Past expectations and present prices - hysteresis in a simple economy[1]

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Abstract. We give an illustration of hysteresis (path-dependence) in a simple economy. In the presence of multiple possible equilibrium prices, we find that past expectations determine present prices. This phenomenon of path-dependence is robust to perturbations of the economy.

Keywords: Hysteresis, Path-dependence, Tatônnement, Equilibrium selection.

JEL classification: C62, D50.

1 Introduction

It is known that expectations play a key role in economics. Not only as deriving from an objective reality, but also as primary causes. This idea underlies what Keynes (1936) designated as "animal spirits", as well as the distinction that financial analysts find useful between the "bull market" and the "bear market". It is well-known that expectations about the future may influence (present) prices - good examples are the interest rate and the inflation rate. What is not so accepted is that past expectations may also be a determinant of present prices.

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Hysteresis is associated to the permanence of effects from a temporary stimulus and to the idea of path dependence. Such phenomena are at the heart of evolutionary models, but do not appear frequently in mainstream economics. Some exceptions are the studies of international trade by Amable (1995) and Ljungqvist (1994), of investment by Dixit (1992), and of unemployment by Roed (1997).[2]

In most equilibrium models, the concern for uniqueness of equilibrium leaves no room for path-dependence. For example, the results of Arrow, Block and Hurwicz (1959) on stability of equilibrium use conditions that guarantee uniqueness of equilibrium. It is only in the presence of multiple equilibrium prices that the problem of selection arises and path-dependence may become an issue.

To study equilibrium selection, we need to make assumptions on the way market prices are formed. A famous process of reaching equilibrium is based on "the law of demand and supply": if demand exceeds supply, prices rise; and if supply exceeds demand, prices fall. In the lines of Arrow and Hurwicz (1958), we restrict our analysis to a price formation process of this kind, the "instantaneous adjustment process".[3]

We study an economy with two agents, a single consumption good, and two states of nature. There is subjective uncertainty about which of the two possible states of nature will occur, and a problem of risk sharing arises. Agents are subjective expected utility maximizers, having the same state-dependent utility functions but different prior beliefs. When agents have very asymmetric beliefs, the economy exhibits multiple equilibria, and this opens the way for the study of path-dependence.

We know, from Aumann's (1976) classic, that if the agents have the same prior beliefs, they cannot agree to disagree. For agents to have different posteriors (disagree), we need to assume that they have different priors, or, alternatively, to assume that their posteriors are not common information (they are not agreeing). [4]

The beliefs of the agents change during the bargaining process, because of some exogenous signal, and prices adjust instantaneously to reflect the

[2]See Göcke (2002) and Katzner (1999) for surveys.

[3]The same conclusions would be reached if the "lagged adjustment process" was assumed instead (see Arrow and Hurwicz, 1958).

[4]Disagreement is possible for example if the set of states of nature is infinite countable, see Correia-da-Silva (2008).

changes in preferences. This is in the spirit of a strong efficient market hypothesis. When beliefs (preferences) suffer a small perturbation, the economy finds a different equilibrium nearby, as long as it exists. This seems acceptable provided both equilibria are stable, as in our example. The existence of multiple stable equilibria causes the equilibrium price obtained to exhibit path-dependence. Hence, equilibrium prices may depend on past (transitory) expectations, and we prove also that this path-dependence is robust to perturbations of the economy under study.

Our analysis generalizes Bala's (1997) study of a pitchfork bifurcation in the tatônnement process and includes an example by Kehoe (1991) and Berliant and Dakhlia (2002). A symmetry breaking perturbation of the problem leads to more complex configurations of the equilibrium price set. We do what is technically known as a universal unfolding of the pitchfork bifurcation.

The rest of the paper is organized as follows: in section 2 we present the model of the economy; in section 3 we make a qualitative study of price trajectories and equilibrium dynamics; and in section 4 we give a numerical example of hysteresis.

