 + \epsilon^{Bt}) \leq (\tfrac{1}{4} - \delta, \tfrac{1}{4} + \delta) \cdot (1 + \epsilon^{As}, \epsilon^{Bs}) \Leftrightarrow$$

$$\Leftrightarrow (\tfrac{1}{4} - \delta)\epsilon^{At} + (\tfrac{1}{4} + \delta)(1 + \epsilon^{Bt}) \leq (\tfrac{1}{4} - \delta)(1 + \epsilon^{As} + (\tfrac{1}{4} + \delta)\epsilon^{Bs}) \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{At} + 1 + \epsilon^{Bt} - 1 - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + 1 + 1 + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{4}(\epsilon^{At} + \epsilon^{Bt} - \epsilon^{As} - \epsilon^{Bs}) + \delta(-\epsilon^{At} + \epsilon^{Bt} + \epsilon^{As} - \epsilon^{Bs}) \leq -2\delta.$$

Adding the two inequalities, we obtain:

$$(1+2)\ \ \delta(\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt}) \leq -2\delta \Leftrightarrow \epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \leq -2.$$

Which is impossible, because $\epsilon^{As} - \epsilon^{Bs} - \epsilon^{At} + \epsilon^{Bt} \geq -4\epsilon > -\frac{4}{10}$.

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