2 The model

Consider a simple economy with two agents, a single good, and two states of nature. Agents trade in contingent goods (Arrow, 1953), with the objective of maximizing subjective expected utility (Savage, 1954).

Denote consumption of agent A in state 1 by x_{A1} and consumption of agent A in state 2 by x_{A2}. Similarly, consumption of agent B is $x_B = (x_{B1}, x_{B2})$. For simplicity, we assume that agents have the same state-specific utility functions, and that these are equal for both states of nature:

$$u_{A1} = -x_{A1}^{1-\mu}, \;\; u_{A2} = -x_{A2}^{1-\mu}, \;\; u_{B1} = -x_{B1}^{1-\mu}, \text{ and } \;\; u_{B2} = -x_{B2}^{1-\mu}.$$

These utility functions imply that agents have a constant relative risk aversion equal to μ.[5] This parameter is greater than 1, thus preferences are monotonic.

The agents have different, strictly positive, prior beliefs about the probabilities of occurrence of the two states of nature. Agent A assigns a probability

[5] The Arrow-Pratt measure of Relative Risk Aversion is: $RRA = -\frac{xu''(x)}{u'(x)} = \mu$.

$q_{A1} > 0$ to the occurrence of state 1 and $1 - q_{A1}$ to that of state 2. This is represented by a probability vector $q_A = (q_{A1}, 1 - q_{A1})$. Analogously, the prior beliefs of agent B are given by $q_B = (q_{B1}, 1 - q_{B1})$. As a result of this subjective uncertainty, the subjective expected utility functions are different when prior beliefs differ:

$$\begin{aligned} U_A(x_{A1}, x_{A2}) &= -q_{A1} x_{A1}^{1-\mu_A} - (1 - q_{A1}) x_{A2}^{1-\mu_A} \\ U_B(x_{B1}, x_{B2}) &= -q_{B1} x_{B1}^{1-\mu} - (1 - q_{B1}) x_{B2}^{1-\mu}. \end{aligned}$$

Notice that everything applies to a general economy with two goods, since consumption of a single good in two states of nature can also be seen as consumption of two goods.

Let their initial endowments of contingent goods be:

$$e_A = (e_{A1}, e_{A2}) \text{ and } e_B = (0, 1).$$

Solving the maximization problem for both agents, we find that in the market the excess demand for good 1 is (see the Appendix for the detailed calculations using Lagrange Multipliers):

$$z_1(p) = \frac{p e_{A1} + e_{A2}}{p + p^{\frac{1}{\mu}} (\frac{1-q_{A1}}{q_{A1}})^{\frac{1}{\mu}}} + \frac{1}{p + p^{\frac{1}{\mu}} (\frac{1-q_{B1}}{q_{B1}})^{\frac{1}{\mu}}} - e_{A1}. \quad (1)$$

We have normalized $p_2 = 1$ and made $p_1 = p$.

The budget restrictions are obviously active. As suggested by Walras' Law, equilibrium in one market implies equilibrium in the two markets.[6]

We consider the classical tatônnement process or "instantaneous adjustment process" as it is called by Arrow and Hurwicz (1958) and Arrow, Block and Hurwicz (1959). This process is modelled by the differential equation $\dot{p} = z_1(p)$.

The case considered by Bala (1997), which results in a pitchfork bifurcation, is a particular case in which $e_A = (1, 0)$, and $q_{B1} = 1 - q_{A1}$. We call this setup *symmetric*. This means that the probability that agent A attributes to state 1 is equal to the probability that agent B attributes to state 2. Note that, in this case, a high value of q_{A1} corresponds to highly asymmetric beliefs.

[6]It is enough to consider the market for good 1, as excess demand for good 2 is $z_2 = -p z_1$.

Writing $a = \frac{1-q_{A1}}{q_{A1}} = \frac{q_{B1}}{1-q_{B1}}$, we find the excess demand of good 1 to be

$$z_1(p) = \frac{p}{p + p^{\frac{1}{\mu}} a^{\frac{1}{\mu}}} + \frac{1}{p + p^{\frac{1}{\mu}} a^{-\frac{1}{\mu}}} - 1.$$

When q_{A1} varies in $(0, 1)$, a takes all positive values since $a = \frac{1}{q_{A1}} - 1$. As q_{A1} approaches the origin, a tends to infinity. The analogous occurs for q_{B1} and a but, in this instance, a tends to infinity as q_{B1} approaches unity.

2.1 Existence and stability of equilibria

Observe that $p = 1$ is always an equilibrium price:

$$z_1(1) = \frac{1}{1 + a^{\frac{1}{\mu}}} + \frac{1}{1 + a^{-\frac{1}{\mu}}} - 1 = \frac{1 - a^{\frac{1}{\mu}} a^{-\frac{1}{\mu}}}{(1 + a^{\frac{1}{\mu}})(1 + a^{-\frac{1}{\mu}})} = 0.$$

However this equilibrium is not stable for all values of the parameter a. To see this we need to differentiate the excess demand of good 1 with respect to price p:

$$\frac{dz_1(1)}{dp} = \frac{(\mu - 2)a^{(1/\mu)} - \mu a^{(2/\mu)}}{\mu(1 + a^{(1/\mu)})^2}.$$

When the sign of $\frac{dz_1(1)}{dp}$ is negative we know that the law of demand holds, or, equivalently, that the equilibrium is stable. This sign is given by that of the numerator and is negative provided that

$$a < a^* = \left(\frac{\mu - 2}{\mu}\right)^{\mu}.$$

Note that for $a = a^*$ the stability of the equilibrium price $p = 1$ is not determined by the sign of $\frac{dz_1(1)}{dp}$ as this is zero. We say that a bifurcation occurs for this value of a and hence, use a as a bifurcation parameter in what follows.

The value found for a^* above only makes sense for $\mu \geq 2$. For $\mu < 2$, a^* is either negative or does not exist. In either case, the sign of $\frac{dz_1(1)}{dp}$ remains constant and negative ensuring the validity of the law of demand. The law of demand is not satisfied when $a > a^*$ for $\mu \geq 2$. Since for $\mu = 2$ the bifurcation occurs for $a = 0$ and the parameter a only takes positive values, we consider henceforth only value of $\mu > 2$.

We also have

$$\frac{\partial^2 z_1}{\partial a \partial p}(1, a^*) < 0, \quad \frac{\partial^2 z_1}{\partial p^2}(1, a^*) = 0 \text{ and } \frac{\partial^3 z_1}{\partial p^3}(1, a^*) < 0 \;\; \forall \mu > 2.$$

Hence, from Golubitsky and Schaeffer (1985, Prop. II, 9.2) we know that, near $p = 1$ and $a = a^*$, the dynamics of the tatônnement are qualitatively equivalent to those of a subcritical pitchfork, for all values of $\mu > 2$ (see figure 1). Qualitative equivalence means that there is a coordinate system in which these dynamics are described by $\dot{x} = x^3 + \lambda x$. It is not our concern to establish the change of coordinates that takes equation (1) into this form. We are satisfied with describing the consequences of the qualitative behavious of solutions to (1) and for that it suffices that these are qualitatively equivalent to those of a pitchfork. We simply remark that the change of coordinates involved may be chosen so that the bifurcation occurs for the same parameter value.

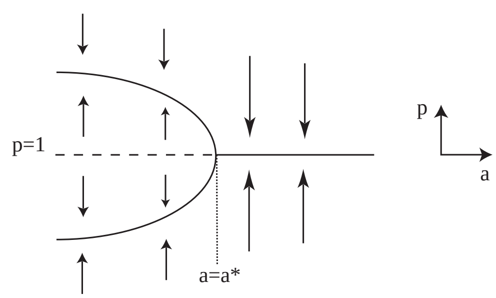

Figure 1: The pitchfork occurs for $a^* = (\frac{\mu-2}{\mu})^\mu$. Stability of branches is indicated both by solid lines and by arrows representing price movements.

It is worthwhile describing the situations which support the existence of a pitchfork bifurcation. When μ approaches infinity, a^* tends to, and remains below, $1/e^2$. The dependence of a^* on μ is drawn in figure 2. When μ tends to infinity, a^* corresponds to a belief $q_{A1} = \frac{1}{1+e^2} \simeq 0.88$. The line of stability change corresponds to parameter values for which a pitchfork bifurcation occurs.

We observe from figure 2 that for small values of a, two stable equilibrium prices exist and choice between these will be an issue for the dynamics. In terms of beliefs, such values of a correspond to q_{A1} close to unity, and therefore q_{B1} close to the origin. Hence, agents have very asymmetric beliefs. We shall return to this issue later. On the other hand, if agents A and B have similar beliefs regarding state 1 and state 2, respectively, then the Law of Demand holds and the equilibrium price is uniquely determined.

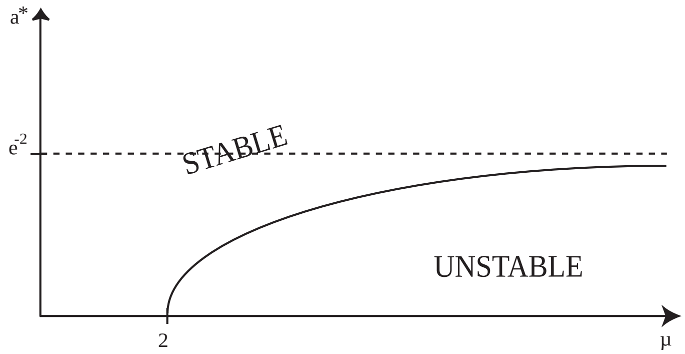

Figure 2: The stability of the equilibrium price $p = 1$ varies with μ and a.

2.2 Robustness of the model

The particular case described in the previous subsection is not robust to perturbations in the initial endowments or to perturbations that make the beliefs of agents independent from each other. These perturbations correspond to breaking the symmetry of the setup of the previous subsection. The symmetry is described by the fact that endowments are $e_A = (1, 0)$ and $e_B = (0, 1)$, and that beliefs are $q_A = (\frac{1}{1+a}, \frac{a}{1+a})$ and $q_B = (\frac{a}{1+a}, \frac{1}{1+a})$. This means that each agent possesses only one good (not the same) and that the beliefs are symmetric with respect to each good. The pitchfork bifurcation described is robust only under perturbations that do not destroy this symmetry.

Let us consider two new parameters, ε and $\theta \geq 0$, and use them to define generic non-symmetric endowments and beliefs by writing $e_A = (1, \theta)$ and $q_A = (\frac{1}{1+a-\varepsilon}, \frac{a-\varepsilon}{1+a-\varepsilon})$, maintaining the values for e_B and q_B. That is, agent A now has both goods and the probability assigned by agent A to the occurrence of state 1 no longer is equal to that assigned by agent B to state 2. The tatônnement process defined by equation (1) describing the original economy is now transformed into $\dot{p} = Z_1(p)$ where

$$Z_1(p) = \frac{p + \theta}{p + p^{1/\mu}(a - \varepsilon)^{1/\mu}} + \frac{1}{p + p^{1/\mu}a^{-1/\mu}} - 1. \tag{2}$$

It may be confirmed that, for $\mu > 2$, our economy (2) is an *universal unfolding* of the symmetric problem, that is, it includes all the perturbations, in endowments and beliefs, using as few parameters as possible. To see this, construct a matrix of partial derivatives as required to apply Golubitsky and

Schaeffer (1985, Prop. III, 4.4). This is given by

$$\begin{pmatrix} 0 & 0 & \frac{\partial^2 z_1}{\partial p \partial a} & \frac{\partial^3 z_1}{\partial p^3} \\ 0 & \frac{\partial^2 z_1}{\partial a \partial p} & \frac{\partial^2 z_1}{\partial a^2} & \frac{\partial^3 z_1}{\partial a \partial p^2} \\ \frac{\partial Z_1}{\partial \varepsilon} & \frac{\partial^2 Z_1}{\partial \varepsilon \partial p} & \frac{\partial^2 Z_1}{\partial \varepsilon \partial a} & \frac{\partial^3 Z_1}{\partial \varepsilon \partial p^2} \\ \frac{\partial Z_1}{\partial \theta} & \frac{\partial^2 Z_1}{\partial \theta \partial p} & \frac{\partial^2 Z_1}{\partial \theta \partial a} & \frac{\partial^3 Z_1}{\partial \theta \partial p^2} \end{pmatrix}$$

and, for $p = 1$, $a = a^*$ and $\theta = \varepsilon = 0$, has non-zero determinant. The determinant was calculated using Maple. It is easily checked, using Maple, that the determinant (a long expression depending on μ) is always positive for the range of parameters we are considering. For example, substituting $\mu = 5$ as we shall do in section 4 the determinant equals $1220703125/41278242816 \neq 0$. Therefore, the perturbations introduced by the unfolding parameters ε and θ are robust. This remains true if, instead of perturbing only agent A, we perturb either only agent B or both agents. The crucial point is that we must perturb both endowments and beliefs.

3 Robust Bifurcation Diagrams and Hysteresis

In this section, we describe the bifurcation diagrams that can be found in a robust way (i.e., persistent under additional perturbation) in the tatônnement defined by equation (2) and interpret the dynamics of the solutions. The introduction of the new parameters ε and θ allows us to perturb the bifurcation diagram of figure 1. This is recovered when both ε and θ are zero. From standard bifurcation theory (cf. Golubitsky and Schaeffer (1985) and Golubitsky *et al.* (1988)), the bifurcation diagrams present in a universal unfolding of a pitchfork such as equation (2) are those drawn in figure 3. These are presented in convenient coordinates but the stability and qualitative behaviour of the solutions may be made independent of the coordinate changes.

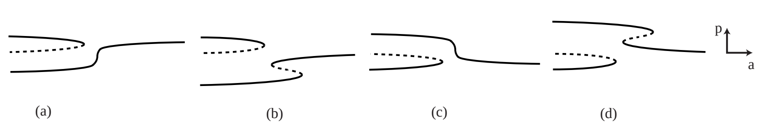

Figure 3: Bifurcation diagrams found in a universal unfolding of a pitchfork. We focus on case (b).

Varying the unfolding parameters ε and θ takes us through all the possible bifurcation diagrams. For our parameter range ($\theta \geq 0$), we do not observe bifurcation diagram (d). For an example where diagrams (a) and (c) occur see Dakhlia (1999, figures 5 and 6). The transition between the diagrams occurs when either a line of bifurcation, $\mathcal{B}$, or a line of hysteresis, $\mathcal{H}$, is crossed. These lines may be computed numerically since they are described by a set of equations as follows (the index p or a indicates a derivative with respect to that variable):

$$\mathcal{B} = \{(\varepsilon, \theta) \in \mathbf{R}^2 : \exists (p, a) : Z_1 = Z_{1p} = Z_{1a} = 0\}$$

and

$$\mathcal{H} = \{(\varepsilon, \theta) \in \mathbf{R}^2 : \exists (p, a) : Z_1 = Z_{1p} = Z_{1pp} = 0\}.$$

Transition between (a) and (b) corresponds to crossing a line of hysteresis. The bifurcation diagram in (b) in figure 3 exhibits hysteresis since a variation in the bifurcation parameter a produces permanent effects in the equilibrium price p. Hence, the value of past expectations may condition present prices. We develop this idea further by taking a closer look at bifurcation diagram (b) with the aid of figure 4.

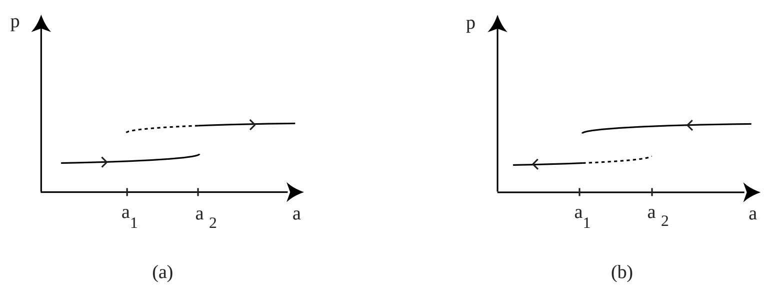

Figure 4: The equilibrium price changes at a_1 if a decreases and at a_2 if a increases.

From figure 4 we see that a high initial value of a sustains a high price for longer than a low initial value. In this instance, if the present expectation is between a_1 and a_2 then the present price will depend on whether this present expectation was reached through an increase or a decrease in expectations. note also that a decrease in a corresponds to an increase in the belief by agent A that state $A1$ will occur. Hence, an increase in this belief supports a higher present price for good 1.

4 An Illustration

For concreteness, consider $\mu = 5$ and $\theta = 0.1$. Let expectations about the future state of nature be extremely asymmetric: use $q_A = (93.5\%, 6.5\%)$ and $q_B = (1.5\%, 98.5\%)$, which is approximately equivalent to $a = 0.0152$ and $\epsilon = -0.0543$.

Solving for equilibrium prices, we obtain three solutions:

$$\check{p} = 0.282, \quad \bar{p} = 1.873 \quad \text{and} \quad \hat{p} = 5.045.$$

The intermediate solution ($\bar{p} = 1.873$) is unstable. This means that the law of demand is not satisfied: a small increase (decrease) in demand is amplified by the effect of decreasing (increasing) prices. Prices tend to move either to $\check{p} = 0.282$ or to $\hat{p} = 5.045$.

Recall that to select equilibrium allocations and prices from the multiple possibilities, we assume that the adjustment process follows continuous trajectories, according to the "instantaneous adjustment process". Notice that $\bar{p} = 1.873$ determines the basin of attraction of the remaining equilibria. In fact, $p < \bar{p}$ will converge to $\check{p} = 0.282$ whereas $p > \bar{p}$ will converge to $\hat{p} = 5.045$. See figure 5.

Let us assume two different past expectations that differ from those above only in parameter a. Recall that this parameter determines, essentially, the degree of asymmetry between the beliefs of the two agents. We choose $\hat{a} = 0.02041$ in order to get $\hat{q}_A = (93.05\%, 6.95\%)$ and $\hat{q}_B = (2\%, 98\%)$; and $\check{a} = 0.01010$ for agent's beliefs to be $\check{q}_A = (6.05\%, 93.95\%)$ and $\check{q}_B = (1\%, 99\%)$.

In both cases, equilibrium is unique: $\hat{p}_0 = 5.979$ and $\check{p}_0 = 0.137$, respectively. If expectations change from $\hat{a}$ to a prices adjust to $\hat{p} = 5.045$. On the other hand, if expectations change from $\check{a}$ to a, prices adjust to $\check{p} = 0.282$.

This example illustrates the behaviour depicted in figure 4. The economy exhibits hysteresis (path-dependence). Past expectations are a determinant of equilibrium prices.

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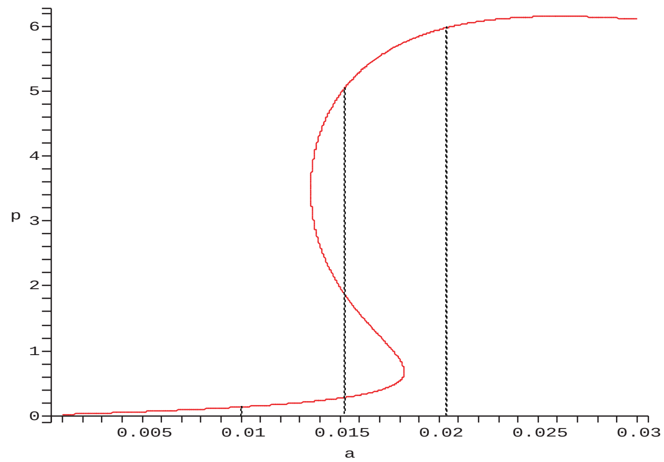

Figure 5: Equilibrium prices for different expectations: at expectations given by $a = 0.0152$, there are three equilibrium prices, $\check{p}$, $\bar{p}$ and $\hat{p}$. At expectations given by $\hat{a} = 0.02041$ and $\check{a} = 0.01010$, there is a unique equilibrium price. If expectations change from $\check{a}$ to a, then prices adjust to $\check{p}$. If expectations change from $\hat{a}$ to a, then prices adjust to $\hat{p}$.

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References

Amable, B., J. Henry, F. Lordon and R. Topol: Weak and strong hysteresis: an application to foreign trade. Economic Notes **24** (2), 353-374 (1995).

Arrow, K.J: The Role of Securities in the Optimal Allocation of Risk-Bearing. Econometrie, translated and reprinted in 1964, Review of Economic Studies, Vol. 31, pp. 91-96 (1953).

Arrow, K.J. and L. Hurwicz: On the Stability of Competitive Equilibrium, I. Econometrica **26**, 522-552 (1958).

Arrow, K.J., H.D. Block and L. Hurwicz: On the Stability of Competitive Equilibrium, II. Econometrica **27**, 82-109 (1959).

Aumann, R. J.: Agreeing to disagree. Annals of Statistics, **4**, 1236-1239 (1976).

Bala, V.: A pitchfork bifurcation in the tatônnement process. Economic Theory **10**, 521-530 (1997).

Berliant, M. and S. Dakhlia: Sensitivity analysis for applied general equilibrium models in the presence of multiple Walrasian equilibria. Economic Theory **19**, 459-476 (2002).

Correia-da-Silva, J.: Agreeing to disagree in a countable space of equiprobable states. FEP-Working Paper 260, January 2008.

Dakhlia, S.: Testing for a unique equilibrium in applied general equilibrium models. Journal of Economic Dynamics and Control **23**, 1281-1297 (1999).

Dixit, A.: Investment and Hysteresis. Journal of Economic Perspectives **6** (1), 107-132 (1992).

Göcke, M.: Various concepts of hysteresis applied in economics. Journal of Economic Surveys **16** (2), 167-188 (2002).

Golubitsky, M. and D. Schaeffer: Singularities and groups in bifurcation theory, vol. I. Springer, New York (1985).

Golubitsky, M., I. Stewart and D. Schaeffer: Singularities and groups in bifurcation theory, vol. II. Springer, New York (1988).

Katzner, D.W.: Hysteresis and the modeling of economic phenomena. Review of Political Economy **11** (2), 171-181 (1999).

Kehoe, T. J.: Computation and multiplicity of equilibria. Handbook of Mathematical Economics, vol. IV. Hildenbrand, W. and H. Sonnenschein (eds.), Elsevier, Amsterdam (1991).

Keynes, J.M.: The General Theory of Employment, Interest and Money. Macmillan, London (1936).

Ljungqvist, L.: Hysteresis in international trade: a general equilibrium analysis. Journal of International Money and Finance **13**, 387-399 (1994).

Roed, K.: Hysteresis in Unemployment. Journal of Economic Surveys **11** (4), 389-418 (1997).

Savage, L. J.: Foundations of Statistics. Wiley, New York (1954).

Appendix

In this appendix we derive equation (1) in section 2. The agents' subjective expected utility functions are:

$$\begin{aligned} U_A(x_{A1}, x_{A2}) &= -q_{A1}x_{A1}^{1-\mu_A} - (1-q_{A1})x_{A2}^{1-\mu_A} \\ U_B(x_{B1}, x_{B2}) &= -q_{B1}x_{B1}^{1-\mu} - (1-q_{B1})x_{B2}^{1-\mu}. \end{aligned}$$

Given the initial endowments and normalizing prices to $p_1 = p$ and $p_2 = 1$, we can set up the Lagrangian to solve for the demand of good 1:

$$\begin{aligned} \mathcal{L}_A &= -q_{A1}x_{A1}^{1-\mu} - (1-q_{A1})x_{A2}^{1-\mu} - \lambda_A(px_{A1} + x_{A2} - pe_{A1} - e_{A2}); \\ \mathcal{L}_B &= -q_{B1}x_{B1}^{1-\mu} - (1-q_{B1})x_{B2}^{1-\mu} - \lambda_B(px_{B1} + x_{B2} - 1). \end{aligned}$$

First order conditions are:

$$\begin{cases} \frac{\partial \mathcal{L}_A}{\partial x_{A1}} = -q_{A1}(1-\mu)x_{A1}^{-\mu} - \lambda_A p = 0 \\ \frac{\partial \mathcal{L}_A}{\partial x_{A2}} = -(1-q_{A1})(1-\mu)x_{A2}^{-\mu} - \lambda_A = 0 \\ px_{A1} + x_{A2} - pe_{A1} - e_{A2} = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_A p = q_{A1}(1-\mu)x_{A1}^{-\mu} \\ \lambda_A = (1-q_{A1})(1-\mu)x_{A2}^{-\mu} \\ px_{A1} + x_{A2} = pe_{A1} + e_{A2} \end{cases}$$

From this set of equations, demand for good 1 can be obtained. The price can be obtained by dividing the first two equations and from it the quantities demanded by agent A can be obtained as follows:

$$p = \frac{q_{A1}}{1-q_{A1}}\left(\frac{x_{A1}}{x_{A2}}\right)^{-\mu} \Leftrightarrow \left(\frac{x_{A2}}{x_{A1}}\right)^{-\mu} = \frac{1}{p}\frac{q_{A1}}{1-q_{A1}} \Leftrightarrow x_{A2} = p^{\frac{1}{\mu}}\left(\frac{1-q_{A1}}{q_{A1}}\right)^{\frac{1}{\mu}}x_{A1}.$$

Replacing this value in the restriction (third equation) we obtain the quantity of good 1 demanded by agent A:

$$\begin{aligned} px_{A1} + x_{A2} = pe_{A1} + e_{A2} \quad &\Rightarrow \quad x_{A1}\left[p + p^{\frac{1}{\mu}}\left(\frac{1-q_{A1}}{q_{A1}}\right)^{\frac{1}{\mu}}\right] = pe_{A1} + e_{A2} \Rightarrow \\ &\Rightarrow \quad x_{A1} = \frac{pe_{A1} + e_{A2}}{p + p^{\frac{1}{\mu}}\left(\frac{1-q_{A1}}{q_{A1}}\right)^{\frac{1}{\mu}}}. \end{aligned}$$

Similarly, for agent B, we obtain:

$$x_{B1} = \frac{pe_{B1} + e_{B2}}{p + p^{\frac{1}{\mu}}\left(\frac{1-q_{B1}}{q_{B1}}\right)^{\frac{1}{\mu}}} = \frac{1}{p + p^{\frac{1}{\mu}}\left(\frac{1-q_{B1}}{q_{B1}}\right)^{\frac{1}{\mu}}}.$$

In the market, excess demand for good 1 is:

$$z_1(p) = x_{A1} + x_{B1} - e_{A1} - e_{B1} = x_{A1} + x_{B1} - e_{A1}.$$

Substituting the quantities above in this equation we obtain equation (1).